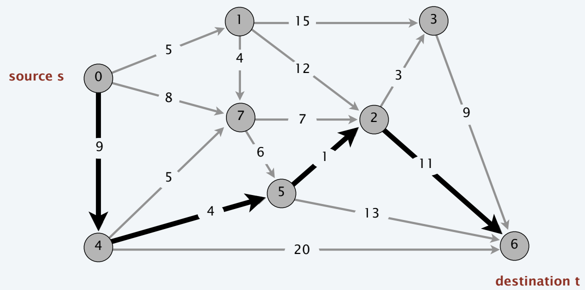
CSCI 355: Algorithm Design and Analysis
5. Greedy Algorithms II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Borůvka*
- *single-link clustering*

Single-pair shortest path problem

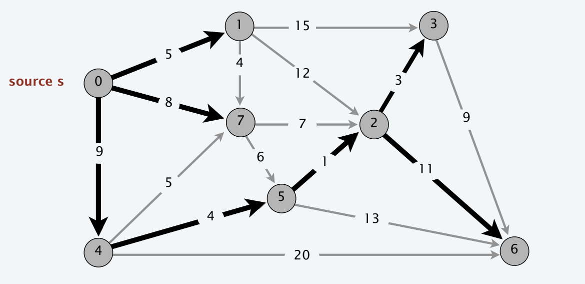
Problem. Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, and destination $t \in V$, find a shortest directed path from s to t.



length of path = 9 + 4 + 1 + 11 = 25

Single-source shortest paths problem

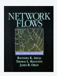
Problem. Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, find a shortest directed path from s to every node.



shortest-paths tree

Shortest path applications

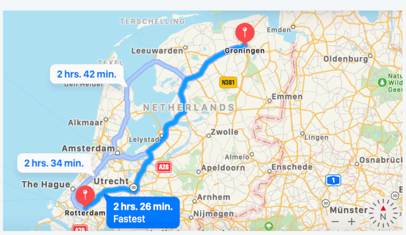
- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in LaTeX.
- Urban traffic planning.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Optimal truck routing through given traffic congestion pattern.



Network Flows: Theory, Algorithms, and Applications,
by Ahuja, Magnanti, and Orlin, Prentice Hall, 1993.

Edsger Dijkstra

" *What's the shortest way to travel from Rotterdam to Groningen? It is the algorithm for the shortest path, which I designed in about 20 minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path.* " — Edsger Dijkstra



Dijkstra's algorithm: single-source shortest paths

see 5A

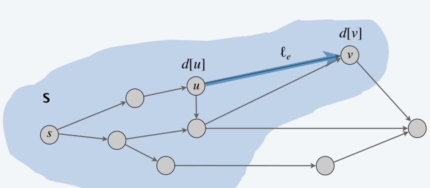
Greedy approach. Maintain a set of explored vertices S for which the algorithm has determined $d[u]$ = the length of a shortest $s \rightsquigarrow u$ path.

- Initialize $S \leftarrow \{ s \}$, $d[s] \leftarrow 0$.
- Repeatedly choose an unexplored vertex $v \notin S$ which minimizes

$$\pi(v) \;=\; \min_{e\,=\,(u,v)\,:\,u \in S} d[u] + \ell_e$$

the length of a shortest path from s to some vertex u in the explored part S, followed by a single edge $e = (u, v)$

  add v to S, and set $d[v] \leftarrow \pi(v)$.
- To recover the path, set $pred[v] \leftarrow e$ that achieves the minimum.



Dijkstra's algorithm: proof of correctness

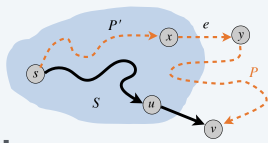
Invariant. For each vertex $u \in S$, $d[u]$ = length of a shortest $s \rightsquigarrow u$ path.

Pf. [by induction on $|S|$]

Base case: $|S| = 1$ is easy since $S = \{ s \}$ and $d[s] = 0$.

Inductive hypothesis: Assume true for $|S| \geq 1$.

