Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Problem Set 1: Proof Techniques
Winter 2019

This problem set is provided to you for the purposes of self-study; your solutions will not be marked. Each of these problems, as well as hints and some solutions, can be found in the course textbook.

4.2. Below you will find pairs of statements A and B. For each pair, please indicate which of the following three sentences are true and which are false:

- If A, then B.
- If B, then A.
- A if and only if B.

Note: you do not need to prove your assertions.

(a) A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a square.

(b) A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a parallelogram.

(c) A: Joe is a grandfather. B: Joe is male.

(d) A: Ellen resides in Los Angeles. B: Ellen resides in California.

(e) A: This year is divisible by 4. B: This year is a leap year.

(f) A: Lines ℓ_1 and ℓ_2 are parallel. B: Lines ℓ_1 and ℓ_2 are perpendicular.

5.4. Prove that the product of two even integers is even.

5.6. Prove that the product of two odd integers is odd.

5.11. Suppose a, b, d, x, and y are integers. Prove that if $d \mid a$ and $d \mid b$, then $d \mid (ax + by)$.

6.5. Disprove: If p and q are prime, then $p + q$ is composite.

6.8. An integer is a *palindrome* if it reads the same forwards and backwards when expressed in base-10. For example, 1331 is a palindrome. Disprove: All palindromes with two or more digits are divisible by 11.

6.9. Consider the polynomial $n^2 + n + 41$.

(a) Calculate the value of this polynomial for $n = 1, 2, 3, \ldots, 10$. Notice that all the numbers you computed are prime.

(b) Disprove: If n is a positive integer, then $n^2 + n + 41$ is prime.

20.1. Please state the contrapositive of each of the following statements:

(a) If x is odd, then x^2 is odd.

(b) If p is prime, then $2^p - 2$ is divisible by p.

(c) If x is nonzero, then x^2 is positive.

(d) If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

(e) If the battery is fully charged, the car will start.

(f) If A or B, then C.

20.9. Prove by contradiction: If a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$.

20.12. Prove by contradiction: A positive integer is divisible by 10 if and only if its last (one's) digit (when written in base ten) is a zero. You may assume that every positive integer N can be expressed as follows:

$$N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10 + d_0$$

where the numbers d_0 through d_k are in the set $\{0, 1, \ldots, 9\}$ and $d_k \neq 0$. In this notation, d_0 is the one's digit of N's base ten representation.

21.6. For all positive integers n, we have

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1.$$

Prove this using the smallest counterexample/well-ordering proof technique.

22.4. Prove the following equations by induction. In each case, n is a positive integer.

(a) $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$.

(b) $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

(c) $9 + 9 \times 10 + 9 \times 100 + \cdots + 9 \times 10^{n-1} = 10^n - 1$.

(d) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.

(e) $1 + x + x^2 + x^3 + \cdots + x^n = (1 - x^{n+1})/(1 - x)$. You should assume $x \neq 1$. What is the correct right hand side when $x = 1$?

22.12. The *Tower of Hanoi* is a puzzle consisting of a board with three dowels and a collection of n disks of n different sizes (radii). The disks have holes drilled through their centers so that they can fit on the dowels on the board. Initially, all the disks are on the first dowel and are arranged in size order (from the largest on the bottom to the smallest on the top).

The object is to move all the disks to another dowel in as few moves as possible. Each move consists of taking the top disk off one of the stacks and placing it on another stack, with the added condition that you may not place a larger disk atop a smaller one.

Prove: For every positive integer n, the Tower of Hanoi puzzle (with n disks) can be solved in $2^n - 1$ moves.