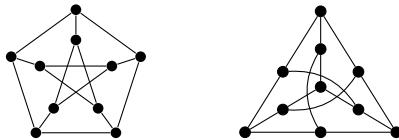


4 Isomorphisms

Thinking back to Definition 1, recall that a graph is nothing more than a set of vertices and a set of edges. Therein lies a crucial observation that new students of graph theory often miss: graphs are defined solely by their vertices and edges. The way we draw a graph on paper does not matter!

As an example, consider the following pair of graphs: the Petersen graph and some other strange-looking graph. Although these two graphs appear to be different, if we consider the number of vertices and the edges between vertices, we find that they are in fact the exact same graph. Try to prove this yourself by labelling the vertices and edges.



The property of two graphs being identical up to the way they are drawn is an important one, as it allows us to convert strange-looking or complicated graphs (such as the one at right) into more familiar graphs (such as the one at left). In turn, this conversion could allow us to reduce certain problems to equivalent problems with known solutions, or to simplify problems by showing that a given graph related to that problem possesses a certain structure. If two graphs are identical in this way, then we call them **isomorphic**.

Definition 33 (Isomorphism). Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijective function $f : V_1 \rightarrow V_2$ that preserves the property of vertex adjacency; that is, if u and v are adjacent in G_1 , then $f(u)$ and $f(v)$ are adjacent in G_2 .

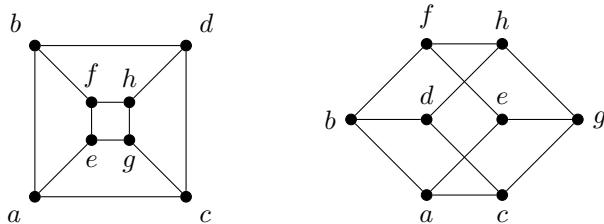
Formally speaking, the function f itself is the isomorphism to which the definition refers. Isomorphisms can apply to any mathematical set, but they most often appear in the context of graph theory. The term “isomorphism” comes from the Greek *isos*, meaning “equal”, and *morphe*, meaning “form”. Thus, isomorphisms produce “equal forms” of graphs.

If graphs G_1 and G_2 are isomorphic, then we often denote this by the shorthand notation $G_1 \simeq G_2$.

Isomorphisms abide by every property required to be an equivalence relation: given graphs G_1 , G_2 , and G_3 ,

- $G_1 \simeq G_1$;
- if $G_1 \simeq G_2$, then $G_2 \simeq G_1$; and
- if $G_1 \simeq G_2$ and $G_2 \simeq G_3$, then $G_1 \simeq G_3$.

Example 34. The following two graphs are isomorphic, as evidenced by the labels of vertices in both graphs.



How do we determine when two graphs are isomorphic? We could do so by inspection, but this approach is suboptimal when dealing with graphs that contain more than a handful of vertices and edges. Taking a more formalized approach, we shift our focus away from the graphs G_1 and G_2 and toward the isomorphism f . If we can show that f is **edge-preserving** (which is just a more concise way of saying that f preserves the property of vertex adjacency), then we can conclude that f is an isomorphism.

We can determine whether f is edge-preserving by representing both G_1 and G_2 as adjacency matrices. Then, G_1 and G_2 are isomorphic if and only if there exists a permutation matrix \mathbf{P} such that $\mathbf{A}_{G_1} = \mathbf{P}\mathbf{A}_{G_2}\mathbf{P}^\top$. In

other words (or if you don't remember your linear algebra lectures), G_1 and G_2 are isomorphic if and only if we can rearrange the rows and columns of \mathbf{A}_{G_2} to get \mathbf{A}_{G_1} .

Remark. Note that we choose to use adjacency matrices here because the property of edge-preservation affects pairs of vertices, and the entries of an adjacency matrix relate pairs of vertices. Using an adjacency matrix over an adjacency list also gives us the benefit of being able to use matrix algebra to solve a problem.

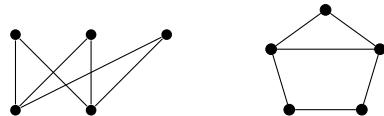
At this point, the natural question arises: if we know how to determine when two graphs are isomorphic, then how can we determine when two graphs are *not* isomorphic? We could take the same approach we just saw, where we essentially try to prove that we cannot permute the rows and columns of one adjacency matrix to obtain another, but this can be tedious and difficult to do by hand.