- Let v be the next vertex added to S, and let (u, v) be the final edge.
- A shortest $s \rightsquigarrow u$ path plus (u, v) is an $s \rightsquigarrow v$ path of length $\pi(v)$.
- Consider any other $s \rightsquigarrow v$ path P. We show that it is no shorter than $\pi(v)$.
- Let $e = (x, y)$ be the first edge in P that leaves S, and let P' be the subpath from s to x.
- The length of P is already $\geq \pi(v)$ as soon as it reaches y:



$$\ell(P) \;\geq\; \ell(P') + \ell_e \;\geq\; d[x] + \ell_e \;\geq\; \pi(y) \;\geq\; \pi(v) \quad \blacksquare$$

↑ non-negative lengths — ↑ inductive hypothesis — ↑ definition of $\pi(y)$ — ↑ Dijkstra's alg. chose v instead of y

Dijkstra's algorithm: efficient implementation

see 5A

Critical optimization 1. For each unexplored vertex $v \notin S$: explicitly maintain $\pi[v]$ instead of computing directly from definition

$$\pi(v) \;=\; \min_{e = (u,v)\,:\, u \in S} d[u] + \ell_e$$

- For each $v \notin S$: $\pi(v)$ can only decrease (because set S increases).
- More specifically, suppose u is added to S and there is an edge $e = (u, v)$ leaving u. Then, it suffices to update:

$$\pi[v] \leftarrow \min \{ \pi[v],\ \pi[u] + \ell_e) \}$$

↑ recall: for each $u \in S$, $\pi[u] = d[u]$ = length of shortest $s \rightsquigarrow u$ path

Critical optimization 2. Use a min-oriented priority queue (PQ) to choose an unexplored vertex that minimizes $\pi[v]$.

Dijkstra's algorithm: efficient implementation

Implementation.

- Algorithm maintains $\pi[v]$ for each node v.
- Priority queue stores unexplored vertices, using $\pi[\cdot]$ as priorities.
- Once u is deleted from the PQ, $\pi[u]$ = length of a shortest $s \rightsquigarrow u$ path.

```
DIJKSTRA (V, E, ℓ, s)
FOREACH v ≠ s : π[v] ← ∞, pred[v] ← null; π[s] ← 0.
Create an empty priority queue pq.
FOREACH v ∈ V : INSERT(pq, v, π[v]).
WHILE (IS-NOT-EMPTY(pq))
  u ← DEL-MIN(pq).
  FOREACH edge e = (u, v) ∈ E leaving u:
    IF (π[v] > π[u] + ℓe)
      DECREASE-KEY(pq, v, π[u] + ℓe).
      π[v] ← π[u] + ℓe ; pred[v] ← e.
```

Dijkstra's algorithm: which priority queue?

Performance. Depends on PQ: n INSERT, n DELETE-MIN, $\leq m$ DECREASE-KEY.

- Array implementation is optimal for dense graphs. ⟵ $\Theta(n^2)$ edges
- Binary heap is much faster for sparse graphs. ⟵ $\Theta(n)$ edges
- 4-way heap is worth the trouble in performance-critical situations.

priority queue	INSERT	DELETE-MIN	DECREASE-KEY	total
node-indexed array (A[i] = priority of i)	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman–Tarjan 1984)	$O(1)$	$O(\log n)$ †	$O(1)$ †	$O(m + n \log n)$
integer priority queue (Thorup 2004)	$O(1)$	$O(\log \log n)$	$O(1)$	$O(m + n \log \log n)$

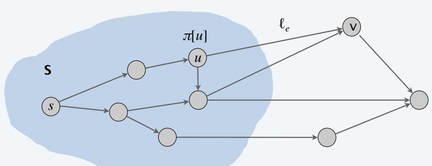
(assuming $m \geq n$)
† = amortized

Extensions of Dijkstra's algorithm

Dijkstra's algorithm and proof extend to several related problems:

- Shortest paths in undirected graphs: $\pi[v] \leq \pi[u] + \ell(u, v)$.
- Maximum capacity paths: $\pi[v] \geq \min \{ \pi[u],\ c(u, v) \}$.
- Maximum reliability paths: $\pi[v] \geq \pi[u] \times \gamma(u, v)$.

