Module 9: Tries and String Matching

CS 240 - Data Structures and Data Management

Sajed Haque Veronika Irvine Taylor Smith
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Pattern Matching

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The **text** (or **haystack**) being searched within
- $P[0..m-1]$ – The **pattern** (or **needle**) being searched for
- Strings over **alphabet** Σ
- Return the first i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \le j \le m-1$$

- This is the first **occurrence** of P in T
- If P does not **occur** in T, return `FAIL`
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching

Example:

- $T =$ "Where is he?"
- $P_1 =$ "he"
- $P_2 =$ "who"

Definitions:

- **Substring** $T[i..j]$ $0 \le i \le j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \dots T[j]$ in order
- A **prefix** of T:
  a substring $T[0..i]$ of T for some $0 \le i < n$
- A **suffix** of T:
  a substring $T[i..n-1]$ of T for some $0 \le i \le n-1$

General Idea of Algorithms

Pattern matching algorithms consist of **guesses** and **checks**:

- A **guess** is a position i such that P might start at $T[i]$.
  Valid guesses (initially) are $0 \le i \le n - m$.
- A **check** of a guess is a single position j with $0 \le j < m$ where we compare $T[i+j]$ to $P[j]$. We must perform m checks of a single **correct** guess, but may make (many) fewer checks of an **incorrect** guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.

```
BruteforcePM(T[0..n − 1], P[0..m − 1])
T: String of length n (text), P: String of length m (pattern)
for i ← 0 to n − m do
    match ← true
    j ← 0
    while j < m and match do
        if T[i + j] = P[j] then
            j ← j + 1
        else
            match ← false
    if match then
        return i
return FAIL
```


Example

- Example: T = abbbababbab, P = abba

a	b	b	b	a	b	a	b	b	a	b
a	b	b	**a**							
	a									
		a								
			a							
				a	b	**b**				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b$, $T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- $m \le n/2 \Rightarrow \Theta(mn)$

Pattern Matching

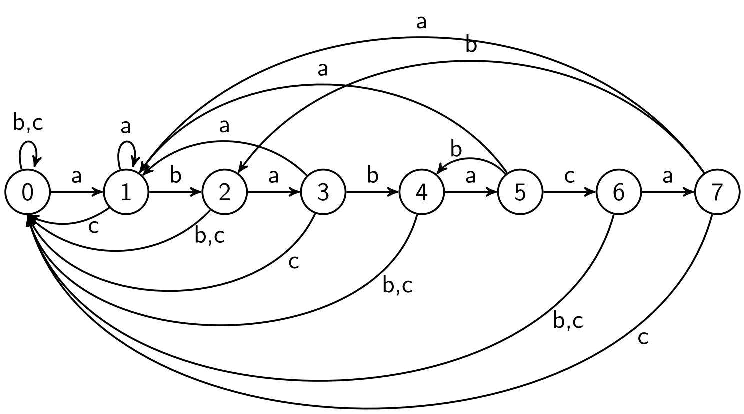
More sophisticated algorithms

- Deterministic finite automata (**DFA**)
- **KMP**, **Boyer-Moore** and **Rabin-Karp**
- Do extra **preprocessing** on the pattern P
- We **eliminate guesses** based on completed matches and mismatches.

String matching with finite automata

There is a string-matching automaton for every pattern P. It is constructed from the pattern in a preprocessing step before it can be used to search the text string.

Example: Automaton for the pattern $P =$ `ababaca`



String matching with finite automata

Let P the pattern to search, of length m. Then

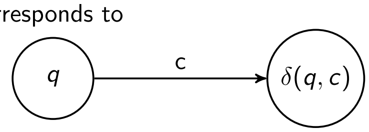
- the *states* of the automaton are $0, \ldots, m$
- the *transition function* δ of the automaton is defined as follows, for a state q and a character c in Σ:

$$\delta(q, c) = \ell(P[0..q-1]c),$$

where

- $P[0..q-1]c$ is the concatenation of $P[0..q-1]$ and c
- for a string s, $\ell(s) \in \{0, \ldots, m\}$ is *the length of the longest prefix of P that is also a suffix of s*.

