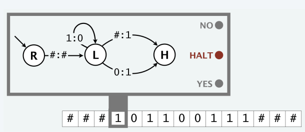
CSCI 355: Algorithm Design and Analysis
2. Algorithm Analysis

- *computational tractability*
- *asymptotic order of growth*
- *survey of common running times*

Models of computation: Turing machines

Deterministic Turing machine. Simple and idealistic model.



Running time. Number of steps.
Memory. Number of tape cells utilized.

Caveat. No random access of memory.
- Single-tape TM requires $\geq n^2$ steps to detect n-bit palindromes.
- Easy to detect palindromes in $\leq cn$ steps on a real computer.

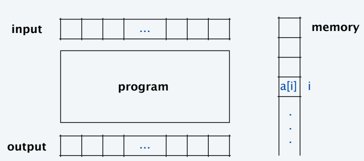
Models of computation: word RAM

Word RAM.

assume $w \geq \log_2 n$

- Each memory location and input/output cell stores a w-bit integer.
- Primitive operations: arithmetic/logic operations, read/write memory, array indexing, following a pointer, conditional branch, …

constant-time C-style operations ($w = 64$)



Running time. Number of primitive operations.
Memory. Number of memory cells utilized.

Caveat. At times, need more refined model (e.g., multiplying n-bit integers).

Brute force

Brute force. For many nontrivial problems, there is a natural brute-force search algorithm that checks every possible solution.

- Typically takes 2^n steps (or worse) for inputs of size n.
- Unacceptable in practice.

Ex. Stable matching problem: test all $n!$ perfect matchings for stability.

Polynomial running time

Desirable scaling property. When the input size doubles, the algorithm should slow down by at most some multiplicative constant factor C.

Def. An algorithm is poly-time if the above scaling property holds.

There exist constants a > 0 and b > 0 such that, for every input of size n, the algorithm performs ≤ a n^b primitive computational steps. ← corresponds to $C = 2^b$

Cobham (1964)

Edmonds (1965)

Polynomial running time

We say that an algorithm is efficient if it has a polynomial running time.

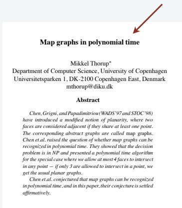
Theory. Definition is (relatively) insensitive to model of computation.

Practice. It really works!

- The poly-time algorithms that people develop have both small constants and small exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions. Some poly-time algorithms in the wild have galactic constants and/or huge exponents.

n^{120} ↙



Map graphs in polynomial time

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Abstract

Chen, Grigni, and Papadimitriou (WADS'97 and STOC'98) have introduced a modified notion of planarity, where two faces are considered adjacent if they share at least one point. The corresponding abstract graphs are called map graphs. *Chen et.al. raised the question of whether map graphs can be recognized in polynomial time. They showed that the decision problem is in NP and presented a polynomial time algorithm for the special case where we allow at most 4 faces to intersect in any point — if only 3 are allowed to intersect in a point, we get the usual planar graphs.*

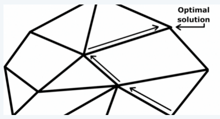
Chen et.al. conjectured that map graphs can be recognized in polynomial time, and in this paper, their conjecture is settled affirmatively.

Q. Which would you prefer: $20\,n^{120}$ or $n^{1\,+\,0.02\,\ln n}$?

Worst-case analysis

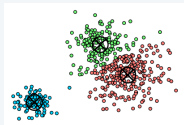
Worst case. Running time guarantee for any input of size n.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Exceptions. Some exponential-time algorithms are used widely in practice because the worst-case instances don't arise.



simplex algorithm

Linux grep



k-means algorithm

Other types of analyses

Probabilistic. Expected running time of a randomized algorithm.
Ex. The expected number of compares to quicksort n elements is $\sim 2n \ln n$.

Amortized. Worst-case running time for any sequence of n operations.
Ex. Starting from an empty stack, any sequence of n push and pop operations takes $O(n)$ primitive computational steps using a resizing array.

Also. Average-case analysis, smoothed analysis, competitive analysis, ...

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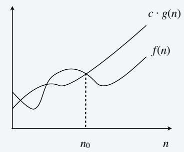
Big O notation

Upper bounds. $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $O(n^2)$. ⟵ choose $c = 50$, $n_0 = 1$
- $f(n)$ is neither $O(n)$ nor $O(n \log n)$.

Typical usage. Insertion sort makes $O(n^2)$ compares to sort n elements.

Big O notational abuses

One-way "equality." $O(g(n))$ is a set of functions, but computer scientists often write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

Ex. Consider $g_1(n) = 5n^3$ and $g_2(n) = 3n^2$.

- We have $g_1(n) = O(n^3)$ and $g_2(n) = O(n^3)$.
- But, do not conclude $g_1(n) = g_2(n)$.

