St. Francis Xavier University
Department of Computer Science

CSCI 544: Computational Logic
Lecture 8: Predicate Logic III—Deductive Systems
Winter 2023

1 Extending Natural Deduction

Recall that *natural deduction* is a deductive system that allows us to make inferences about logical formulas using a set of inference rules. When we first introduced natural deduction in the context of propositional logic, we had a number of inference rules to handle the standard logical connectives of conjunction, disjunction, and so on.

Now that we're working with predicate logic, we can still use natural deduction to infer things about formulas, but we must add to our inference rule set. Strictly speaking, we must develop an entirely "new" set of inference rules for every logical symbol in predicate logic, but the rules for logical connectives in propositional logic will be identical to those in predicate logic. (In a sense, we are extending or overloading the existing rules to work for the semantics of predicate logic as well as propositional logic.) For this reason, we will skip the definitions of these inference rules and focus our attention on the inference rules for symbols unique to predicate logic.

Just like before, our inference rules will have both an *introduction* form and an *elimination* form. Some inference rules will also require us to bring back the notion of subproofs, and each new rule we study in this lecture will be accompanied by some illustrative examples.

Remark. In what follows, we will occasionally be using substitutions. Whenever we write a substitution $A[x/t]$, we will implicitly assume that t is free for x in A.

1.1 Universal Quantification

The introduction and elimination inference rules for universal quantification are a lot like the earlier inference rules we saw for the conjunction connective. Recall the forms of those rules:

$$\frac{A_1 \qquad A_2}{A_1 \land A_2} \land \text{ i} \qquad \frac{A_1 \land A_2}{A_1} \land \text{ e}_1 \qquad \frac{A_1 \land A_2}{A_2} \land \text{ e}_2$$

Each of these inference rules operate on two conjuncts: A_1 and A_2. In the introduction case, we know that if A_1 and A_2 hold, then $A_1 \land A_2$ must also hold. For elimination, we know that we can get either of A_1 or A_2 from the formula $A_1 \land A_2$.

We can think of the inference rules for universal quantification as being a lot like the generalized form of these conjunction inference rules. Instead of having two conjuncts, we will consider an arbitrary number of conjuncts (precisely speaking, one for each possible substitution of the quantified variable).

To eliminate a universal quantifier, we will make the observation that, given the formula $\forall x\ A$, we can deduce the substitution $A[x/t]$ for whichever term t we like while ensuring that this substitution holds by the property of universal quantification.

$$\frac{\forall x\ A}{A[x/t]} \forall \text{ e}$$

Example 1. Let's prove that the sequent $\forall x\ (P(x) \Rightarrow \neg Q(x)), P(t) \vdash \neg Q(t)$ is valid. The proof, using our universal quantification elimination inference rule, is as follows:

1.	$\forall x\ (P(x) \Rightarrow \neg Q(x))$	premise
2.	$P(t)$	premise
3.	$P(t) \Rightarrow \neg Q(t)$	$\forall$ e 1
4.	$\neg Q(t)$	$\Rightarrow$ e 3, 2

Line 3 makes use of the universal quantification elimination rule to replace all occurrences of x in the first premise's formula with the term t. We can then apply the implication elimination rule to that resultant formula together with our second premise to obtain our desired conclusion.

Our rule for introducing a universal quantifier is a bit more complex, though again we can relate it to our rule for conjunction introduction. In order to introduce conjunction, we had to show that both subformulas A_1 and A_2 held so that we could then write $A_1 \wedge A_2$.

To introduce a universal quantifier, we must show that some formula A holds for all values x_0 we can substitute into the variable x. We say that x_0 is a *dummy variable*, used to represent any of the possible values we could have chosen to substitute. The premise part of our rule uses a subproof box similar to what we used previously for assumptions, but in this context the box refers to the scope of the dummy variable x_0.

In summary, if we start with a "fresh" dummy variable x_0 and we can derive $A[x/x_0]$, then we can consequently derive $\forall x\ A$. Since we made no assumptions about x_0 besides the fact that it exists, we can draw the conclusion that any value works in the substitution, thus leading us to the universal quantification we desire.

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ A[x/x_0] \end{array}}}{\forall x\ A}\ \forall\ \text{i}$$

If this inference rule doesn't immediately make sense, or if you don't immediately see how we can go from the specific case (involving x_0) to the general case (involving $\forall x$), then consider the following analogy. Suppose that a computer technician says that they can repair any broken computer you give them. It's easy enough to give the technician a broken computer and verify that they can, in fact, repair it. You couldn't possibly give the technician every broken computer on the planet, though. However, assuming that the computer you gave the technician wasn't special or pre-prepared in any way—that is, the broken computer was arbitrarily chosen—then the technician repairing that computer is sufficient evidence to convince you that they're capable of repairing any broken computer.