An easier approach for showing that two graphs are not isomorphic is to compare the **invariants** of each graph. An invariant of a graph is a property that identical graphs must share; if this property does not match between the graphs, then we can conclude that the graphs are not isomorphic. Three common invariants we can use to rule out isomorphism are:

- the number of vertices (since the isomorphism f must be a bijection, both graphs must contain the same number of vertices);
- the number of edges (since the isomorphism f must be edge-preserving, both graphs must contain the same number of edges); and
- the degrees of each vertex (since the isomorphism must be an edge-preserving bijection, there must be a correspondence between vertices of the same degree in each graph).

However, we must be careful: although we can rule out isomorphism if an invariant does not hold, we cannot conclude that two graphs are isomorphic if invariants do hold.

Example 35. Consider the following pair of graphs:



Both of these graphs contain 5 vertices and 6 edges, and within each graph there are two vertices of degree 3 and three vertices of degree 2. However, these graphs are not isomorphic! (To see why, observe that the right graph contains C_3 as a subgraph, while the left graph does not. Indeed, no bipartite graph contains C_3 as a subgraph.)

Finally, it is worth mentioning another famous problem in computer science called (appropriately) the **graph isomorphism** problem. This problem asks whether two given graphs are isomorphic to one another, but it has a number of applications outside of graph theory: for instance, the graph isomorphism problem can determine matching patterns in images, calculate matchings and measures on text strings, and construct layouts of electrical circuits on boards. Although the graph isomorphism problem has been solved for certain types of graphs, not much is known about the general problem. In fact, the general problem is one of the few problems suspected to lie *between* the classes of “easy” and “hard” problems.

5 Planarity

We often informally refer to the process of depicting a graph on paper as “drawing”. However, simply saying that we have “drawn” a graph allows some undesirable properties to sneak in that might make our graph confusing to look at. Thus, we want to formalize the notion of drawing in some way to ensure that our depictions of graphs are clear.

Mathematicians have a name for such a formalism: an **embedding** of a graph G on a surface (for our purposes, a two-dimensional plane) is a representation of the vertices and edges of G that meet the following criteria:

- each vertex $u \in V$ is assigned a point on the surface, and no two vertices share the same point;
- each edge $e \in E$ is assigned a curve on the surface, and no two edges share the same curve;
- the endpoints of some curve e are exactly the two incident vertices u and v ; and
- no vertex other than the two incident vertices u and v lie on the edge e .

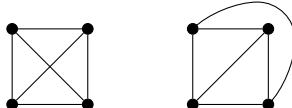
Essentially, an embedding ensures that we are able to see all of the vertices and edges of a graph and that no vertices or edges overlap one another.

Observe that edges overlapping (that is, one edge covering another edge entirely) is different from edges crossing (that is, one edge intersecting another edge at some point on the surface, which includes an edge intersecting itself). Many graphs have edges that cross, and we have seen such graphs throughout these notes. But a special subset of graphs have edges that do not cross, and that subset is the topic of this section.

Definition 36 (Planar graph). A graph $G = (V, E)$ is planar if G has an embedding in the plane that is crossing-free.

Often, if we are given a graph with crossing edges, we can redraw the edges in such a way that no edges cross one another. If we can redraw a graph G into a crossing-free graph G' , then we say that G' is a **planar representation** of G .

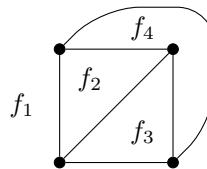
Example 37. The embedding of the graph K_4 at left is not planar. However, the embedding of the graph at right is both planar and isomorphic to K_4 . Therefore, K_4 is a planar graph.



In the same way that borders on Earth divide land into countries, the edges of a planar graph divide a surface into **faces**. Faces need not be confined to the interior of a graph; there always exists one exterior, unbounded face that surrounds the graph.

Just like vertices, faces have degrees. The degree of a face is equal to the number of edges making up the boundary of that face. Intuitively, the degree of a face can be thought of as the number of lines you would need to draw, without lifting your pencil, to trace out the face on paper and end up at the point where you started.

Example 38. Recall the planar embedding of K_4 from Example 37. This planar graph consists of four faces, each labelled in the following figure. Observe that one face, f_1 , is an exterior face. Further observe that each face of the graph has degree 3, since each face is bounded by three edges.



