Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Problem Set 6: Trees
Winter 2019

This problem set is provided to you for the purposes of self-study; your solutions will not be marked. Each of these problems, as well as hints and some solutions, can be found in the course textbook.

50.1. Let G be a graph in which every vertex has degree 2. Is G necessarily a cycle?

50.2. Let T be a tree. Prove that the average degree of a vertex in T is less than 2.

Note. The average degree of a vertex in a graph $G = (V, E)$ is defined as $\frac{1}{|V|}\sum_{v\in V}\deg(v)$.

50.3. There are exactly three trees with vertex set $\{1, 2, 3\}$. Note that all these trees are paths; the only difference is which vertex has degree 2.

How many trees have vertex set $\{1, 2, 3, 4\}$?

50.7. Let T be a tree with at least two vertices and let $v \in V$. Prove that if $T - v$ is a tree, then v is a leaf.

50.9. Let G be a forest with n vertices and c components. Find and prove a formula for the number of edges in G.

50.14. Prove:

(a) Every cycle is connected.

(b) Every cycle is 2-regular.

50.19. Let G be a connected graph. The *Weiner index* of G, denoted $W(G)$, is the sum of the distances between all pairs of vertices in G. In other words, if $V = \{1, 2, 3, \ldots, n\}$, then

$$W(G) = \sum_{1\le i<j\le n} d(i, j),$$

where $d(i, j)$ is the distance between vertices i and j. For example, on a path on four vertices we have

$$W(P_4) = (1 + 2 + 3) + (1 + 2) + 1 = 10.$$

In this problem we ask that you show a star graph (a tree with one vertex adjacent to all the other vertices which are, consequently, leaves) is the tree with the smallest Weiner index of all trees. For this problem, let S_n denote the star graph with n vertices.

(a) Calculate $W(S_n)$ in simplest possible terms.

(b) Prove that if T is any tree on n vertices, then $W(T) \ge W(S_n)$.

(c) Prove that if T is any tree on n vertices and $W(T) = W(S_n)$, then T must be a star graph.