Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Problem Set 3: Discrete Probability
Winter 2019

This problem set is provided to you for the purposes of self-study; your solutions will not be marked. Each of these problems, as well as hints and some solutions, can be found in the course textbook.

30.2. Let $(\Omega, \mathbb{P})$ be the probability space in which $\Omega = \{1, 2, 3, 4\}$. Suppose $\mathbb{P}[1] = x$, $\mathbb{P}[2] = 2x$, $\mathbb{P}[3] = 3x$, and $\mathbb{P}[4] = 4x$. Find x.

30.7. An experiment is performed in which a coin is flipped and a die is rolled. Describe this experiment as a sample space. Explicitly list all elements of the set Ω and the value of $\mathbb{P}[\omega]$ for each element of Ω.

31.10. Two dice are rolled. Let A denote the event that the number on the first die is greater than the number on the second die.

- (a) Explicitly write down A as a set.
- (b) Evaluate $\mathbb{P}[A]$.

31.13.
- (a) What is the probability that a poker hand is a three of a kind? (A *three of a kind* has three cards of the same value and two other cards of different values, such as three 10s, a 7, and a jack.)
- (b) What is the probability that a poker hand is a full house? (A *full house* has three cards with one common value and two other cards of another common value, such as three queens and two 4s.)
- (c) What is the probability that a poker hand has one pair? (*One pair* means two cards have the same value and three other cards have three other values, such as two 9s, a king, an 8, and a 5.)
- (d) What is the probability that a poker hand has two pairs? (*Two pairs* means two cards have one common value, two more cards have another common value, and a fifth card has yet another value, such as two jacks, two 8s, and a 3.)
- (e) What is the probability that a poker hand is a flush? (A *flush* means all five cards have the same suit.)

31.18. Suppose A and B are events in a sample space. Please prove: if $A \subseteq B$, then $\mathbb{P}[A] \leq \mathbb{P}[B]$.

31.20. Suppose that A and B are events in a sample space for which $\mathbb{P}[A] > 1/2$ and $\mathbb{P}[B] > 1/2$. Prove that $\mathbb{P}[A \cap B] \neq 0$.

32.3. A pair of dice are rolled. What is the probability that neither die shows a 2 given that they sum to 7?

32.4. A pair of dice are rolled. What is the probability that they sum to 7 given that neither die shows a 2?

32.22. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbb{P}[\omega] = 1/6$ for all $\omega \in \Omega$. For this sample space, find a pair of events A and B such that (a) $0 < \mathbb{P}[A] < 1$, (b) $0 < \mathbb{P}[B] < 1$, and (c) A and B are independent.

Next, show that if we were to change the probability space to $(\Omega, \mathbb{P})$ with $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathbb{P}[\omega] = 1/5$ for all ω, then no such pair of events can be found.

32.31. An unfair coin shows heads with probability p and tails with probability $1 - p$. Suppose this coin is tossed twice. Let A be the event that the coin comes up first heads and then tails, and let B be the event that the coin comes up first tails and then heads.

- (a) Calculate $\mathbb{P}[A]$.

(b) Calculate $\mathbb{P}[B]$.

(c) Calculate $\mathbb{P}[A \mid A \cup B]$.

(d) Calculate $\mathbb{P}[B \mid A \cup B]$.

(e) Explain how to use an unfair coin to make a fair decision (choose between two alternatives with equal probability).

33.2. Let $(\Omega, \mathbb{P})$ be the probability space with $\Omega = \{1, 2, 3, \ldots, 10\}$ and $\mathbb{P}[\omega] = 1/10$ for all $\omega \in \Omega$. For this sample space, define the random variables X and Y by $X[\omega] = 2\omega$ and $Y[\omega] = \omega^2$ for all $\omega \in \Omega$.

(a) Evaluate $\mathbb{P}[X < 10]$.

(b) Evaluate $\mathbb{P}[Y < 10]$.

(c) Evaluate $(X + Y)[\omega]$.

(d) Evaluate $\mathbb{P}[X + Y < 10]$.

(e) Evaluate $\mathbb{P}[X > Y]$.

(f) Evaluate $\mathbb{P}[X = Y]$.

(g) Are X and Y independent? Justify your answer.

33.5. A pair of dice are rolled. Let X be the (absolute value of the) difference between the numbers on the dice.

(a) What is $X[\{2, 5\}]$?

(b) Evaluate $\mathbb{P}[X = a]$ for all integers a.

34.2. Let $(\Omega, \mathbb{P})$ be the probability space with $\Omega = \{a, b, c\}$ and $\mathbb{P}[\omega] = 1/3$ for all $\omega \in \Omega$. Find the expected value of each of the following random variables:

(a) X, where $X[a] = 1$, $X[b] = 2$, and $X[c] = 10$.

(b) Y, where $Y[a] = Y[b] = -1$ and $Y[c] = 2$.

(c) Z, where $Z = X + Y$.

34.4. You play a game in which you roll a die and you win (in dollars) the square of the number on the die. For example, if you roll a 5, then you win \$25. On average, how much money would you expect to receive per play of this game?

34.15. Let X and Y be real-valued random variables defined on a probability space $(\Omega, \mathbb{P})$. Suppose $X[\omega] \leq Y[\omega]$ for all $\omega \in \Omega$. Prove that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Challenge Questions

These questions are harder than the questions you will typically see in this course. Attempt these questions for enrichment only after you have a good understanding of the material in this lecture.

34.17. *Markov's inequality.* Let $(\Omega, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{N}$ be a nonnegative-integer-valued random variable. Let a be a positive integer. Prove that

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

34.24. *Chebyshev's inequality.* Let X be a nonnegative-integer-valued random variable. Suppose $\mathbb{E}[X] = \mu$ and $\mathbb{V}[X] = \sigma^2$. Let a be a positive integer. Prove:

$$\mathbb{P}[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}.$$