St. Francis Xavier University
Department of Computer Science

CSCI 544: Computational Logic
Assignment 1
Due February 3, 2023 at 11:15am

[6 marks] 1. Recall the biconditional logical connective, "if and only if". We know that $p \Leftrightarrow q$ is true whenever both p and q are true or both p and q are false.

This logical connective is called "if *and* only if" because it is true when both the "if" connective *and* the "only if" connective are true; that is, when $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is true.

By contrast, what are the truth conditions for the following fantasy logical connectives, and do we really need to define them? Draw truth tables for each, and show that each is logically equivalent to some existing logical connective.

(a) "If *if* only if".

(b) "If *only if* only if".

(c) "If *if and only if* only if".

[6 marks] 2. Given a set C of logical connectives, we say that the set is *adequate* if all other logical connectives can be expressed as some combination of only the connectives in C.

(a) Show that we can express the connective $\vee$ in terms of the set of connectives $\{\neg, \wedge\}$.

(b) Explain why the set $\{\neg, \wedge, \vee\}$ is an adequate set of connectives.

Hint. Suppose we have a truth table. Can we convert the truth table to a propositional formula sharing the same truth values using only these three connectives?

(c) Explain why the set $\{\wedge, \Rightarrow\}$ is *not* an adequate set of connectives.

Hint. Suppose we have an interpretation assigning true to every formula. What will the truth value be of a formula using only $\wedge$ and $\Rightarrow$, and what does this imply?

[7 marks] 3. Suppose we want to prove that

$$\vDash ((p \wedge q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \vee (q \Rightarrow r)).$$

Proving it in this form may appear strange, since one could misinterpret this as saying that if r follows from $p \wedge q$, then r follows from either p or q alone. This is not always correct. To remove any uncertainty, prove that

$$\{p \wedge q \Rightarrow r\} \vDash (p \Rightarrow r) \vee (q \Rightarrow r),$$

but

$$\{p \wedge q \Rightarrow r\} \nvDash p \Rightarrow r \text{ and}$$
$$\{p \wedge q \Rightarrow r\} \nvDash q \Rightarrow r.$$

[6 marks] 4. Using the method of semantic tableaux, determine whether each of the following formulas are valid. If a formula is not valid, give an example of an interpretation demonstrating this. (Remember that a formula A is valid if and only if $\neg A$ is unsatisfiable.)

(a) $A = (p \wedge q) \Rightarrow (p \vee r)$.

(b) $B = (p \vee \neg(q \wedge r)) \Rightarrow ((p \Leftrightarrow r) \vee q)$.

[10 marks] 5. Prove the validity of each of the following sequents using natural deduction.

(a) $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$.

(b) $p \Rightarrow q, r \Rightarrow s \vdash (p \vee r) \Rightarrow (q \vee s)$

[5 marks] 6. Using natural deduction, prove the following:

$$\vdash (p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow q) \Rightarrow q).$$

Hint. You may use the law of excluded middle (i.e., $p \vee \neg p$) as an intermediate "lemma" step in your proof.