For this reason, we must choose a "fresh" dummy variable x_0 that has no assumptions associated with it whatsoever. In other words, an application of this inference rule is correct only if we arrive at $A[x/x_0]$ in such a way that no assumptions contain x as a free variable. Any assumptions involving x place a constraint on the quantification, which is something that we don't want.

Example 2. Let's prove that the sequent $\forall x\ (P(x) \Rightarrow Q(x)), \forall x\ P(x) \vdash \forall x\ Q(x)$ is valid. The proof, using our universal quantification introduction inference rule, is as follows:

1.		$\forall x\ (P(x) \Rightarrow Q(x))$	premise
2.		$\forall x\ P(x)$	premise
3.	x_0	$P(x_0) \Rightarrow Q(x_0)$	$\forall$ e 1
4.		$P(x_0)$	$\forall$ e 2
5.		$Q(x_0)$	$\Rightarrow$ e 3, 4
6.		$\forall x\ Q(x)$	$\forall$ i 3–5

Here, we make use of the universal quantification introduction rule to obtain the conclusion, and so we must introduce a "fresh" dummy variable x_0 in our subproof immediately before the conclusion. The remainder of the proof is a straightforward application of our other inference rules: we use universal quantification elimination on lines 3–4 of the subproof, followed by implication elimination on line 5, and this is enough to give us our conclusion.

Before we move on, observe that each of our inference rules for universal quantification involve a substitution $A[x/t]$, yet this notation doesn't appear in any of our proofs. This gets back to a point we made regarding substitutions in an earlier lecture: $A[x/t]$ is not itself a formula, but instead a notation referring to the resultant formula obtained by replacing all occurrences of x in A with t. In our proofs, we simply apply the substitution to obtain that resultant formula directly, and we use that formula in the proof.

1.2 Existential Quantification

You may recall from our earlier discussion of natural deduction that our inference rule for disjunction introduction was quite similar to the rule for conjunction elimination, but "reversed"; that is, the $\vee$ i rule acted like the dual of the $\wedge$ e rule. Looking at one form of each rule, we can see this duality clearly:

$$\frac{A_1 \wedge A_2}{A_1} \wedge \text{e}_1 \qquad\qquad \frac{A_1}{A_1 \vee A_2} \vee \text{i}_1$$

We now know from the previous section that our universal quantification inference rules act like a generalized form of our conjunction rules, applying to n conjuncts instead of two. Thus, following the connection between our conjunction and disjunction rules, it follows that the dual of our universal quantification elimination rule should behave somewhat like a rule for existential quantification introduction.

What would such a dual rule look like? Well, recall that our universal quantification elimination rule takes a quantified formula $\forall x\ A$ and removes the quantifier by performing a substitution: if A holds for all values of x, then surely it must hold for some particular value t substituted into x. Flipping this around, we see that if A holds for some particular value t, then naturally there exists some value we can assign to x that satisfies A. Thus, our existential quantification introduction rule will take the following form:

$$\frac{A[x/t]}{\exists x\ A} \exists \text{ i}$$

As we can see, we've essentially "reversed" our $\forall$e rule and replaced the $\forall$ quantifier with a $\exists$ quantifier. In our premise, we state that A holds for some value t substituted into every occurrence of x. That, of course, implies that there exists a value (namely, t) for which A is satisfied, and so we can draw the conclusion $\exists x\ A$.

Example 3. Let's prove that the sequent $\neg P(y) \vdash \exists x\ (P(x) \Rightarrow Q(y))$ is valid. The proof, using our existential quantification introduction rule, is as follows:

1.	$\neg P(y)$	premise
2.	$P(y)$	assumption
3.	$\bot$	$\neg$ e 2, 1
4.	$Q(y)$	$\bot$ e 3
5.	$P(y) \Rightarrow Q(y)$	$\Rightarrow$ i 2–4
6.	$\exists x\ (P(x) \Rightarrow Q(y))$	$\exists$ i 5

Here, we employ a subproof to arrive at a contradiction on line 3, from which we can then conclude anything we want—namely, $Q(y)$. Introducing implication on line 5 gets us close to our goal, and we use our new inference rule on line 6 to arrive at the desired conclusion.

Note that, in this example, the formula to which we applied our $\exists$ i rule was $A[x/y] = P(y) \Rightarrow Q(y)$. We can then see that we took the original formula to be $A = P(x) \Rightarrow Q(y)$. However, knowing what $A[x/t]$ evaluates to does not always determine what A is itself. We could have instead taken the formula to be $A = P(x) \Rightarrow Q(x)$, and in doing so, we would have arrived at the conclusion $\exists x\ (P(x) \Rightarrow Q(x))$, which is not what we wanted to prove.

