St. Francis Xavier University
Department of Computer Science

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1 Systems and Proofs

One of the primary uses of formal logical systems is to prove statements or propositions that we care about. Indeed, all of mathematics—and every mathematical proof—is formed on the foundation of some logical system. But what constitutes that foundation? We need a starting point: some set of statements that we take to be true, together with some set of rules that tell us what we're allowed to do at any given step of a proof.

Recall from an earlier lecture the notion of *logical consequence*: if we have a set of formulas $S = \{A_1, A_2, \dots, A_n\}$ and a separate formula A, we say that A is a logical consequence of S if and only if every model of S is also a model of A, and we write $S \vDash A$. The notion of logical consequence nicely encompasses this "foundation" we were talking about: the set of formulas S contains formulas that we know to be true by our model, and $S \vDash A$ indicates that $\vDash (A_1 \wedge A_2 \wedge \cdots \wedge A_n) \Rightarrow A$ is also true. By the definition of the implication logical connective, this allows us to conclude that A must be true.

We've already seen a decision procedure to determine if the truth of some formula A follows from the truth of a given set of formulas S: the method of semantic tableaux. We have an algorithm to construct a semantic tableau, which we can then use to test the validity of a formula. Why, then, is the course continuing? Aren't we done?

Well, the method of semantic tableaux has a few shortcomings. For example, the method of semantic tableaux is a decision procedure, so the most information we can glean from it is a yes/no answer. We aren't able to gain any insight into *how* a formula can be proved true, only that it can be proved true. Additionally, the method of semantic tableaux assumes that our set of formulas S is finite; there's no way to handle infinitely many formulas, and as we know from certain techniques like mathematical induction, the notion of infinity appears often in proofs. Lastly, and most importantly, we're still living in the world of propositional logic. We remarked in the introductory lecture that propositional logic is a very simple system, which makes it rather easy to develop decision procedures for this system. However, for the more complicated logical systems, we won't be able to use these nice decision procedures; indeed, such nice procedures might not even exist.

Thus, we need to develop another technique to handle proofs that overcomes each of the aforementioned shortcomings. This technique needs to be transparent and mechanical, so that we can see what is happening at each step and understand which rule to apply in which order. The technique also needs to be able to handle infinite formulas, even if only a finite number of these formulas is used in any given proof. Finally, the technique needs to be adaptable to different logical systems.

Members of the family of such techniques are known as *deductive systems* or, sometimes, *proof systems*.

Definition 1 (Deductive system)**.** A deductive system consists of a set of formulas called axioms and a set of rules of inference. A proof in a deductive system consists of a sequence of formulas $S = \{A_1, A_2, \dots, A_n\}$ such that each formula A_i in S is either

- an axiom; or
- inferrable from previous formulas $A_{j_1}, \dots, A_{j_k}$ in S, where $j_1 < \cdots < j_k < i$, using some rule of inference.

For some sequence of formulas $S = \{A_1, A_2, \ldots, A_n\}$, we say that the last formula in the sequence, A_n, is a *theorem*, and the sequence S itself is a *proof* of that theorem. If some theorem A_n has a proof S, then we say that A_n is *provable*, and we denote this by $\vdash A_n$. Additionally, if $\vdash A$ for some formula A, then we can use A as an axiom in future proofs.

There are many different kinds of deductive systems, each with their own specializations and benefits. In this lecture, we will focus on one particular deductive system that is the most "human-friendly" in terms of its approach; the structure of proofs in this deductive system is very similar to how humans prove something by starting with some premises and arriving at a conclusion. Other deductive systems exist that are more amenable to being implemented on a computer, and these systems form the basis of tools such as proof assistants and automated theorem provers.

2 Natural Deduction

The method of *natural deduction* is a deductive system that places an emphasis on inference rules over axioms. This system was developed by mathematicians and logicians who were displeased with the strong focus on the axiomatization of mathematics championed by people like Gottlob Frege, David Hilbert, Bertrand Russell, and Alfred Whitehead in the late 19th and early 20th centuries. In an effort to build a more *natural* method of proof (hence the name) that more closely approximated the human process of reasoning, the German mathematician Gerhard Gentzen proposed the method of natural deduction in his 1934 dissertation, though similar work was presented by the Polish logicians Jan Łukasiewicz and Stanisław Jaśkowski in the late 1920s.

