St. Francis Xavier University
Department of Computer Science

CSCI 554: Matrix Computation
Assignment 1
Due February 17, 2022 at 2:15pm

Assignment Regulations.

- This assignment may be completed individually or in a group of up to four people. If you are collaborating on an assignment as a group, your group must submit exactly one joint set of answers.
- Please include your full name and email address on your submission. For groups, every member must include their full name and email address on the joint submission.
- You may either handwrite or typeset your submission. If your submission is handwritten, please ensure that the handwriting is neat and legible.

[4 marks] 1. Using forward or backward substitution (whichever method is most appropriate), solve the system corresponding to the matrix equation

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 1 \\ 12 \end{bmatrix}.$$

Show all intermediate steps. You may modify the provided code on the course website to verify your solution, but giving the vector x by itself is not a complete answer.

[10 marks] 2. Consider the matrix

$$A = \begin{bmatrix} 25 & 31 \\ 31 & 41 \end{bmatrix}.$$

(a) Prove that A is a positive definite matrix.
Hint. Consider the result of multiplying A by an arbitrary vector x.

(b) Compute the Cholesky factor R of A. Show all intermediate steps. You may modify the provided code on the course website to verify your solution, but giving the factor R by itself is not a complete answer.

(c) The Cholesky factor of a positive definite matrix is unique, but there exist other non-Cholesky factorizations where $A = R^\intercal R$. Find three other upper triangular matrices R such that $A = R^\intercal R$. What does this imply about the number of upper triangular matrices R that work as a factor of an arbitrary $n \times n$ matrix A?

[5 marks] 3. Consider a positive definite block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Further suppose that A_{11} is of dimension $i \times i$ and A_{22} is of dimension $j \times j$. Since A is positive definite, we can show that both A_{11} and A_{22} are positive definite.

Let R_{11} denote the Cholesky factor of A_{11}, let $R_{12} = (R_{11}^{-1})^\intercal A_{12}$, and let $\widetilde{A}_{22} = A_{22} - R_{12}^\intercal R_{12}$. We say that $\widetilde{A}_{22}$ is the *Schur complement* of A_{11} in the matrix A.

(a) Prove that $\widetilde{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

(b) Show that the matrix A can be decomposed into three parts, where the first part is lower triangular, the middle part involves $\widetilde{A}_{22}$, and the last part is upper triangular.

Hint. Since A is positive definite, think about how the Cholesky factors of each block matrix relate to this decomposition.

[10 marks] 4. Consider the matrix/vector pair

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \text{ and } b = \begin{bmatrix} 9 \\ 9 \\ 16 \end{bmatrix}.$$

(a) Compute the LU decomposition of A. Show all intermediate steps. You may modify the provided code on the course website to verify your solution, but giving the matrices L and U by themselves is not a complete answer.

(b) Using your matrices L and U from part (a), solve the linear system $Ax = b$. Show all intermediate steps. You may modify the provided code on the course website to verify your solution, but giving the vector x by itself is not a complete answer.

[6 marks] 5. Recall the properties of vector norms. We proved that the 2-norm (i.e., the Euclidean norm) is a norm by showing that it satisfied all of the necessary properties.

(a) Prove that the 1-norm is a norm.

(b) Prove that the ∞-norm is a norm.

[5 marks] 6. Recall the definition of the condition number $\kappa(A)$ of a matrix A.

(a) Prove that, for any matrix A, the condition numbers of A and its inverse are the same; that is, $\kappa(A) = \kappa(A^{-1})$.

(b) Prove that, for any matrix A and scalar value $c > 0$, multiplying A by c does not affect the condition number; that is, $\kappa(cA) = \kappa(A)$.

(c) Consider the matrix

$$H_4 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}.$$

This matrix is also known as a *Hilbert matrix*. Assume the norm we are using is the 1-norm. Compute $\|H_4\|_1$, $\|H_4^{-1}\|_1$, and $\kappa_1(H_4)$. What does the condition number tell you about H_4?

(You may write code to compute these values if you like, but please include your code with your assignment submission.)