Let's stay with our analogy to our disjunction inference rules to develop a rule for existential quantification elimination. Recall that our rule for disjunction elimination was as follows:

$$\frac{A_1 \vee A_2 \qquad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \qquad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \vee \text{ e}$$

Since we didn't know in advance which of A_1 or A_2 held, we had to consider two subproofs taking either of A_1 or A_2 as assumptions. If, in both subproofs, we arrived at the same conclusion B, then we took B to be our overall conclusion.

Likewise, when we try to eliminate an existential quantifier, we don't know in advance for which values of x does the formula A hold. Thus, we must introduce a subproof as one of our premises. We know by the premise $\exists x\ A$ that A holds for at least one value of x, so our subproof will introduce a "fresh" dummy variable x_0 to represent a generic, arbitrary value that we assign to x. Then, starting from the assumption $A[x/x_0]$, if we can arrive at some conclusion B, we know that this formula B must hold no matter the value of x_0. Thus, we take B to be our overall conclusion.

Remark. Note that we require the usual assumption that x_0 does not occur outside of its subproof; namely, x_0 cannot occur in the formula B we derive.

Having made this observation, our inference rule for existential quantification elimination takes the following form:

$$\frac{\exists x\ A \qquad \boxed{\begin{array}{cc} x_0 & A[x/x_0] \\ & \vdots \\ & B \end{array}}}{B} \exists \text{ e}$$

Example 4. Let's prove that the sequent $\exists x\ P(x), \forall x\ (P(x) \Rightarrow Q(x)) \vdash \exists x\ Q(x)$ is valid. The proof, using our existential quantification elimination rule, is as follows:

1.		$\forall x\ (P(x) \Rightarrow Q(x))$	premise
2.		$\exists x\ P(x)$	premise
3.	x_0	$P(x_0)$	assumption
4.		$P(x_0) \Rightarrow Q(x_0)$	$\forall$ e 1
5.		$Q(x_0)$	$\Rightarrow$ e 4, 3
6.		$\exists x\ Q(x)$	$\exists$ i 5
7.		$\exists x\ Q(x)$	$\exists$ e 2, 3–6

The structure of this proof is relatively straightforward, though lines 6 and 7 look a bit strange since they contain the same formulas! As it turns out, even though the formulas are identical, we obtain them in different ways and use them for different purposes. We need to perform the step on line 6 in order to remove the occurrence of the dummy variable x_0 in Q, as that dummy variable isn't allowed to leave the scope of our subproof box. We therefore use the $\exists$ i rule on line 6 to remove x_0, and we then separately apply the $\exists$ e rule on line 7 to both $\exists x\ P(x)$ and the subproof involving $P[x/x_0]$ to arrive at our conclusion.

1.3 Equality

Taking our logical system one step further, it's possible for us to extend predicate logic to take into account the notion of *equality.* Note that we aren't talking about equality in the semantic sense—that is, logical equivalence or *intensional* equality—but rather about equality in terms of the result of a computation. Thus, when we talk about equality in this section, we will not focus on *formulas* being equal; our focus will instead be on equality between *terms*.

Formally speaking, we say that *predicate logic with equality* is the logical system that follows all of the syntactic and semantic rules of predicate logic with the added restriction that the symbol = is interpreted as equality over the domain $\{(t,t) \mid t \in D\}$, where D is the domain of some interpretation $\mathscr{I}$.

We will therefore need to define inference rules to handle the notion of equality in predicate logic. Definitionally, if we take any term t, then that term must be equal to itself. This is a property known as *reflexivity*, and we can express this as our equality introduction inference rule:

$$\frac{}{t = t} = \mathrm{i}$$

In other words, we can introduce equality simply by taking any term t and setting it to be equal to itself. Since this inference rule doesn't require any premises, it acts as an axiom.

Clearly, this inference rule makes sense, but we don't have much use for it on its own. If we combined this rule with substitution, on the other hand, then we would have a way of substituting equal terms in a formula. Indeed, this is how we express the equality elimination inference rule:

$$\frac{t_1 = t_2 \qquad A[x/t_1]}{A[x/t_2]} = \mathrm{e}$$

Note, as always, that both t_1 and t_2 must be free for x in A in order to apply this inference rule.

