

Perfect matching

Def. A **matching** M is a set of ordered pairs $h-s$ with $h \in H$ and $s \in S$ s.t.

- Each hospital $h \in H$ appears in at most one pair of M .
- Each student $s \in S$ appears in at most one pair of M .

Def. A matching M is **perfect** if $|M| = |H| = |S| = n$.

	1 st	2 nd	3 rd		1 st	2 nd	3 rd
Atlanta	Xavier	Yolanda	Zeus	Xavier	Boston	Atlanta	Chicago
Boston	Yolanda	Xavier	Zeus	Yolanda	Atlanta	Boston	Chicago
Chicago	Xavier	Yolanda	Zeus	Zeus	Atlanta	Boston	Chicago

a perfect matching $M = \{ A-Z, B-Y, C-X \}$

5

Unstable pair

Def. Given a perfect matching M , hospital h and student s form an **unstable pair** if both:

- h prefers s to matched student.
- s prefers h to matched hospital.

Key point. An unstable pair $h-s$ could each improve by joint action.

	1 st	2 nd	3 rd		1 st	2 nd	3 rd
Atlanta	Xavier	Yolanda	Zeus	Xavier	Boston	Atlanta	Chicago
Boston	Yolanda	Xavier	Zeus	Yolanda	Atlanta	Boston	Chicago
Chicago	Xavier	Yolanda	Zeus	Zeus	Atlanta	Boston	Chicago

A-Y is an unstable pair for matching $M = \{ A-Z, B-Y, C-X \}$

6

Stable matching problem

Def. A **stable matching** is a perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n hospitals and n students, find a stable matching (if one exists).

	1 st	2 nd	3 rd		1 st	2 nd	3 rd
Atlanta	Xavier	Yolanda	Zeus	Xavier	Boston	Atlanta	Chicago
Boston	Yolanda	Xavier	Zeus	Yolanda	Atlanta	Boston	Chicago
Chicago	Xavier	Yolanda	Zeus	Zeus	Atlanta	Boston	Chicago

a stable matching $M = \{ A-X, B-Y, C-Z \}$

9

Proof of correctness: perfect matching

Claim. Gale-Shapley outputs a matching.

Pf.

- Hospital proposes only if unmatched. \rightarrow matched to ≤ 1 student
- Student keeps only best hospital. \rightarrow matched to ≤ 1 hospital ■

Claim. In Gale-Shapley matching, all hospitals get matched.

Pf. [by contradiction]

- Suppose, for sake of contradiction, that some hospital $h \in H$ is unmatched upon termination of Gale-Shapley algorithm.
- Then some student, say $s \in S$, is unmatched upon termination.
- By Observation 2, s was never proposed to.
- But, h proposes to every student, since h ends up unmatched. ■

Claim. In Gale-Shapley matching, all students get matched.

Pf. [by counting]

- By previous claim, all n hospitals get matched.
- Thus, all n students get matched. ■

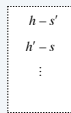
14

Proof of correctness: stability

Claim. In Gale-Shapley matching M^g , there are no unstable pairs.

Pf. Consider any pair $h-s$ that is not in M^g .

- Case 1: h never proposed to s .
 \rightarrow h prefers its Gale-Shapley partner s' to s . \leftarrow hospitals propose in decreasing order of preference
 \rightarrow $h-s$ is not unstable.
- Case 2: h proposed to s .
 \rightarrow s rejected h (either right away or later)
 \rightarrow s prefers Gale-Shapley partner h' to h . \leftarrow students only trade up
 \rightarrow $h-s$ is not unstable.
- In either case, the pair $h-s$ is not unstable. ■



Gale-Shapley matching M^g

15

Summary

Stable matching problem. Given n hospitals and n students, and their preference lists, find a stable matching if one exists.

Theorem. [Gale-Shapley 1962] The Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* and L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q . Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive q acceptances, it will generally have to offer to admit more than q applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how he ranks the colleges to which he has applied; even if this is known it will not be known (c) which of the other colleges will offer to admit him. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota, and be reasonably close to the attainable optimum in quality.

16

CSCI 355: ALGORITHM DESIGN AND ANALYSIS

1. STABLE MATCHING

- ▶ *stable matching problem*
- ▶ *Gale-Shapley algorithm*
- ▶ *hospital optimality*
- ▶ *context*

Understanding the solution

For a given problem instance, there may be several stable matchings.

	1 st	2 nd	3 rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1 st	2 nd	3 rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: $S = \{A-X, B-Y, C-Z\}$ and $S' = \{A-Y, B-X, C-Z\}$

19

Understanding the solution

Def. Student s is a **valid partner** for hospital h if there exists any stable matching in which h and s are matched.

Ex.

- Both X and Y are valid partners for A.
- Both X and Y are valid partners for B.
- Z is the only valid partner for C.