Long ago, Euler discovered a relationship between the numbers of vertices, edges, and faces of certain planar graphs. Summing these numbers in a particular way produces a constant value, known as the **characteristic**. Although Euler's result is easy to remember, it was given the rather uninspired name of "Euler's formula", which isn't particularly helpful once you realize exactly how many formulas Euler discovered throughout his life. Nonetheless, Euler's formula for connected planar graphs is as follows.

Theorem 39 (Euler's formula). *Given a connected planar simple graph $G = (V, E)$ with v vertices, e edges, and f faces,*

$$v - e + f = 2.$$

Proof. Let G be a connected planar simple graph. We will prove the claim using the principle of mathematical induction on the number of edges of G . Let $P(e)$ be the statement “given a graph G with v vertices and e edges, $v - e + f = 2$ ”.

When $e = 0$, we know that G is an edgeless graph. Since G is connected, it must consist of only one vertex, and as a result, G contains one exterior face. This gives $v - e + f = 1 - 0 + 1 = 2$. Therefore, $P(0)$ is true.

Assume that $P(e')$ is true for some $e' \in \mathbb{N}$.

We now show that $P(e'+1)$ is true; that is, we add one edge to the graph G and check whether the statement holds. The new edge can be added in one of two ways:

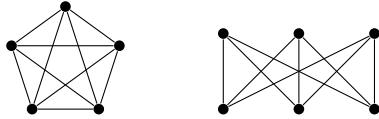
- If the edge is incident to one existing vertex, then we must add a new vertex to G . This increases both the number of vertices and the number of edges by one, which gives $(v+1) - (e'+1) + f = v+1 - e' - 1 + f = v - e' + f = 2$.
- If the edge connects two existing vertices to one another, then we divide some face within G into two faces. This increases both the number of edges and the number of faces by one, which gives $v - (e'+1) + (f+1) = v - e' - 1 + f + 1 = v - e' + f = 2$.

In either case, $P(e'+1)$ is true. By the principle of mathematical induction, $P(e)$ is true for all $e \in \mathbb{N}$. \square

Note that the fact that G is connected is crucial to the correctness of Theorem 39. Graphs that are not connected have a slightly different formula.

So, why do we care about Euler’s formula? Using this result, we can derive a number of conditions that must be met in order for a graph to be planar.

If a graph has no planar representation, then we say that it is **nonplanar**. Proving the nonplanarity of a graph often reduces to showing that the graph does not meet the conditions set out by Euler’s formula (or by some corollary of Euler’s formula). As illustrative examples of such proofs, we will focus on proving the nonplanarity of two graphs with which we are already familiar: K_5 and $K_{3,3}$.



Theorem 40. *The complete graph K_5 is nonplanar.*

Proof. Assume by contradiction that K_5 is planar. Then K_5 satisfies Euler’s formula. Since the graph consists of 5 vertices and 10 edges, we know that there should exist $f = 2 - 5 + 10 = 7$ faces in the graph.

Let b denote the number of “boundary edges” surrounding each face of the graph. Any face must be bounded by at least three edges, so we know that $b \geq 3f$. However, since each edge in the graph is a boundary edge for exactly two faces, we also know that $b = 2e$. Therefore, we have that $2e \geq 3f$.

However, since $e = 10$ and $f = 7$, this inequality asserts that $2(10) \geq 3(7)$, which is impossible. Therefore, K_5 is nonplanar. \square

Theorem 41. *The complete bipartite graph $K_{3,3}$ is nonplanar.*

Proof. Assume by contradiction that $K_{3,3}$ is planar. Then $K_{3,3}$ satisfies Euler’s formula. Since the graph consists of 6 vertices and 9 edges, we know that there should exist $f = 2 - 6 + 9 = 5$ faces in the graph.

Again, let b denote the number of “boundary edges” surrounding each face of the graph. We know once more that $b = 2e$. Since $K_{3,3}$ is a complete bipartite graph, we also know that C_3 is not a subgraph (by Example 35), so each face must be bounded by at least four edges, giving $b \geq 4f$.

However, since $e = 9$ and $f = 5$, this inequality asserts that $2(9) \geq 4(5)$, which is impossible. Therefore, $K_{3,3}$ is nonplanar. \square

At this point, you might wonder why we chose to prove that these two specific graphs were nonplanar. There was a hidden motive for our choices of proofs: we actually require these results to prove the final result of this section. First, however, we require just one more bit of terminology.

It is possible to define an operation on a graph that, given an edge e incident to vertices u and v , divides the edge into two edges e_1 and e_2 , where e_1 is incident to u and a new vertex w and e_2 is incident to w and v . This operation changes nothing about the graph apart from the number of vertices and edges and, in particular, it does not affect the planarity of the graph. We call this operation a **subdivision**.