In natural deduction, we have a set of inference rules that allows us to infer formulas from other formulas. Starting with a set of formulas $\{A_1, A_2, \ldots, A_n\}$, which we call *premises*, we apply inference rules to these premises in order to arrive at another formula B, which we call the *conclusion*. The overall process of going from premises to conclusion is denoted by a *sequent* of the form

$$A_1, A_2, \ldots, A_n \vdash B.$$

We say that a sequent is *valid* if a proof for that sequent exists.

Example 2. Suppose we have the following propositions:

$$\begin{aligned} p &= \text{The student is enrolled in the course.} \\ q &= \text{The student passes the final exam.} \\ r &= \text{The student fails the course.} \end{aligned}$$

The following valid sequent models the scenario where the student in the course does well on the exam:

$$p \wedge \neg q \Rightarrow r, \neg r, p \vdash q.$$

The proof of a sequent consists of a numbered list of n steps, where premises can appear wherever they are needed and the conclusion appears as the nth step. Each step from 1 to $(n-1)$ can be either a premise or an intermediate formula obtained by applying an inference rule. Inference rules are used to add or remove various logical connectives; therefore, most inference rules we will see have both an *introduction* form and an *elimination* form.

Proofs may also contain *subproofs*, which can be thought of as mini-proofs that allow us to obtain mini-conclusions that we can then use in intermediate steps of our main proof. Subproofs are indicated by placing a box around the lines constituting the subproof. While not every inference rule requires the use of subproofs, they will appear later in this section, so it's best to become aware of them now.

Since natural deduction places such a heavy emphasis on inference rules, the majority of our discussion in this section will consist of defining these rules and seeing how each rule is applied in a proof. Although it isn't always obvious how the proof for a given sequent should be structured, familiarity with the inference rules together with sufficient practice will bolster your comfort with natural deduction.

2.1 Conjunction

Our first set of inference rules will focus on the conjunction connective. To introduce conjunction, we must start with two formulas A_1 and A_2 that we have previously concluded are true (either by taking them as premises or by using other inference rules to obtain them). We write these two formulas in the "top half" of our inference rule. The resultant formula, $A_1 \land A_2$, is then written in the "bottom half". Our first introduction inference rule, therefore, takes the following form:

$$\frac{A_1 \qquad A_2}{A_1 \land A_2} \land \text{i}$$

The label on the right is our shorthand to denote "conjunction introduction". Using this inference rule gives us the formula $A_1 \land A_2$, which we can then use in subsequent applications of inference rules.

What if we already have the formula $A_1 \land A_2$, and we want to use just one of the subformulas? For this, we need to eliminate conjunction. Depending on the subformula we want, we can eliminate conjunction in one of two ways:

$$\frac{A_1 \land A_2}{A_1} \land \text{e}_1 \qquad \frac{A_1 \land A_2}{A_2} \land \text{e}_2$$

Again, the label on the right of each inference rule is shorthand for "conjunction elimination". Both of these inference rules are quite natural: if we know that $A_1 \land A_2$ holds, then we can take either of A_1 or A_2 and know that each will hold as well.

Example 3. Let's prove that the sequent $p \land q, r \vdash q \land r$ is valid. The proof, using our conjunction inference rules, is as follows:

1.	$p \land q$	premise
2.	r	premise
3.	q	$\land$ e$_2$ 1
4.	$q \land r$	$\land$ i 3, 2

Both premises on lines 1 and 2 appeared on the left-hand side of our sequent, so we could use them anywhere in our proof. The notation "$\land$ e$_2$ 1" on line 3 of our proof indicates that we applied the rule $\land$ e$_2$ to line 1 of our proof to obtain q from $p \land q$. Similarly, the notation "$\land$ i 3, 2" on line 4 of our proof indicates that we applied the rule $\land$ i to lines 3 and 2 of our proof to obtain $q \land r$ from both q and r.

2.2 Disjunction

When we defined our inference rule for conjunction introduction, we needed to be sure that both of our subformulas A_1 and A_2 held in order to obtain the formula $A_1 \land A_2$. With disjunction, on the other hand, we don't require *both* subformulas to be true; we only need at least one subformula to be true. Therefore, if we want to obtain a formula $A_1 \lor A_2$, we can consider two cases: the case where A_1 is true, or the case where A_2 is true. This gives rise to two introduction inference rules:

$$\frac{A_1}{A_1 \lor A_2} \lor \text{i}_1 \qquad \frac{A_2}{A_1 \lor A_2} \lor \text{i}_2$$

Observe that the first inference rule is valid for any choice of A_2, since we already know that A_1 holds and so $A_1 \lor A_2$ must also hold. Likewise, the second inference rule is valid for any choice of A_1.