	1 st	2 nd	3 rd
A	X	Y	Z
B	Y	X	Z
C	X	Y	Z

	1 st	2 nd	3 rd
X	B	A	C
Y	A	B	C
Z	A	B	C

an instance with two stable matchings: $S = \{A-X, B-Y, C-Z\}$ and $S' = \{A-Y, B-X, C-Z\}$

20

Understanding the solution

Def. Student s is a **valid partner** for hospital h if there exists any stable matching in which h and s are matched.

Hospital-optimal assignment. Each hospital receives best valid partner.

- Is it a perfect matching?
- Is it stable?

Claim. All executions of Gale–Shapley yield **hospital-optimal** assignment.

Corollary. Hospital-optimal assignment is a stable matching!

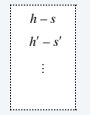
23

Hospital optimality

Claim. Gale–Shapley matching M^* is hospital-optimal.

Pf. [by contradiction]

- Suppose a hospital is matched with student other than best valid partner.
- Hospitals propose in decreasing order of preference.
→ some hospital is rejected by a valid partner during Gale–Shapley
- Let h be first such hospital, and let s be the first valid partner that rejects h .
- Let M be a stable matching where h and s are matched.
- When s rejects h in Gale–Shapley, s forms (or re-affirms) commitment to a hospital, say h' .



stable matching M

- s prefers h' to h . ← students only trade up
- Let s' be partner of h' in M .
- h' had not been rejected by any valid partner (including s') at the point when h is rejected by s . ← because this is the first rejection by a valid partner
- Thus, h' had not yet proposed to s' when h' proposed to s .
- h' prefers s to s' . ← hospitals propose in decreasing order of preference
- Thus, $h'-s$ is unstable in M , a contradiction. ▀

24

Student pessimality

Q. Does hospital-optimality come at the expense of the students?

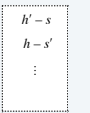
A. Yes.

Student-pessimal assignment. Each student receives worst valid partner.

Claim. Gale–Shapley finds **student-pessimal** stable matching M^* .

Pf. [by contradiction]

- Suppose $h-s$ matched in M^* but h is not the worst valid partner for s .
- There exists stable matching M in which s is paired with a hospital, say h' , whom s prefers less than h .
→ s prefers h to h' .
- Let s' be the partner of h in M .
- By hospital-optimality, s is the best valid partner for h .
→ h prefers s to s' .
- Thus, $h-s$ is an unstable pair in M , a contradiction. ▀



stable matching M

25

Extensions

Extension 1. Some agents declare others as unacceptable.

Extension 2. Some hospitals have more than one position.

Extension 3. Unequal number of positions and students.

med-school student
unwilling to work
in Cleveland

≥ 43K med-school students;
only 31K positions

Def. Matching M is **unstable** if there is a hospital h and student s such that:

- h and s are acceptable to each other; and
- Either s is unmatched, or s prefers h to assigned hospital; and
- Either h does not have all its places filled, or h prefers s to at least one of its assigned students.

Theorem. There exists a stable matching.

Pf. Straightforward generalization of Gale-Shapley algorithm.

27

CSCI 355: ALGORITHM DESIGN AND ANALYSIS 1. STABLE MATCHING

- ▶ *stable matching problem*
- ▶ *Gale-Shapley algorithm*
- ▶ *hospital optimality*
- ▶ *context*

Historical context

National resident matching program (NRMP).

- Centralized clearinghouse to match med-school students to hospitals.
- Began in 1952 to fix unraveling of offer dates.
- Originally used the "Boston Pool" algorithm.
- Algorithm overhauled in 1998.
 - med-school student optimal
 - deals with various side constraints (e.g., allow couples to match together)

hospitals began making
offers earlier and earlier,
up to 2 years in advance

stable matching no longer
guaranteed to exist

The Redesign of the Matching Market for American Physicians:
Some Engineering Aspects of Economic Design

By ALVIN E. ROTH AND ELLIOTT PERLSSON*

We report on the design of the new clearinghouse adopted by the National Resident Matching Program, which annually fills approximately 20,000 jobs for new physicians. Because the market has complementarities between applicants and between positions, the theory of simple matching markets does not apply directly. However, computational experiments show the theory provides good approximations. Furthermore, the set of stable matchings, and the opportunities for strategic manipulation, are surprisingly small. A new kind of "core convergence" result explains this: that each applicant receives only a small fraction of available positions is important. We also describe engineering aspects of the design process. JEL: C78, B41, J44



29

2012 Nobel Prize in Economics

Lloyd Shapley. Stable matching theory and Gale-Shapley algorithm.

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original applications:
college admissions and
opposite-sex marriage

Alvin Roth. Applied Gale-Shapley to matching med-school students with hospitals, students with schools, and organ donors with patients.



Lloyd Shapley

Alvin Roth



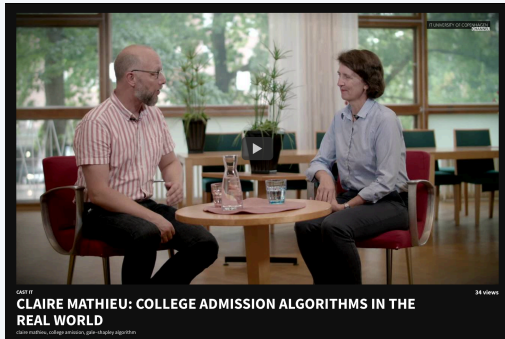
30

College admissions in France

French student. Applies to 10 college programs.