Domain and codomain. f and g are real-valued functions.

- The domain is typically the natural numbers: $\mathbb{N} \to \mathbb{R}$. (input size, recurrence relations)
- Sometimes we extend to the reals: $\mathbb{R}_{\geq 0} \to \mathbb{R}$. (plotting, limits, calculus)
- Or restrict to a subset.

Bottom line. OK to abuse notation in this way; not OK to misuse it.

Big O notation: properties

Reflexivity. f is $O(f)$.

Constants. If f is $O(g)$ and $c > 0$, then $c f$ is $O(g)$.

Products. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 f_2$ is $O(g_1 g_2)$.

Pf.

- $\exists\, c_1 > 0$ and $n_1 \geq 0$ such that $0 \leq f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$.
- $\exists\, c_2 > 0$ and $n_2 \geq 0$ such that $0 \leq f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$.
- Then, $0 \leq f_1(n) \cdot f_2(n) \leq \underbrace{c_1 \cdot c_2}_{c} \cdot g_1(n) \cdot g_2(n)$ for all $n \geq \underbrace{\max\{n_1, n_2\}}_{n_0}$. ▪

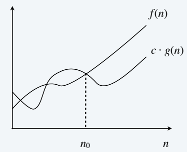
Sums. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$. (ignore lower-order terms)

Transitivity. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

Ex. $f(n) = 5n^3 + 3n^2 + n + 1234$ is $O(n^3)$.

Big Omega notation

Lower bounds. $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n) \geq 0$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

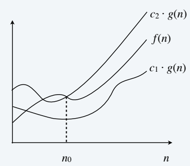
- $f(n)$ is both $\Omega(n^2)$ and $\Omega(n)$. ⟵ choose $c = 32$, $n_0 = 1$
- $f(n)$ is not $\Omega(n^3)$.

Typical usage. Any compare-based sorting algorithm requires $\Omega(n \log n)$ compares in the worst case.

Vacuous statement. Any compare-based sorting algorithm requires at least $O(n \log n)$ compares in the worst case.

Big Theta notation

Tight bounds. $f(n)$ is $\Theta(g(n))$ if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $\Theta(n^2)$. ⟵ choose $c_1 = 32$, $c_2 = 50$, $n_0 = 1$
- $f(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$.

Typical usage. Mergesort makes $\Theta(n \log n)$ compares to sort n elements.

between $\frac{1}{2}\, n \log_2 n$ and $n \log_2 n$

Asymptotic bounds and limits

Proposition. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < \infty$ then $f(n)$ is $\Theta(g(n))$.

Pf.

- By definition of the limit, for any $\varepsilon > 0$, there exists n_0 such that

  $$c - \epsilon \leq \frac{f(n)}{g(n)} \leq c + \epsilon$$

  for all $n \geq n_0$.
- Choose $\varepsilon = \frac{1}{2}\, c > 0$.
- Multiplying by $g(n)$ yields $1/2\, c \cdot g(n) \leq f(n) \leq 3/2\, c \cdot g(n)$ for all $n \geq n_0$.
- Thus, $f(n)$ is $\Theta(g(n))$ by definition, with $c_1 = 1/2\, c$ and $c_2 = 3/2\, c$. ▪

Proposition. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f(n)$ is $O(g(n))$ but not $\Omega(g(n))$.

Proposition. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f(n)$ is $\Omega(g(n))$ but not $O(g(n))$.

Asymptotic bounds for some common functions

Polynomials. Let $f(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then, $f(n)$ is $\Theta(n^d)$.

Pf. $\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0$

Logarithms. $\log_a n$ is $\Theta(\log_b n)$ for every $a > 1$ and every $b > 1$.

Pf. $\frac{\log_a n}{\log_b n} = \frac{1}{\log_b a}$

no need to specify base (assuming it is a constant)

Logarithms and polynomials. $\log_a n$ is $O(n^d)$ for every $a > 1$ and every $d > 0$.

Pf. $\lim_{n \to \infty} \frac{\log_a n}{n^d} = 0$

Exponentials and polynomials. n^d is $O(r^n)$ for every $r > 1$ and every $d > 0$.

Pf. $\lim_{n \to \infty} \frac{n^d}{r^n} = 0$

Factorials. $n!$ is $2^{\Theta(n \log n)}$.

Pf. Stirling's formula: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

Big O notation with multiple variables

Upper bounds. $f(m, n)$ is $O(g(m, n))$ if there exist constants $c > 0$, $m_0 \geq 0$, and $n_0 \geq 0$ such that $0 \leq f(m, n) \leq c \cdot g(m, n)$ for all $n \geq n_0$ and $m \geq m_0$.

Ex. $f(m, n) = 32mn^2 + 17mn + 32n^3$.

- $f(m, n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$.
- $f(m, n)$ is $O(n^3)$ if a precondition to the problem implies $m \leq n$.
- $f(m, n)$ is neither $O(n^3)$ nor $O(mn^2)$.