Key algebraic structure. Closed semiring (min-plus, bottleneck, Viterbi, …).



$$
\begin{aligned}
a + b &= b + a \\
a + (b + c) &= (a + b) + c \\
a + 0 &= a \\
a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\
a \cdot 0 = 0 \cdot a &= 0 \\
a \cdot 1 = 1 \cdot a &= a \\
a \cdot (b + c) &= a \cdot b + a \cdot c \\
(a + b) \cdot c &= a \cdot c + b \cdot c \\
a^* = 1 + a \cdot a^* &= 1 + a^* \cdot a
\end{aligned}
$$

CSCI 355: ALGORITHM DESIGN AND ANALYSIS
5. GREEDY ALGORITHMS II

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- *minimum spanning trees*
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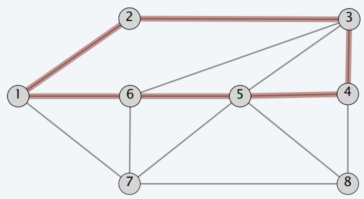
Paths and cycles

Def. A path is a sequence of edges which connects a sequence of vertices.

Def. A cycle is a path with no repeated vertices or edges other than the starting and ending vertices.



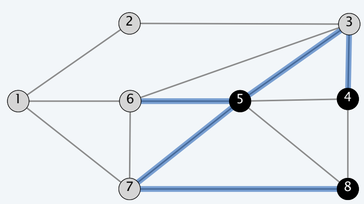
path P = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6) }

cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

Cuts

Def. A cut is a partition of vertices into two nonempty subsets S and $V - S$.

Def. The cutset of a cut S is the set of edges with exactly one endpoint in S.

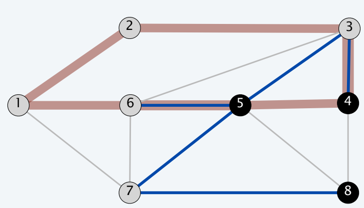


cut S = { 4, 5, 8 }

cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

Cycle–cut intersection

Proposition. A cycle and a cutset intersect in an even number of edges.



cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

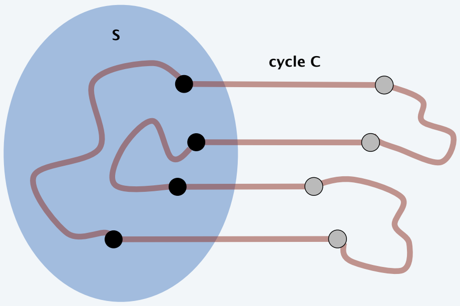
cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

intersection C ∩ D = { (3, 4), (5, 6) }

Cycle–cut intersection

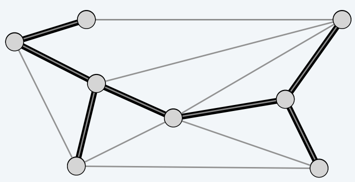
Proposition. A cycle and a cutset intersect in an even number of edges.

Pf. [by picture]



Spanning trees

Def. Let $H = (V, T)$ be a subgraph of an undirected graph $G = (V, E)$.
H is a spanning tree of G if H is both acyclic and connected.

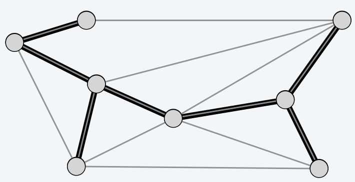


graph G = (V, E)
spanning tree H = (V, T)

Spanning trees: properties

Proposition. Let $H = (V, T)$ be a subgraph of an undirected graph $G = (V, E)$.
Then, the following statements are equivalent:

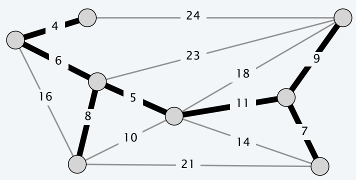
- H is a spanning tree of G.
- H is acyclic and connected.
- H is connected and has $|V| - 1$ edges.
- H is acyclic and has $|V| - 1$ edges.
- H is minimally connected: the removal of any edge disconnects H.
- H is maximally acyclic: the addition of any edge creates a cycle in H.



graph G = (V, E)
spanning tree H = (V, T)

Minimum spanning trees (MSTs)

Def. Given a connected, undirected graph $G = (V, E)$ with edge costs c_e, a minimum spanning tree (V, T) is a spanning tree of G such that the sum of the edge costs in T is minimized.