French college. Ranks applicants; starts sending out offers on May 21.

Goal. Match 1M students to 10K college programs.



CLAIRE MATHIEU: COLLEGE ADMISSION ALGORITHMS IN THE REAL WORLD

31

A modern application

Content delivery networks. Distribute much of world's content on web.

User. Preferences based on latency and packet loss.

Web server. Preferences based on costs of bandwidth and co-location.

Goal. Assign billions of users to servers, every 10 seconds.



Algorithmic Nuggets in Content Delivery

Bruce M. Maggs
Duke and Akamai
bmm@cs.duke.edu

Ramesh K. Sittaraman
UMass, Akamai and Akamai
ramesh@cs.umass.edu

This article is an editorial note submitted to CCR. It has NCFP been peer reviewed. The authors take full responsibility for the article's technical content. Comments can be posted through CCR Online.

ABSTRACT

This paper "peeks under the cover" at the algorithms that provide the basic functionality of a leading content delivery network. Based on our experience in building one of the largest distributed systems in the world, we illustrate how fundamental algorithmic research has been adapted to balance the load between and within server clusters, manage the caches on servers, select paths through an overlay routing network, and elect leaders in various contexts. In each instance, we first explain the theory underlying the algorithms, then introduce practical considerations not captured by the theoretical analysis, and finally describe what an implementer in practice. Through these examples, we highlight the role of algorithmic research in the design of complex distributed systems. The paper also illustrates the close synergy that exists between research and industry where research ideas drive new and more products and product requirements drive future research.



32

CSCI 355: ALGORITHM DESIGN AND ANALYSIS

1. REPRESENTATIVE PROBLEMS

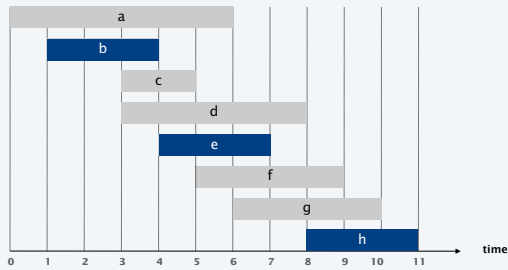
▶ five representative problems

Interval scheduling

Input. Set of jobs with start times and finish times.

Goal. Find maximum cardinality subset of mutually compatible jobs.

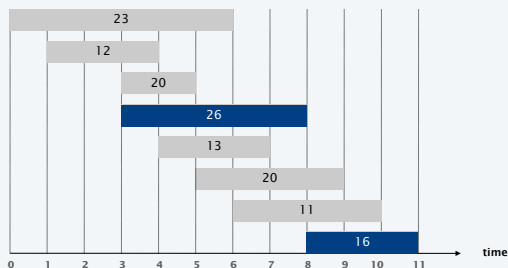
↑
jobs don't overlap



Weighted interval scheduling

Input. Set of jobs with start times, finish times, and weights.

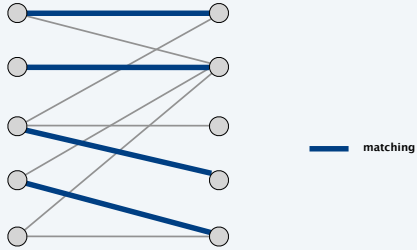
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite matching

Problem. Given a bipartite graph $G = (L \cup R, E)$, find a max cardinality matching.

Def. A subset of edges $M \subseteq E$ is a **matching** if each node appears in exactly one edge in M .

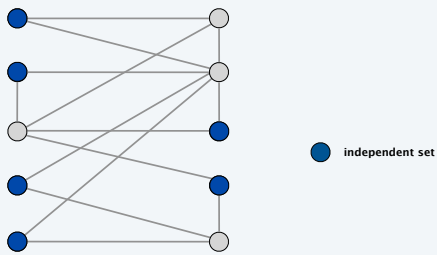


36

Independent set

Problem. Given a graph $G = (V, E)$, find a max cardinality independent set.

Def. A subset $S \subseteq V$ is **independent** if for every $(u, v) \in E$, either $u \notin S$ or $v \notin S$ (or both).



37

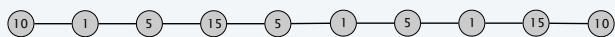
Competitive facility location

Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes.

Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a **maximum weight** subset of nodes.



Second player can guarantee 20, but not 25.

38

Five representative problems

Variations on a theme: independent set.

Interval scheduling: $O(n \log n)$ greedy algorithm.

Weighted interval scheduling: $O(n \log n)$ dynamic programming algorithm.

Bipartite matching: $O(n^3)$ max-flow based algorithm.

Independent set: **NP**-complete.

Competitive facility location: **PSPACE**-complete.