Typical usage. In the worst case, breadth-first search takes $O(m + n)$ time to find a shortest path from s to t in a digraph with n nodes and m edges.

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Constant time

Constant time. Running time is $O(1)$.

bounded by a constant, which does not depend on input size n

Examples.

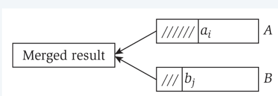
- Conditional branch.
- Arithmetic/logic operation.
- Declare/initialize a variable.
- Follow a link in a linked list.
- Access element i in an array.
- Compare/exchange two elements in an array.
- …

Linear time

Linear time. Running time is $O(n)$.

Merge two sorted lists. Combine two sorted linked lists $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ into a sorted whole.

$O(n)$ algorithm. Merge in mergesort.




```
i ← 1; j ← 1.
WHILE (both lists are nonempty)
   IF (a_i ≤ b_j) append a_i to output list and increment i.
   ELSE           append b_j to output list and increment j.
Append remaining elements from nonempty list to output list.
```


Logarithmic time

Logarithmic time. Running time is $O(\log n)$.

Search in a sorted array. Given a sorted array A of n distinct integers and an integer x, find index of x in array.

$O(\log n)$ algorithm. Binary search.

remaining elements

- Invariant: If x is in the array, then x is in $A[lo \mathinner{..} hi]$.
- After k iterations of WHILE loop, $(hi - lo + 1) \le n / 2^k \Rightarrow k \le 1 + \log_2 n$.

```
lo ← 1; hi ← n.
WHILE (lo ≤ hi)
   mid ← ⌊(lo + hi) / 2⌋.
   IF      (x < A[mid])  hi ← mid – 1.
   ELSE IF (x > A[mid])  lo ← mid + 1.
   ELSE RETURN mid.
RETURN –1.
```

Linearithmic time

Linearithmic time. Running time is $O(n \log n)$.

Sorting. Given an array of n elements, rearrange them in ascending order.

$O(n \log n)$ algorithm. Mergesort.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Quadratic time

Quadratic time. Running time is $O(n^2)$.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest to each other.

$O(n^2)$ algorithm. Enumerate all pairs of points (with $i < j$).

$min \leftarrow \infty$.
FOR $i = 1$ TO n
  FOR $j = i + 1$ TO n
    $d \leftarrow (x_i - x_j)^2 + (y_i - y_j)^2$.
    IF $(d < min)$
      $min \leftarrow d$.

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. [see §5.4]

Cubic time

Cubic time. Running time is $O(n^3)$.

3-SUM. Given an array of n distinct integers, find three that sum to 0.

$O(n^3)$ algorithm. Enumerate all triples (with $i < j < k$).

FOR $i = 1$ TO n
  FOR $j = i + 1$ TO n
    FOR $k = j + 1$ TO n
    IF $(a_i + a_j + a_k = 0)$
      RETURN (a_i, a_j, a_k).

Remark. $\Omega(n^3)$ seems inevitable, but $O(n^2)$ is not hard. [see next slide]

Polynomial time

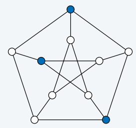
Polynomial time. Running time is $O(n^k)$ for some constant $k > 0$.

Independent set of size k. Given a graph, find k nodes such that no two are joined by an edge.

k is a constant

$O(n^k)$ algorithm. Enumerate all subsets of k nodes.

```
FOREACH subset S of k nodes:
   Check whether S is an independent set.
   IF (S is an independent set)
      RETURN S.
```



independent set of size 3

- Check whether S is an independent set of size k takes $O(k^2)$ time.
- Number of k-element subsets = $\binom{n}{k} = \frac{n(n-1)(n-2)\times\cdots\times(n-k+1)}{k(k-1)(k-2)\times\cdots\times 1} \leq \frac{n^k}{k!}$
- $O(k^2\, n^k / k!) = O(n^k)$.

poly-time for $k = 17$, but not practical

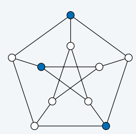
Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

Independent set. Given a graph, find independent set of max size.

$O(n^2\, 2^n)$ algorithm. Enumerate all subsets of n elements.

```
S* ← ∅.
FOREACH subset S of n nodes:
   Check whether S is an independent set.
   IF (S is an independent set and |S| > |S*|)
      S* ← S.
RETURN S*.
```



independent set of max size

Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

Euclidean TSP. Given n points in the plane, find a tour of minimum length.

$O(n \times n!)$ algorithm. Enumerate all permutations of length n.

```
π* ← ∅.
FOREACH permutation π of n points:
   Compute length of tour corresponding to π.
   IF (length(π) < length(π*))
      π* ← π.
RETURN π*.
```

for simplicity, we'll assume Euclidean distances are rounded to nearest integer (to avoid issues with infinite precision)