MST cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

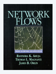
Cayley's theorem. The complete graph on n nodes has n^{n-2} spanning trees.

can't solve by brute force

MST applications

MST is a fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Model locality of particle interactions in turbulent fluid flows.
- Reducing data storage in sequencing amino acids in a protein.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).



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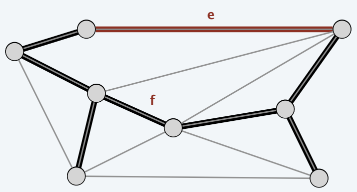
Fundamental cycles

Fundamental cycle. Let $H = (V, T)$ be a spanning tree of $G = (V, E)$.

- For any non-tree edge $e \in E$, $T \cup \{ e \}$ contains a unique cycle, say C.
- For any edge $f \in C$, $(V, T \cup \{ e \} - \{ f \})$ is a spanning tree.



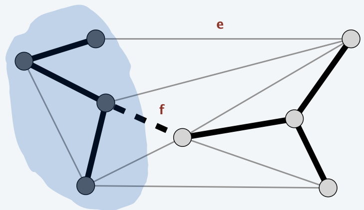
graph G = (V, E)
spanning tree H = (V, T)

Observation. If $c_e < c_f$, then (V, T) is not an MST.

Fundamental cutsets

Fundamental cutset. Let $H = (V, T)$ be a spanning tree of $G = (V, E)$.

- For any tree edge $f \in T$, $(V, T - \{ f \})$ has two connected components.
- Let D denote the corresponding cutset.
- For any edge $e \in D$, $(V, T - \{ f \} \cup \{ e \})$ is a spanning tree.



Observation. If $c_e < c_f$, then (V, T) is not an MST.

Greedy algorithm: MSTs

see 5B

Red rule.

- Let C be a cycle with no red edges.
- Select an uncoloured edge of C of max cost and colour it red.

Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncoloured edge in D of min cost and colour it blue.

Greedy algorithm.

- Apply the red and blue rules (nondeterministically!) until all edges are coloured. The blue edges form an MST.
- Note: we can stop once we have $n - 1$ edges coloured blue.

Greedy algorithm: proof of correctness

Colour invariant. There exists an MST (V, T^*) containing every blue edge and no red edge.

Pf. [by induction on number of iterations]

Base case. No edges coloured ⇒ every MST satisfies the invariant.

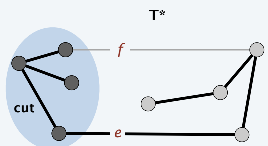
Greedy algorithm: proof of correctness

Colour invariant. There exists an MST (V, T^*) containing every blue edge and no red edge.

Pf. [by induction on number of iterations]

Induction step (blue rule). Suppose the colour invariant is true before applying the blue rule.

- let D be the chosen cutset, and let f be the edge coloured blue.
- if $f \in T^*$, then T^* still satisfies the invariant.
- Otherwise, consider the fundamental cycle C by adding f to T^*.
- let $e \in C$ be another edge in D.
- e is uncoloured and $c_e \geq c_f$ since
  - $e \in T^* \Rightarrow e$ is not red
  - blue rule $\Rightarrow e$ is not blue and $c_e \geq c_f$
- Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies the invariant.



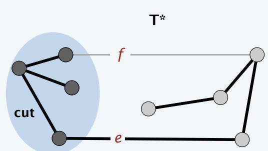
Greedy algorithm: proof of correctness

Colour invariant. There exists an MST (V, T^*) containing every blue edge and no red edge.

