Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Assignment 2
Due February 14, 2019 at 11:30am

[5 marks] 1. The Online Encyclopedia of Integer Sequences (`https://oeis.org`) is a valuable computer resource for discovering and exploring patterns or hidden connections in integer sequences. For instance, sequence A000040 corresponds to the sequence of prime numbers, and sequence A000045 corresponds to terms of the Fibonacci sequence. In this problem, we will investigate a sequence that arises from partitioning the elements of a particular set.

Let $p(n)$ denote the number of ways to partition the set $S = \{1, 2, \ldots, 2n\}$ into n subsets each of size 2.

(a) Determine $p(1)$ and $p(2)$ directly.

(b) Develop a formula to calculate $p(n)$ for all $n \geq 2$.

(c) Visit `https://oeis.org` and enter a few terms of the sequence $p(n)$ as given by your formula. Which sequence is generated by your formula? Give the name and number of the sequence. (You will need to enter four or five terms to narrow your search sufficiently.)

[5 marks] 2. Analyze the following probabilistic argument. Is it correct? Explain why or why not.

> A member of the tech support staff is updating equipment in the server room. The probability of successfully updating the equipment is $1/3$ and each update attempt is independent. The staff member will make three attempts to update the equipment before quitting. Since the probability of a successful update is $1/3 = 0.33\ldots$, the probability of a successful update after three attempts is $3(1/3) = 0.99\ldots = 1$, so the staff member is certain to succeed in updating the equipment.

[5 marks] 3. In this question, we will consider the following problem in computational complexity theory:

<u>MAX-SAT</u> Given: a Boolean formula in conjunctive normal form Determine: an assignment of truth values that maximizes the number of true clauses in the formula

The MAX-SAT (maximum satisfiability) problem is similar to the Boolean satisfiability problem SAT, except instead of finding an assignment that makes every clause true, we want to find an assignment that maximizes the number of true clauses. Thus, MAX-SAT is a generalization of SAT.

As an example, consider the formula $(x_0 \vee x_1) \wedge (x_0 \vee \neg x_1) \wedge (\neg x_0 \vee x_1) \wedge (\neg x_0 \vee \neg x_1)$. This formula is in conjunctive normal form; each clause contains only variables and $\vee$s, and all clauses are joined only by $\wedge$s. There is no way to satisfy all four clauses of this formula at once, but it is possible to satisfy three out of the four clauses.

(a) Suppose a formula in conjunctive normal form consists of n variables, and we assign truth values to each variable randomly by flipping a fair coin. If we assign T after flipping tails and F after flipping heads, what is the probability of each possible assignment of truth values to the n variables?

(b) Suppose a formula in conjunctive normal form is such that each clause is a disjunction of exactly two distinct variables or their negations; that is, each clause is of the form $(\_ \vee \_)$, where the blank spaces contain variables. What is the probability that a given clause is true, if we assign truth values randomly by flipping a fair coin as in part (a)?

(c) Suppose a formula in conjunctive normal form consists of c clauses as defined in part (b). What is the expected number of true clauses in the formula, if we assign truth values randomly by flipping a fair coin as in part (a)?

[5 marks] 4. The "all-pegs" Tower of Hanoi puzzle is a variant of the classical puzzle where we must transfer n disks from the first peg to the third peg, but without jumping directly from peg 1 to peg 3. In other words, our solution must use peg 2 whenever we move a disk from peg 1 to peg 3. All other aspects of the puzzle variant are identical to the original.

(a) Find a recurrence relation for H'_n, the number of moves needed to solve the "all-pegs" Tower of Hanoi puzzle with n disks.

(b) Solve your recurrence relation from part (a) to obtain a formula for the number of moves needed to solve the "all-pegs" Tower of Hanoi puzzle with n disks.