Now, we are ready to see a fascinating theorem due to the Polish mathematician Kazimierz Kuratowski. This theorem gives us a pair of conditions that identify when certain graphs are nonplanar. Put simply, if we can obtain a graph G by taking subdivisions of either of the nonplanar graphs K_5 or $K_{3,3}$, then G itself is nonplanar.

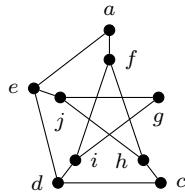
Theorem 42 (Kuratowski's theorem). *A graph G is nonplanar if and only if G contains a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph.*

Proof. (\Rightarrow): Omitted.

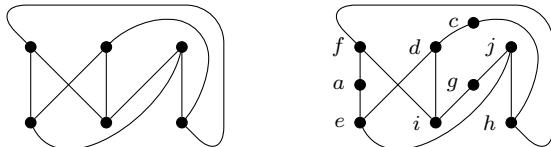
(\Leftarrow): If G contains a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph, then G is nonplanar by Theorem 40 or Theorem 41, respectively. \square

Kuratowski's theorem is biconditional, so we must prove two statements in order to prove the overall theorem. Although one direction of the proof (\Leftarrow) followed immediately from other results we proved, the other direction (\Rightarrow) is a long and detailed proof. We omit it here, but many textbooks dedicated to graph theory contain the full proof.

Example 43. We can show that the Petersen graph is nonplanar by using Kuratowski's theorem. First, delete one vertex from the Petersen graph to obtain the following subgraph, then label the vertices.



Next, consider the graph at left, which is isomorphic to the complete bipartite graph $K_{3,3}$. Take a subdivision of this graph to obtain the graph at right, and again label the vertices.



Finally, observe that the given labelings establish an isomorphism between the two graphs, meaning that the Petersen graph contains a subdivision of $K_{3,3}$ as a subgraph. By Kuratowski's theorem, the Petersen graph is nonplanar.

6 Colourings

In this section, we will be doing something that we don't usually get to do in a university-level course: colouring. (Hopefully you remembered to bring your crayons!)

In some of the graphs we have seen, we labelled vertices with symbols like a , b , u , and v . Importantly, each of these symbols had to be unique, since we wanted to distinguish between vertices. A **graph colouring**, on the other hand, is like a method of nonunique labelling. We can colour any aspect of a graph, such as vertices, edges, or faces.

Remark. For the remainder of this section, we will focus on vertex colourings. Similar definitions and results exist for both edge colourings and face colourings, but we will not discuss either topic here.

Instead of giving each vertex a symbolic label, we can assign a “colour” to each vertex that distinguishes it from nearby vertices. These colours need not be unique; we can label more than one vertex with the same colour.

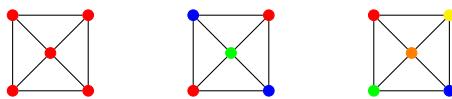
In our formal definition, we use numerical values to represent colours.

Definition 44 (Vertex colouring). Given a graph $G = (V, E)$ and a set of k colours $\{1, 2, \dots, k\}$, a vertex colouring of G is a function $f : V \rightarrow \{1, 2, \dots, k\}$.

If a vertex colouring f has the property that, for all edges $\{u, v\} \in E$, $f(u) \neq f(v)$, then we say that the vertex colouring is **proper**. A proper vertex colouring is one in which no adjacent vertices share the same colour.

If we can construct a proper vertex colouring of a graph using at most k colours, then we say that the graph is **k -colourable**.

Example 45. Each of the following graphs depicts a vertex colouring. However, only the two rightmost graphs depict a proper vertex colouring. Since we can obtain a proper vertex colouring using three colours, this graph is 3-colourable.



6.1 Chromatic Numbers

Given a graph G with n vertices, if all we care about is colouring the vertices in some way, then we can use as many as n colours (if we give each vertex a unique colour) or as few as one colour (if we give each vertex the same colour). Occasionally, however, we might want to see how many colours we require in order to construct a proper vertex colouring of G . This idea leads to the notion of the **chromatic number** of G .

Definition 46 (Chromatic number). Given a graph G , the chromatic number of G , denoted by $\chi(G)$, is equal to the smallest number of colours required to construct a proper vertex colouring of G .