Now, how do we handle disjunction elimination? The very same observations we just made about the truth of each subformula turn out to be a dilemma when it comes to the elimination inference rule. If we have a formula $A_1 \lor A_2$, we can't say conclusively which of A_1 and A_2 are true: are they both true, or is only one true, and which one should we use?

To get around this problem, we must introduce the notion of *assumptions* in our proofs. An assumption is the first step of a subproof in which we work toward obtaining an intermediate conclusion, which will later serve to prove our main conclusion. Subproofs can make use of any premises or intermediate conclusions that appeared in our proof prior to the subproof, and we can even nest subproofs. However, once we finish a subproof, we cannot reuse anything appearing in that subproof later in the main proof, besides the intermediate conclusion we obtained.

In an inference rule, a subproof is indicated by a box where the assumption is at the top and the intermediate conclusion is at the bottom. In our disjunction elimination inference rule, we will require two subproofs to handle two cases: in our first subproof, we will assume that the subformula A_1 is true and we will conclude our goal; and in our second subproof, we will assume that the subformula A_2 is true and we will conclude the same goal. Our elimination inference rule is therefore:

$$\frac{A_1 \lor A_2 \qquad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \qquad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \lor \mathrm{e}$$

Therefore, if we assume $A_1 \lor A_2$ holds, and if we show both that (i) assuming A_1 leads to a proof for B, and (ii) assuming A_2 leads to a proof for B, then we can conclude that, in any case, B follows.

It's important to keep in mind a few considerations to follow when using the $\lor$ e inference rule. First, in order for an application of the inference rule to be valid, the two conclusions in the two subproofs must match each other, and they must match the overall conclusion in the "bottom half" of the inference rule. In addition, one subproof cannot use the assumption or any intermediate steps found in the other subproof, unless those steps appeared in the overall proof prior to the subproof.

Example 4. Let's prove that the sequent $p \land (q \lor r) \vdash ((p \land q) \lor (p \land r))$ is valid. The proof, using our disjunction inference rules, is as follows:

1.	$p \land (q \lor r)$	premise
2.	p	$\land$ e$_1$ 1
3.	$q \lor r$	$\land$ e$_2$ 1
4.	q	assumption
5.	$p \land q$	$\land$ i 2, 4
6.	$(p \land q) \lor (p \land r)$	$\lor$ i$_1$ 5
7.	r	assumption
8.	$p \land r$	$\land$ i 2, 7
9.	$(p \land q) \lor (p \land r)$	$\lor$ i$_2$ 8
10.	$(p \land q) \lor (p \land r)$	$\lor$ e 3, 4–6, 7–9

Lines 4–6 and 7–9 constitute two separate subproofs. In line 3, we have the disjunction $q \lor r$, and each subproof underneath line 3 uses one of the subformulas in that disjunction as its assumption. Within the first subproof on lines 4–6, we begin by assuming q holds, and we apply two inference rules to arrive at the conclusion $(p \land q) \lor (p \land r)$. Likewise, within the second subproof on lines 7–9, we begin by assuming r holds, and we reach the same conclusion. As a result, we finish the overall proof on line 10 with that conclusion.

Note that, on lines 6 and 9, one of the subformulas in the disjunction comes from the previous line, but it seems as though we pulled the other subformula out of thin air. Indeed, that's an apt description of what we did; in the inference rule $\lor$ i$_1$, by assuming that A_1 is true, we can conclude $A_1 \lor A_2$ for any choice of A_2 we want. A similar observation applies for the inference rule $\lor$ i$_2$. Therefore, we simply chose the other subformula to suit our purposes and to help us reach our conclusion.

2.3 Implication

An implication is a powerful connective that shows that one formula follows from another, and as such, it can be used to great effect in a natural deduction proof. But how do we introduce an implication into a proof?

Like we did with disjunction elimination, we can use assumptions and subproofs to get what we desire.

In order to prove an implication $A \Rightarrow B$, we need to demonstrate that if A is true, then B is also true. We can do exactly that in a subproof where we take A to be our assumption and arrive at the intermediate conclusion of B. This gives us a rather unique inference rule where the only premise is the assumption we make for A and the resulting subproof:

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{B} \Rightarrow \mathrm{i}$$

Note that, just like we indicated for the disjunction elimination inference rule, we must ensure that the intermediate conclusion of the subproof matches the overall conclusion in the "bottom half" of the inference rule. As for the assumption, on the other hand, we can choose anything we like for A as long as some sequence of inference rule applications leads us to conclude B in the subproof. In fact, we could even take A and B to be the same formula, which allows us to express the obvious claim $A \Rightarrow A$.