Graphically, this corresponds to



String matching with finite automata

Let T be the text string of length n,
P the pattern to search of length m and
δ the transition function of a finite automaton for pattern P.

FINITE-AUTOMATON-MATCHER(T, δ, m)
  $n \leftarrow length[T]$
  $q \leftarrow 0$
  for $i \leftarrow 0$ to $n-1$ do
    $q \leftarrow \delta(q, T[i])$
    if $q = m$
      then print "Pattern occurs with shift" $i - (m-1)$

Idea of proof: the state after reading $T[i]$ is $\ell(T[0..i])$.

String matching with finite automata

- Matching time on a text string of length n is $\Theta(n)$
- This does not include the preprocessing time required to compute the transition function δ. There exists an algorithm with $O(m|\Sigma|)$ preprocessing time.
- Altogether, we can find all occurrences of a length-m pattern in a length-n text over a finite alphabet Σ with $O(m|\Sigma|)$ preprocessing time and $\Theta(n)$ matching time.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in **left-to-right**
- **Shifts** the pattern more **intelligently** than the brute-force algorithm
- When a mismatch occurs, how much can we shift the pattern (reusing knowledge from previous matches)?

$T =$ a b c d c a b c ? ? ?

a	b	c	d	c	a	b	a			
					a	b	c	d	c	a

- **KMP Answer**: this depends on the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

KMP Failure Array

T: a b b c a b c d . . .

P: a b b c a b a a

what next slide would match with the text?

KMP Failure Array

Suppose we have a match up to position $T[i-1] = P[j-1]$, but not at the next position.

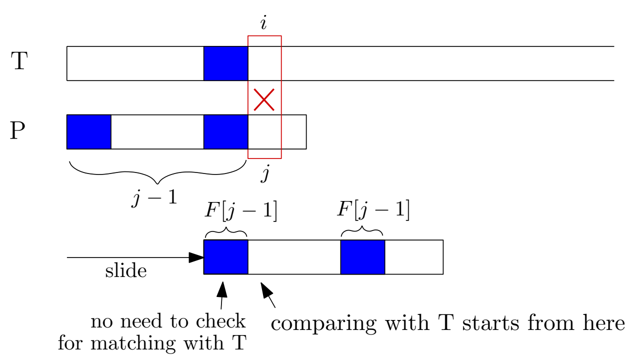
Define $F[j-1]$ as the index we will have to check in P, after we bring the pattern to its next possible position (previous example: $j = 6$, $F[5] = 2$).

This can be computed by trying all sliding positions until finding the first one matching the text (as in previous example). We can do better:

- any possible sliding position corresponds to a prefix of $P[0..j-1]$ that is also a *strict* suffix of it = a suffix of $P[1..j-1]$
- the next possible sliding position corresponds to the **largest** such prefix / suffix
- we let $F[j-1]$ be the length of this prefix / suffix.

KMP Failure Array

Schematically:



KMP Failure Array

- $F[0] = 0$
- $F[j]$, for $j > 0$, is the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- Consider $P =$ abacaba

j	$P[1..j]$	P	$F[j]$
0	—	abacaba	0
1	b	abacaba	0
2	ba	abacaba	1
3	bac	abacaba	0
4	baca	abacaba	1
5	bacab	abacaba	2
6	bacaba	abacaba	3

Computing the Failure Array

```
failureArray(P)
P: String of length m (pattern)
    F[0] ← 0
    i ← 1
    j ← 0
    while i < m do
        if P[i] = P[j] then
            F[i] ← j + 1
            i ← i + 1
            j ← j + 1
        else if j > 0 then
            j ← F[j − 1]
        else
            F[i] ← 0
            i ← i + 1
```


KMP Algorithm

```
KMP(T, P), to return the first match
T: String of length n (text), P: String of length m (pattern)
    F ← failureArray(P)
    i ← 0
    j ← 0
    while i < n do
        if T[i] = P[j] then
            if j = m − 1 then
                return i − j //match
            else
                i ← i + 1
                j ← j + 1
        else
            if j > 0 then
                j ← F[j − 1]
            else
                i ← i + 1
    return −1 // no match
```

