St. Francis Xavier University
Department of Computer Science

CSCI 554: Matrix Computation
Assignment 2
Due April 5, 2022 at 3:15pm

Assignment Regulations.

- This assignment may be completed individually or in a group of up to four people. If you are collaborating on an assignment as a group, your group must submit exactly one joint set of answers.
- Please include your full name and email address on your submission. For groups, every member must include their full name and email address on the joint submission.
- You may either handwrite or typeset your submission. If your submission is handwritten, please ensure that the handwriting is neat and legible.

[6 marks] 1. Suppose we have an orthogonal matrix Q. Orthogonal matrices are perfectly conditioned with respect to the 2-condition number, and we can show this in just three steps: prove that $\|Q\|_2 = 1$, $\|Q^{-1}\|_2 = 1$, and $\kappa_2(Q) = 1$.

You may like to verify these claims by using computer software to find the 2-norms and 2-condition numbers of various orthogonal matrices, but it is not sufficient to give an example matrix as your answer.

[10 marks] 2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Compute the QR decomposition of this matrix using reflection matrices, and give Q and R. Show all intermediate steps.

You may modify the provided code on the course website to verify your solution, but giving Q and R by themselves is not a complete answer.

[10 marks] 3. Consider the pair of vectors

$$v_1 = \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Let $S = \operatorname{span}(v_1, v_2) \in \mathbb{R}^4$.

(a) Apply the Gram–Schmidt process to v_1 and v_2 to obtain an orthonormal basis of S. Show all intermediate steps.

You may modify the provided code on the course website to verify your solution, but giving q_1 and q_2 by themselves is not a complete answer.

(b) Consider the matrix

$$V = \begin{bmatrix} 3 & 1 \\ -3 & 2 \\ 3 & 3 \\ -3 & 4 \end{bmatrix}.$$

Using the Gram–Schmidt process, give a matrix $Q \in \mathbb{R}^{4\times 2}$ and a matrix $R \in \mathbb{R}^{2\times 2}$ such that $V = QR$.

[6 marks] 4. In this question, you will need to use computer software to generate matrices and investigate properties of those matrices. You may choose your favourite software (e.g., Matlab, Octave, etc.). Please include your code along with each of your answers.

(a) Generate a random matrix A of dimension 8×6 where the last two columns are linear combinations of the first four columns. You can do this in Matlab or Octave, for example, by typing the commands

```
A = randn(8,4)                   % generate a random 8 x 4 matrix
A(:,5:6) = A(:,1:2) + A(:,3:4) % set col 5 = col 1 + col 3; set col 6 = col 2 + col 4
[Q,R] = qr(randn(6))             % compute QR decomposition of random 6 x 6 matrix
A = A*Q                          % recombine columns of A by multiplying by Q
```

What matrix A do you obtain, and what is its numerical rank? (You can use the command `rank(A)` to find this value.)

(b) How many singular values does A have, and what are they? (You can use the command `svd(A)` to find and list these values.)

(c) Adjust the tolerance of your computer software's `rank` command until the numerical rank you obtained in part (a) changes. What is the changed numerical rank, and at which tolerance level does this change occur? (You can adjust the tolerance by typing `rank(A, tol)` for various values of tol.)

(d) Following what you learned in parts (a)–(c), what sort of things can you conclude about your matrix A, your computer software, and your computer itself?

[8 marks] 5. (a) Let $A \in \mathbb{C}^{n \times n}$ be a square matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

Hint. You may find the following fact useful: a matrix A is invertible if and only if the zero vector is the only solution to the equation $Ax = 0$.

(b) Consider the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

For both of these matrices, answer the following questions:

- What are the eigenvalues of the matrix?
- What is the eigenvector associated with the largest eigenvalue of the matrix?
- Is the matrix semisimple or defective?