Pf. [by induction on number of iterations]

Induction step (red rule). Suppose the colour invariant is true before applying the red rule.

- let C be the chosen cycle, and let e be the edge coloured red.
- if $e \notin T^*$, then T^* still satisfies the invariant.
- Otherwise, consider the fundamental cutset D by deleting e from T^*.
- let $f \in D$ be another edge in C.
- f is uncoloured and $c_e \geq c_f$ since
  - $f \notin T^* \Rightarrow f$ is not blue
  - red rule $\Rightarrow f$ is not red and $c_e \geq c_f$
- Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies the invariant. ▪

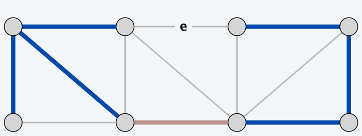


Greedy algorithm: proof of correctness

Theorem. The greedy algorithm terminates, and blue edges form an MST.

Pf. We need to show that either the red or blue rule (or both) applies.

- Suppose edge e is left uncoloured.
- Blue edges form a forest.
- Case 1: both endpoints of e are in the same blue tree.
  $\Rightarrow$ apply the red rule to the cycle formed by adding e to the blue forest.
- Case 2: both endpoints of e are in different blue trees.
  $\Rightarrow$ apply the blue rule to the cutset induced by either of the two blue trees. ▪



Case 2

CSCI 355: Algorithm Design and Analysis
5. Greedy Algorithms II

Review: the greedy MST algorithm

Red rule.
- Let C be a cycle with no red edges.
- Select an uncoloured edge of C of max cost and colour it red.

Blue rule.
- Let D be a cutset with no blue edges.
- Select an uncoloured edge in D of min cost and colour it blue.

Greedy algorithm.
- Apply the red and blue rules (nondeterministically!) until all edges are coloured. The blue edges form an MST.
- Note: we can stop once we have $n - 1$ edges coloured blue.

Theorem. The greedy algorithm is correct.

Special cases of MST algorithms

Special cases. Prim, Kruskal, reverse-delete, Borůvka, ...

Prim's algorithm.
- Adds edges outward from an arbitrary starting vertex.
- Works well on graphs with many edges (dense graphs).

Kruskal's algorithm.
- Adds edges in order from least cost to greatest cost.
- Works well on graphs with few edges (sparse graphs).

Reverse-delete algorithm.
- Deletes edges in order from greatest cost to least cost.

Borůvka's algorithm.
- Finds all min-cost edges incident to each connected component, and adds those edges to a forest.
- Adapts well to parallelization.

Prim's algorithm: MSTs

see 5B

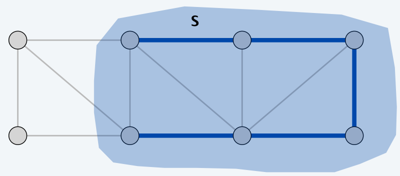
Initialize $S = \{ s \}$ for any vertex s, and set $T = \varnothing$.
Repeat $n - 1$ times:

- Add to T a min-cost edge with exactly one endpoint in S.
- Add the other endpoint of the edge to S.

Theorem. Prim's algorithm computes an MST.

by construction, edges in cutset are uncolored

Pf. Special case of greedy algorithm (blue rule repeatedly applied to S). ▪



Prim's algorithm: implementation

Theorem. Prim's algorithm can be implemented in $O(m \log n)$ time.
Pf. Implementation almost identical to Dijkstra's algorithm.

```
PRIM (V, E, c)
S ← ∅, T ← ∅.
s ← any node in V.
FOREACH v ≠ s : π[v] ← ∞, pred[v] ← null; π[s] ← 0.
Create an empty priority queue pq.
FOREACH v ∈ V : INSERT(pq, v, π[v]).
WHILE (IS-NOT-EMPTY(pq))
   u ← DEL-MIN(pq).
   S ← S ∪ { u }, T ← T ∪ { pred[u] }.
   FOREACH edge e = (u, v) ∈ E with v ∉ S :
      IF (c_e < π[v])
         DECREASE-KEY(pq, v, c_e).
         π[v] ← c_e; pred[v] ← e.
```

$\pi[v]$ = cost of cheapest known edge between v and S

Kruskal's algorithm: MSTs

see 5B

Consider edges in ascending order of cost:

- Add the edge to the tree unless it would create a cycle.

