

Queen's University  
School of Computing

CISC 203: Discrete Mathematics for Computing II  
Problem Set 2: Combinatorics  
Winter 2019

This problem set is provided to you for the purposes of self-study; your solutions will not be marked. Each of these problems, as well as hints and some solutions, can be found in the course textbook.

- 8.9. In how many ways can a black rook and a white rook be placed on different squares of a chess board such that neither is attacking the other? (In other words, they cannot be in the same row or same column of the chess board. A standard chess board is  $8 \times 8$ .)
- 8.13. Let  $n$  be a positive integer. Prove that  $n^2 = P(n, 2) + n$  in two different ways.  
First (and more simply) show this equation is true algebraically.  
Second (and more interestingly) interpret the terms  $n^2$ ,  $P(n, 2)$ , and  $n$  in the context of list counting and use that to argue why the equation must be true.
- 8.16. A padlock has the digits 0 through 9 arranged in a circle on its face. A combination for this padlock is four digits long. Because of the internal mechanics of the lock, no pair of consecutive numbers in the combination can be the same or one place apart on the face. For example 0-2-7-1 is a valid combination, but neither 0-4-4-7 (repeated digit 4) or 3-0-9-5 (adjacent digits 0-9) are permitted. How many combinations are possible?  
*Note.* The use of the word “combination” in this problem is deceiving.
- 19.2. Of the integers between 1 and 100 (inclusive) how many are divisible by 2 or by 5?
- 19.7. How many six-digit numbers do not have three consecutive digits the same? (For this problem, you may consider six-digit numbers whose initial digits might be 0. Thus you should count 012345 and 001122, but not 000987 or 122234.)
- 25.5. Let  $(a_1, a_2, a_3, a_4, a_5)$  be a sequence of five distinct integers. We call such a sequence increasing if  $a_1 < a_2 < a_3 < a_4 < a_5$  and decreasing if  $a_1 > a_2 > a_3 > a_4 > a_5$ . Other sequences may have a different pattern of  $<$ s and  $>$ s. For the sequence  $(1, 5, 2, 3, 4)$  we have  $1 < 5 > 2 < 3 < 4$ . Different sequences may have the same pattern of  $<$ s and  $>$ s between their elements. For example,  $(1, 5, 2, 3, 4)$  and  $(0, 6, 1, 3, 7)$  have the same pattern of  $<$ s and  $>$ s.  
Given a collection of 17 sequences of five distinct integers, prove that 2 of them have the same pattern of  $<$ s and  $>$ s.
- 25.9. Consider a square whose side has length one. Suppose we select five points from this square. Prove that there are two points whose distance is at most  $\sqrt{2}/2$ .
- 27.13. Let  $\pi = (1, 2)(3, 4, 5, 6, 7)(8, 9, 10, 11)(12)$ . Find the smallest positive integer  $k$  for which

$$\pi^{(k)} = \underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{k \text{ times}} = \iota.$$

Generalize. If  $\pi$ 's disjoint cycles have lengths  $n_1, n_2, \dots, n_t$ , what is the smallest integer  $k$  so that  $\pi^{(k)} = \iota$ ?

*Note.*  $\iota$  denotes the identity permutation, which maps elements of a set to themselves.

17.3. Find the coefficient! Answer the following questions with the assistance of the binomial theorem:

- (a) What is the coefficient of  $x^3$  in  $(1 + x)^6$ ?
- (b) What is the coefficient of  $x^3$  in  $(2x - 3)^6$ ?
- (c) What is the coefficient of  $x^3$  in  $(x + 1)^{20} + (x - 1)^{20}$ ?
- (d) What is the coefficient of  $x^3y^3$  in  $(x + y)^6$ ?
- (e) What is the coefficient of  $x^3y^3$  in  $(x + y)^7$ ?

17.11. In how many different ways can we partition an  $n$ -element set into two parts if one part has four elements and the other part has all the remaining elements?

17.16. Prove the following formula:

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

17.17. Let  $n \geq k \geq m \geq 0$  be integers. Prove the following formula:

$$\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}.$$

18.1. Evaluate all 2-combinations with repetition of the set  $A = \{1, 2, 3\}$  and all 3-combinations with repetition of the set  $B = \{1, 2\}$  by explicitly listing all possible multisets of the appropriate size.

18.2. Give a stars-and-bars representation for all the sets you found in the previous problem.

18.5. What multiset is encoded by the stars-and-bars notation  $\star || | \star \star \star$ ?