Observe also that, if we have a sequent of the form $A_1, A_2, \ldots, A_n \vdash B$, we can transform the proof of this sequent into a proof of the theorem

$$\vdash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots))$$

by repeatedly applying the $\Rightarrow$ i inference rule to each of the formulas A_n through A_1 in the original proof.

Example 5. Consider the sequent $p \wedge q \vdash p$ with the following proof:

1.	$p \wedge q$	premise
2.	p	$\wedge$ e$_1$ 1

We can transform this sequent and its proof into a proof of the theorem $\vdash (p \wedge q) \Rightarrow p$ in the following way:

1.	$p \wedge q$	premise
2.	p	$\wedge$ e$_1$ 1
3.	$(p \wedge q) \Rightarrow p$	$\Rightarrow$ i 1–2

The inference rule for eliminating implication from a formula is more commonly known by its Latin name, *modus ponens*, which translates to "method of affirming".[1] The use of modus ponens extends back to antiquity as one of the earliest rules developed in the ancient Greeks' study of formal logic. Put simply, modus ponens states that if A is true, and if A implies B, then B must be true. Unlike the introduction inference rule, we require no assumptions to eliminate an implication. Thus, translating into our inference rule format, we get the following:

$$\frac{A \qquad A \Rightarrow B}{B} \Rightarrow \mathrm{e}$$

[1] Modus ponens is also known as "affirming the antecedent", where "antecedent" is another word for the premise of an implication. Affirming the antecedent means we take the premise to be true.

Example 6. Let's prove that the sequent $p \Rightarrow q \vdash (r \vee p) \Rightarrow (r \vee q)$ is valid. The proof, using our implication inference rules, is as follows:

1.	$p \Rightarrow q$	premise
2.	$r \vee p$	assumption
3.	r	assumption
4.	$r \vee q$	$\vee$ i_1 3
5.	p	assumption
6.	q	$\Rightarrow$ e 1, 5
7.	$r \vee q$	$\vee$ i_2 6
8.	$r \vee q$	$\vee$ e 2, 3–4, 5–7
9.	$(r \vee p) \Rightarrow (r \vee q)$	$\Rightarrow$ i 2–8

Observe that, in this proof, we make use of nested subproofs for the first time. The overall goal of this proof is to show that the conclusion holds by breaking down the implication and showing that it logically follows from our premise, even though the premise doesn't include the proposition r.

In our first subproof from lines 2–8, we assume $r \vee p$ holds, with our goal being to show that the overall implication in the conclusion holds. Since our assumption is a disjunction, we can break it down further into two sub-subproofs. The first, on lines 3–4, assumes r holds and introduces disjunction to obtain the conclusion $r \vee q$. The second, on lines 5–7, assumes p holds and arrives at the same conclusion $r \vee q$ by decomposing the implication in the premise to get q. Since both sub-subproofs have the same conclusion, our subproof arrives at the conclusion $r \vee q$ on line 8 by eliminating the disjunction from line 2. From here, it's a simple matter to obtain the overall conclusion by introducing implication on line 9.

2.4 Negation

In order to introduce negation into a proof, we require the notion of a *contradiction*, which we will denote by the symbol $\bot$. We're already familiar with contradictions from our earlier discussion of semantic tableaux and complementary pairs of literals: having both a literal and its negation in the same set is a contradiction, since we can't possibly satisfy both things at once. The same idea applies here: if we have a formula A in our proof and we want to introduce negation, we must arrive at a contradiction at some point, since we can't possibly infer $\neg A$ from A.