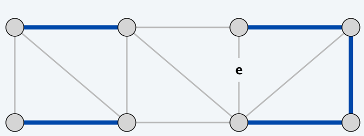
Theorem. Kruskal's algorithm computes an MST.

Pf. Special case of greedy algorithm.

- Case 1: both endpoints of e in same blue tree.
  ⇒ color e red by applying red rule to unique cycle.
- Case 2: both endpoints of e in different blue trees.
  ⇒ color e blue by applying blue rule to cutset defined by either tree. ▪

all other edges in cycle are blue

no edge in cutset has smaller cost (since Kruskal chose it first)



Kruskal's algorithm: implementation

Theorem. Kruskal's algorithm can be implemented in $O(m \log m)$ time.

Pf.

- Sort edges by cost.
- Use union–find data structure to dynamically maintain connected components.

```
KRUSKAL (V, E, c)

SORT m edges by cost and renumber so that c(e1) ≤ c(e2) ≤ … ≤ c(em).
T ← ∅.
FOREACH v ∈ V : MAKE-SET(v).
FOR i = 1 TO m
   (u, v) ← ei.
   IF (FIND-SET(u) ≠ FIND-SET(v))   ← are u and v in same component?
      T ← T ∪ { ei }.
      UNION(u, v).   ← make u and v in same component
RETURN T.
```


Reverse-delete algorithm: MSTs

see 5B

Start with all edges in T and consider them in descending order of cost:

- Delete each edge from T unless doing so would disconnect T.

Theorem. The reverse-delete algorithm computes an MST.

Pf. Special case of greedy algorithm.

- Case 1. [deleting edge e does not disconnect T]
  ⇒ apply red rule to cycle C formed by adding e to another path in T between its two endpoints.
  (no edge in C is more expensive (it would have already been considered and deleted))
- Case 2. [deleting edge e disconnects T]
  ⇒ apply blue rule to cutset D induced by either component. ▪
  (e is the only remaining edge in the cutset (all other edges in D must have been colored red / deleted))

Fact. [Thorup 2000] Reverse-delete can be implemented in $O(m \log n (\log \log n)^3)$ time.

Borůvka's algorithm: MSTs

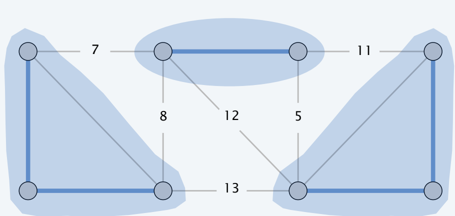
see 5B

Repeat until only one tree remains:

- Apply blue rule to the cutset corresponding to each blue tree.
- Color all selected edges blue.

Theorem. Borůvka's algorithm computes the MST. ← assuming edge costs are distinct

Pf. Special case of greedy algorithm (repeatedly apply blue rule). ▪

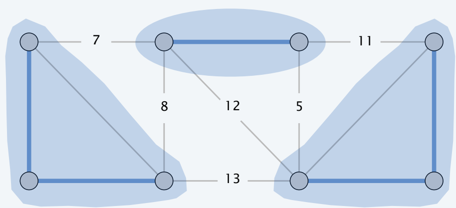


Borůvka's algorithm: implementation

Theorem. Borůvka's algorithm can be implemented in $O(m \log n)$ time.

Pf.