Indeed, we can do quite a bit more with our elimination rule than we could do with our introduction rule alone. For example, starting from the premise $t_1 = t_2$, consider the following proof:

1.	$t_1 = t_2$	premise
2.	$t_1 = t_1$	$=$ i
3.	$t_2 = t_1$	$=$ e 1, 2

Observe that, on line 3, we took A to be the formula $x = t_1$ when we performed the substitution.

If we additionally take the premise $t_2 = t_3$, then we can produce the following proof:

1.	$t_1 = t_2$	premise
2.	$t_2 = t_3$	premise
3.	$t_1 = t_3$	$=$ e 2, 1

Similarly, in this proof, we took A to be the formula $t_1 = x$ to perform the substitution; thus, we effectively have $A[x/t_2]$ on line 1 and $A[x/t_3]$ on line 3 after applying the rule $=$ e.

These two proofs, taken together, show that equality is not only reflexive (as we observed earlier) but also *symmetric* and *transitive*. These three properties are necessary in order for us to discuss the equality of terms as we're doing in this section; that is, *extensional* equality.

For our final natural deduction example, we'll tackle a rather large and lengthy proof that incorporates our inference rules for both quantifiers and equality. In this example, we consider a sequent that models the behaviour of a relation that is antisymmetric but irreflexive: either pair (x, y) or (y, x) belongs to the relation, but the pair (x, x) does not belong to the relation, so x and y cannot be equal.

Example 5. Let's prove that the sequent $\exists x\ \exists y\ (P(x,y) \vee P(y,x)), \neg\exists\ P(x,x) \vdash \exists x\ \exists y\ \neg(x = y)$ is valid. The proof is as follows:

1.			$\exists x\ \exists y\ (P(x,y) \vee P(y,x))$	premise
2.			$\neg\exists x\ P(x,x)$	premise
3.	x_0		$\exists y\ (P(x_0,y) \vee P(y,x_0))$	assumption
4.		y_0	$P(x_0,y_0) \vee P(y_0,x_0)$	assumption
5.			$x_0 = y_0$	assumption
6.			$P(x_0,y_0)$	assumption
7.			$P(y_0,y_0)$	$=$ e 5, 6
8.			$\exists x\ P(x,x)$	$\exists$ i 7
9.			$\bot$	$\neg$ e 8, 2
10.			$P(y_0,x_0)$	assumption
11.			$P(y_0,y_0)$	$=$ e 5, 10
12.			$\exists x\ P(x,x)$	$\exists$ i 11
13.			$\bot$	$\neg$ e 12, 2
14.			$\bot$	$\vee$ e 4, 6–9, 10–13
15.			$\neg(x_0 = y_0)$	$\neg$ i 5–14
16.			$\exists y\ \neg(x_0 = y)$	$\exists$ i 15
17.			$\exists x\ \exists y\ \neg(x = y)$	$\exists$ i 16
18.			$\exists x\ \exists y\ \neg(x = y)$	$\exists$ e 3, 4–17
19.			$\exists x\ \exists y\ \neg(x = y)$	$\exists$ e 1, 3–18

1.4 Soundness and Completeness

Since all we have done in this lecture is add to our natural deduction inference rule set, proving soundness and completeness is done in much the same way as when we proved both conditions in the propositional logic case.

Theorem 6 (Soundness and completeness of natural deduction)**.** *Let A_1, A_2, ..., A_n, and B be formulas. Then $A_1, A_2, \ldots, A_n \vdash B$ is a valid sequent if and only if $A_1, A_2, \ldots, A_n \vDash B$ holds.*

Since the properties of soundness and completeness are unchanged from the last time we studied natural deduction, we will only sketch the ideas of the proofs, as we've already covered the fine details.

1.4.1 Proving Soundness

To establish soundness, we use a proof by deduction to show that any finite number of inference rule applications to the premises of a sequent is sound. In our earlier proof, we saw that we needed to perform a case-based analysis to show that each application of an inference rule is sound. Here, as you might expect, we would need to check additional cases corresponding to our new inference rules for quantifiers and equality.

1.4.2 Proving Completeness

To establish completeness, we can prove the contrapositive: if $A_1, \ldots, A_n \nvdash B$, then $A_1, \ldots, A_n \nvDash B$. This proof can be done in three steps:

1. Show that if $A_1, \ldots, A_n \nvdash B$, then $A_1, \ldots, A_n, \neg B \nvdash B$.
2. Show that if $A_1, \ldots, A_n, \neg B \nvdash B$, then there exists an interpretation $\mathcal{I}$ such that $\vDash A_1, \ldots, A_n, \neg B$.
3. Show that if there exists an interpretation $\mathcal{I}$ such that $\vDash A_1, \ldots, A_n, \neg B$, then $A_1, \ldots, A_n \nvDash B$.