KMP: Example

$P =$ `abacaba`

$T =$ `abaxyabacabbaababacaba`

j	0	1	2	3	4	5	6
$F[j]$	0	0	1	0	1	2	3

0	1	2	3	4	5	6	7	8	9	10	11
a	b	a	x	y	a	b	a	c	a	b	b
a	b	a	c								
		(a)	b								
			a								
				a							
					a	b	a	c	a	b	a
									(a)	(b)	a

Exercise: continue with $T =$ `abaxyabacabbaababacaba`

KMP: Analysis

failureArray

- At each iteration of the while loop, at least one of the following happens:
  1. i increases by one, or
  2. the index $i - j$ increases by at least one ($F[j-1] < j$)
- There are no more than $2m$ iterations of the while loop
- Running time: $\Theta(m)$

KMP

- failureArray can be computed in $\Theta(m)$ time
- At each iteration of the while loop, at least one of the following happens:
  1. i increases by one, or
  2. the index $i - j$ increases by at least one ($F[j-1] < j$)
- There are no more than $2n$ iterations of the while loop
- Running time: $\Theta(n)$

Boyer-Moore Algorithm

Based on three key ideas:

- **Reverse-order searching**: Compare P with a subsequence of T moving backwards
- **Bad character jumps**: When a mismatch occurs at $T[i] = c$
  - If P contains c, we can shift P to align the last occurrence of c in P with $T[i]$
  - Otherwise, we can shift P to align $P[0]$ with $T[i+1]$
- **Good suffix jumps**: If we have already matched a suffix of P, then get a mismatch, we can shift P forward to align with the previous occurence of that suffix (with a mismatch from the suffix we read). If none exists, look for the longest prefix of P that is a suffix of what we read. Similar to failure array in KMP.
- Can skip large parts of T

Bad character examples

P = a l d o
T = w h e r e i s w a l d o

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

6 comparisons (checks)

P = m o o r e
T = b o y e r m o o r e

b	o	y	e	r	m	o	o	r	e
				e					
				(r)	**e**				
					(m)	o	o	r	e

7 comparisons (checks)

Good suffix examples

P = sells␣shells

s	h	e	i	l	a	␣	s	e	l	l	s	␣	s	h	e	l	l	s
							h	e	l	l	s							
							s	(e)	(l)	(l)	(s)	␣	s	h	e	l	l	s

P = odetofood

i	l	i	k	e	f	o	o	d	f	r	o	m	m	e	x	i	c	o
				o	f	o	o	d										
				(e)			(o)	(d)		**d**					**d**			

- Good suffix moves further than bad character for 2nd guess.
- Bad character moves further than good suffix for 3rd guess.
- This is out of range, so **pattern not found**.

Last-Occurrence Function

- **Preprocess** the pattern P and the alphabet Σ
- Build the **last-occurrence function** L mapping Σ to integers
- $L(c)$ is defined as
  - the largest index i such that $P[i] = c$ or
  - -1 if no such index exists
- Example: $\Sigma = \{a, b, c, d\}, P = abacab$

c	a	b	c	d
$L(c)$	4	5	3	-1

- The last-occurrence function can be computed in time $O(m + |\Sigma|)$
- In practice, L is stored in a size-$|\Sigma|$ array.

Good Suffix array

- Again, we **preprocess** P to build a table.
- **Suffix skip array** S of size m: for $0 \leq i < m$, $S[i]$ is the largest index j such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ **and** $P[j] \neq P[i]$.
- **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.
- Computed similarly to KMP failure array in $\Theta(m)$ time.

Good Suffix array

Example: $P =$ bon**obobo**

i	0	1	2	3	4	5	6	7
$P[i]$	b	o	n	o	b	o	b	o
$S[i]$	-6	-5	-4	-3	2	-1	2	6

- Computed similarly to KMP failure array in $\Theta(m)$ time.

Boyer-Moore Algorithm

```
boyer-moore(T,P)
1.  L ← last occurrence array computed from P
2.  S ← good suffix array computed from P
3.  i ← m − 1,     j ← m − 1
4.  while i < n and j ≥ 0 do
5.      if T[i] = P[j] then
6.          i ← i − 1
7.          j ← j − 1
8.      else
9.          i ← i + m − 1 − min(L[T[i]], S[j])
10.         j ← m − 1
11. if j = −1 return i + 1
12. else return FAIL
```

Exercise: Prove that $i - j$ always increases on lines 9–10.