- To implement a phase in $O(m)$ time:
  - compute connected components of blue edges
  - for each edge $(u, v) \in E$, check if u and v are in different components; if so, update each component's best edge in cutset
- $\leq \log_2 n$ phases since each phase (at least) halves total # components. ▪



Does a linear-time comparison-based MST algorithm exist?

year	worst case	discovered by
1975	$O(m \log \log n)$	Yao
1976	$O(m \log \log n)$	Cheriton–Tarjan
1984	$O(m \log^* n)$, $O(m + n \log n)$	Fredman–Tarjan
1986	$O(m \log (\log^* n))$	Gabow–Galil–Spencer–Tarjan
1997	$O(m\, \alpha(n) \log \alpha(n))$	Chazelle
2000	$O(m\, \alpha(n))$	Chazelle
2002	*asymptotically optimal*	Pettie–Ramachandran
20xx	$O(m)$	???

deterministic compare-based MST algorithms

iterated logarithm function

$$\lg^* n = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \lg^*(\lg n) & \text{if } n > 1 \end{cases}$$

n	$\lg^* n$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 2^{16}]$	4
$(2^{16}, 2^{65536}]$	5

(α: inverse Ackermann function)

Theorem. [Fredman–Willard 1990] $O(m)$ in word RAM model.

Theorem. [Dixon–Rauch–Tarjan 1992] $O(m)$ MST verification algorithm.

Theorem. [Karger–Klein–Tarjan 1995] $O(m)$ randomized MST algorithm.

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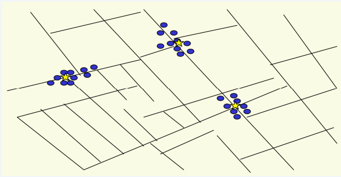
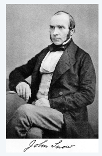
5. Greedy Algorithms II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Borůvka*
- *single-link clustering*

Clustering

Goal. Given a set U of n objects labeled $p_1, \ldots, p_n$, partition the objects into clusters so that objects in different clusters are far apart.





outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.
- Routing in mobile ad-hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases
- Cluster celestial objects into stars, quasars, galaxies.

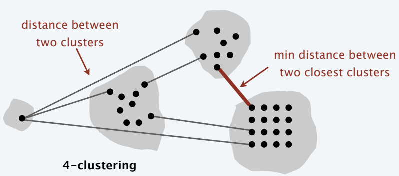
Clustering with maximum spacing

k-clustering. Divide objects into k non-empty groups.

Distance function. Numeric value specifying "closeness" of two objects.
- $d(p_i, p_j) = 0$ iff $p_i = p_j$ [identity of indiscernibles]
- $d(p_i, p_j) \geq 0$ [non-negativity]
- $d(p_i, p_j) = d(p_j, p_i)$ [symmetry]

Spacing. Min distance between any pair of points in different clusters.

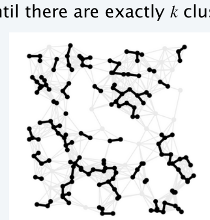
Goal. Given an integer k, find a k-clustering with maximum spacing.



4-clustering

Greedy clustering algorithm

"Well-known" algorithm for single-linkage k-clustering:
- Form a graph on the vertex set U, corresponding to n clusters.
- Find the closest pair of objects such that each object is in a different cluster, and add an edge between them.
- Repeat $n - k$ times (until there are exactly k clusters).

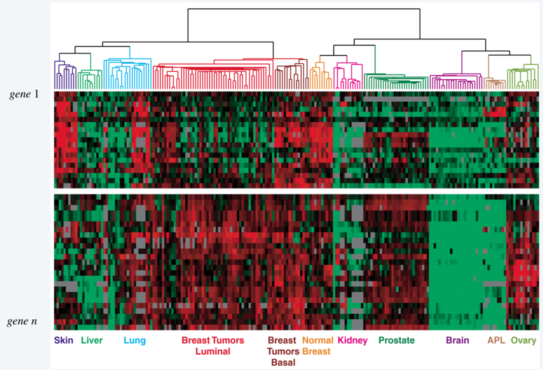


Key observation. This procedure is precisely Kruskal's algorithm (except we stop when there are k connected components).

Alternative. Find an MST and delete the $k - 1$ longest edges.

Dendrogram of cancers in human

Tumors in similar tissues cluster together.



Reference: Botstein & Brown group