We can attain both the upper bound and the lower bound on the chromatic number if we consider particular classes of graphs.

- Given a complete graph K_n with n vertices and $\frac{n(n-1)}{2}$ edges, we have that $\chi(K_n) = n$. This is because each vertex is adjacent to every other vertex in the graph.
- Given an edgeless graph $\overline{K_n}$ with n vertices and zero edges, we have that $\chi(\overline{K_n}) = 1$. This is because no vertices are adjacent to one another in the graph.

Thus, we know that for any graph G , we have that $1 \leq \chi(G) \leq n$.

With a little work, we can obtain a few more results about the chromatic numbers of given graphs. Our first result relates the chromatic number of a graph to the **maximum degree** of any vertex in the graph, denoted by Δ .

Theorem 47. If a graph G has maximum degree Δ , then $\chi(G) \leq \Delta + 1$.

Proof. Assume we have a set of $\Delta + 1$ colours. Choose an arbitrary vertex, say v_1 , from the graph G and assign to it a colour. Repeat this process of assigning colours to arbitrary vertices of G under the condition that, if two vertices v_i and v_j are adjacent, then both vertices should be assigned different colours.

Since the maximum degree of G is Δ and since we have $\Delta + 1$ colours available to use, we will not encounter the issue of running out of colours to assign to adjacent vertices. Therefore, $\chi(G) \leq \Delta + 1$. \square

A stronger result, proved by the English mathematician R. Leonard Brooks, reduces the bound given in Theorem 47 to $\chi(G) \leq \Delta$, unless $G = K_n$ or $G = C_{2n+1}$. However, Brooks' result is harder to prove.

Our next result relates the chromatic number of a graph to the chromatic numbers of its subgraphs.

Theorem 48. *If a graph G contains a subgraph H , then $\chi(H) \leq \chi(G)$.*

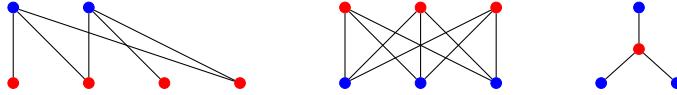
Proof. Given a proper vertex colouring of the graph G , we can copy those colours to the corresponding vertices of H . Thus, a proper vertex colouring of H requires at most $\chi(G)$ colours. \square

Next, recall the notion of a bipartite graph from our discussion on special graph classes. A bipartite graph is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that each edge joins a vertex in V_1 to a vertex in V_2 . The following result characterizes the set of bipartite graphs in terms of their chromatic number.

Theorem 49. *A graph G is a bipartite graph if and only if $\chi(G) = 2$.*

Proof. (\Rightarrow): Given a bipartite graph G , colour each of the vertices in V_1 one colour (say, red), and colour each of the vertices in V_2 another colour (say, blue). No vertices in V_1 are adjacent to vertices in the same set, and the same is true for vertices in V_2 . Thus, $\chi(G) = 2$.

(\Leftarrow): Given a graph G with $\chi(G) = 2$, partition each of the vertices of the first colour into one set V_1 , and partition each of the vertices of the second colour into another set V_2 . Since no two adjacent vertices of G share the same colour, vertices in V_1 are only adjacent to vertices in V_2 and vice versa, making the graph bipartite. \square



Finally, let's investigate chromatic numbers of planar graphs. We do so in the context of maps: a paper map is a plane, land areas are vertices of a graph, and adjacent land areas are connected by non-crossing edges.

In the early days of graph theory, a South African mathematician named Francis Guthrie made a conjecture while trying to colour a map of England. Guthrie asserted that one needs only four colours to colour in any map, no matter how many regions the map depicts. In 1879, an English mathematician named Alfred Kempe claimed to have a proof of Guthrie's conjecture, and for this he received many awards and accolades. However, 11 years later, another English mathematician named Percy Heawood found a flaw in Kempe's proof. Heawood gave his own proof that five colours are in fact sufficient.

From here, work on map colouring lay dormant for many years. Then, in 1976, Kenneth Appel and Wolfgang Haken proved that, indeed, four colours are sufficient to colour any map. The proof of Appel and Haken was a landmark result, since it was one of the first major mathematical theorems to be proved with the aid of a computer; the complete proof consisted of 1936 different cases. Since then, the proof has been shortened considerably—to a mere 633 cases—but it is still too lengthy to reproduce here.

Theorem 50 (Four colour theorem). *If a graph G is planar, then $\chi(G) \leq 4$.*

Proof. Omitted. \square