Boyer-Moore algorithm conclusion

- Worst-case running time $\in O(n + |\Sigma|)$
- This complexity is difficult to prove.
- Worst-case running time $O(nm)$ if we want to report all occurrences
- On typical **English text** the algorithm probes approximately 25% of the characters in T
- Faster than KMP in practice on English text.

Rabin-Karp Fingerprint Algorithm

Idea: use hashing

- Compute hash function for each text position
- No explicit hash table: just compare with pattern hash
- If a match of the hash value of the pattern and a text position found, then compares the pattern with the substring by naive approach

Example:

Hash "table" size = 97
Search Pattern P: 5 9 2 6 5
Search Text T: 3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6
Hash function: $h(x) = x \bmod 97$ and $h(P) = 95$.
31415 mod 97 = 84
14159 mod 97 = 94
41592 mod 97 = 76
15926 mod 97 = 18
59265 mod 97 = 95

Rabin-Karp Fingerprint Algorithm

Guaranteeing correctness

- Need full compare on hash match to guard against collisions
  - 59265 mod 97 = 95
  - 59362 mod 97 = 95

Running time

- Hash function depends on m characters
- Running time is $\Theta(mn)$ for search miss (how can we fix this?)

Rabin-Karp Fingerprint Algorithm

The initial hashes are called **fingerprints**.
Rabin & Karp discovered a way to update these fingerprints in constant time.

Idea:
To go from the hash of a substring in the text string to the next hash value only requires constant time.

- Use previous hash to compute next hash
- $O(1)$ time per hash, except first one

Rabin-Karp Fingerprint Algorithm

Example:

- Pre-compute: 10000 mod 97 = 9
- Previous hash: 41592 mod 97 = 76
- Next hash: 15926 mod 97 = ??

Observation:

$$\begin{aligned} 15926 \bmod 97 &= (41592 - (4 * 10000)) * 10 + 6 \\ &= (76 - (4 * 9)) * 10 + 6 \\ &= 406 \\ &= 18 \end{aligned}$$

Rabin-Karp Fingerprint Algorithm

- Choose table size at random to be huge prime
- Expected running time is $O(m + n)$
- $\Theta(mn)$ worst-case, but this is (unbelievably) unlikely

Main advantage:

- Extends to 2d patterns and other generalizations

Suffix Tries and Suffix Trees

- What if we want to search for **many patterns** P within the same **fixed text** T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.

We will call a trie that stores all suffixes of a text T a **suffix trie**, and the compressed suffix trie of T a **suffix tree**.

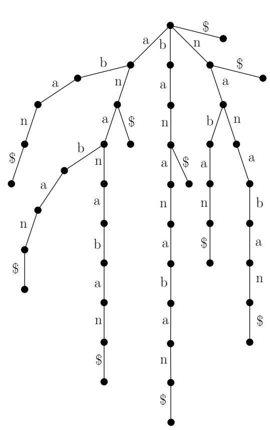
Suffix Trees

- Build the suffix trie, i.e. the trie containing all the suffixes of the text
- Build the suffix tree by compressing the trie above (like in Patricia trees)
- Store two indexes l, r on each node v (both internal nodes and leaves) where node v corresponds to substring $T[l..r]$

Suffix Trie: Example

$T =$ `bananaban`

{`bananaban`, `ananaban`, `nanaban`, `anaban`, `naban`, `aban`, `ban`, `an`, `n`}

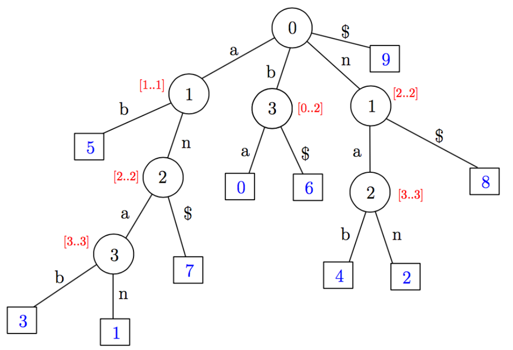


i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree (compressed suffix trie): Example

T =bananaban

{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n}



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Trees: Pattern Matching

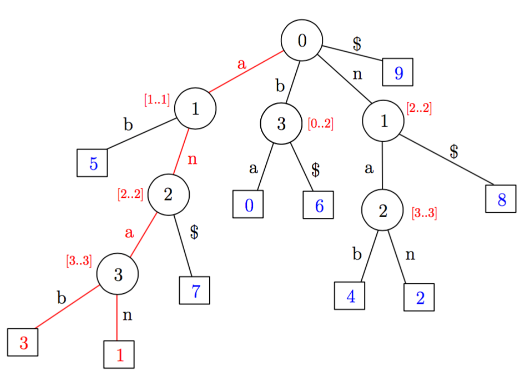
To search for pattern P of length m:

- Similar to Search in compressed trie with the difference that we are looking for a prefix match rather than a complete match
- If we reach a leaf with a corresponding string length less than m, then search is unsuccessful
- Otherwise, we reach a node v (leaf or internal) with a corresponding string length of at least m
- It only suffices to check the first m characters against the substring of the text between indices of the node, to see if there indeed is a match
- We can then visit all children of the node to report all matches

Suffix Tree: Example

T = bananaban

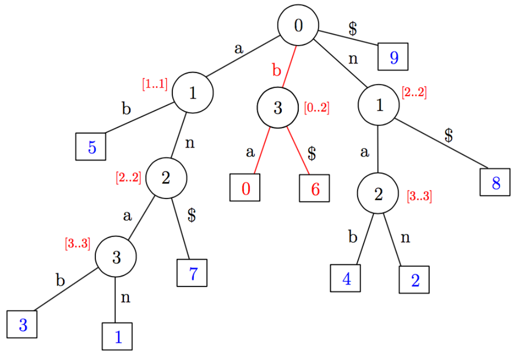
P = ana



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	\$

Suffix Tree: Example

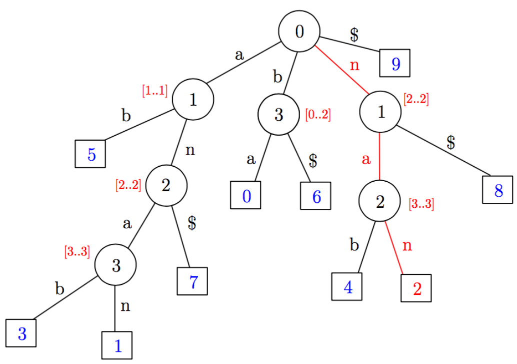
T = bananaban
P = ban



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$

Suffix Tree: Example

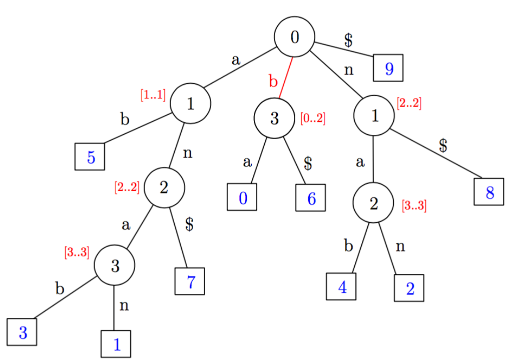
T = bananaban
P = nana



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$

Suffix Tree: Example

T = bananaban
P = bbn **not found**



i	0	1	2	3	4	5	6	7	8	9
$T[i]$	b	a	n	a	n	a	b	a	n	$
	b	b	n							

Pattern Matching Conclusion

	Brute-Force	DFA	KMP	BM	RK	Suffix trees
Preproc.:	–	$O(m\|\Sigma\|)$	$O(m)$	$O(m+\|\Sigma\|)$	$O(m)$	$O(n^2)$ ($\rightarrow O(n)$)
Search time:	$O(nm)$	$O(n)$	$O(n)$	$O(n)$ (often better)	$\widetilde{O}(n+m)$ (expected)	$O(m)$
Extra space:	–	$O(m\|\Sigma\|)$	$O(m)$	$O(m+\|\Sigma\|)$	$O(1)$	$O(n)$