The idea behind introducing negation, then, is to assume that some formula A holds and show in a subproof that this assumption leads to a contradiction. Since our assumption must have been wrong, we can conclude that $\neg A$ holds instead. This leads to the following inference rule where, like the implication introduction inference rule, all we require is our assumption and resulting subproof as the premise:

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A} \; \neg\ \mathrm{i}$$

Contradictions have the interesting property of allowing us to prove anything we want from a contradiction. This makes sense if we recall the definition of the implication connective: both F $\Rightarrow$ T and F $\Rightarrow$ F are true, since we can prove anything by starting with a false premise. This observation leads to a special inference rule for contradiction elimination, where we can conclude any formula A when we take $\bot$ as our only premise:

$$\frac{\bot}{A} \; \bot\ \mathrm{e}$$

By contrast, there is no special inference rule for contradiction introduction specifically. However, we can "introduce" a contradiction as a conclusion by eliminating negation. Since any formula $A \land \neg A$ is inherently contradictory, taking both A and $\neg A$ as premises leads to a contradiction:

$$\frac{A \qquad \neg A}{\bot}\ \neg\text{ e}$$

Example 7. Let's prove that the sequent $p \Rightarrow q, \neg q \vdash \neg p$ is valid. The proof, using our negation inference rules, is as follows:

1.	$p \Rightarrow q$	premise
2.	$\neg q$	premise
3.	p	assumption
4.	q	$\Rightarrow$ e 1, 3
5.	$\bot$	$\neg$ e 2, 4
6.	$\neg p$	$\neg$ i 3–5

After stating our two premises, we begin a subproof with the assumption p on line 3. We choose p as our assumption because it appears in one of our premises, and we can then eliminate the implication from that premise to obtain q in the subproof. However, $\neg q$ appeared earlier in the proof on line 2, and so under our assumption of p, we run into a contradiction. Since our subproof with assumption p led to a contradiction, we conclude that $\neg p$ must, in fact, hold instead.

2.5 Other Useful Rules

The inference rules we've defined for each logical connective will be enough to handle any propositional logic formula we wish to prove. However, there are a number of other less common and more specialized inference rules that find some utility in certain proofs. Here, we survey a handful of these other inference rules, show how to obtain these derived rules from our other rules, and consider a few more examples.

2.5.1 Modus Tollens

The inference rule of *modus tollens*, or "method of denying", is closely related to the modus ponens rule we saw earlier.[2] In contrast to modus ponens, where we show that if A implies B and A is true, then B is true, modus tollens shows that if A implies B and B is *not* true, then A must *not* be true either.

$$\frac{A \Rightarrow B \qquad \neg B}{\neg A}\ \text{MT}$$

We can obtain the inference rule for modus tollens by applying our other inference rules in the following way:

1.	$A \Rightarrow B$	premise
2.	$\neg B$	premise
3.	A	assumption
4.	B	$\Rightarrow$ e 1, 3
5.	$\bot$	$\neg$ e 2, 4
6.	$\neg A$	$\neg$ i 3–5

[2] Modus tollens is also known as "denying the consequent", where "consequent" is another word for the conclusion of an implication. Denying the consequent means we take the conclusion to be false.

Example 8. Let's prove that the sequent $p \Rightarrow \neg(q \vee r), \neg p \Rightarrow \neg(q \vee r) \vdash \neg(q \vee r)$ is valid. The proof, using our modus tollens inference rule, is as follows:

1.	$p \Rightarrow \neg(q \vee r)$	premise
2.	$\neg p \Rightarrow \neg(q \vee r)$	premise
3.	$q \vee r$	assumption
4.	$\neg p$	MT 1, 3
5.	p	MT 2, 3
6.	$\bot$	$\neg$ e 4, 5
7.	$\neg(q \vee r)$	$\neg$ i 3–6

In this proof, we create a subproof with the assumption that the conclusion of both premises is negated, and we then use modus tollens with each premise and the negated conclusion to arrive at a contradiction. From here, we can conclude that our assumption must have been incorrect.

2.5.2 Proof by Contradiction

In our negation introduction inference rule, recall that our lone premise involved assuming that some formula A holds and arriving at a contradiction, thus allowing us to conclude that $\neg A$ actually holds. In a typical proof by contradiction, we take the opposite approach: starting with an assumption that $\neg A$ holds, we work our way toward a contradiction. Since $\neg A$ was the only assumption we made, we conclude that the assumption must have been incorrect, and so A actually holds.

$$\frac{\boxed{\begin{array}{c}\neg A \\ \vdots \\ \bot\end{array}}}{A}\ \text{PBC}$$

In order to derive this inference rule, we require the use of another derived inference rule—double negation elimination—which we will introduce in the following section. As we will see, the proof of the double negation elimination inference rule itself uses the proof by contradiction inference rule, creating a dependency loop of sorts. This is because the principle of proof by contradiction can be written as a proposition of the form $\neg\neg A \Rightarrow A$, which exactly models the elimination of double negation operators. Both the principle of proof by contradiction and the elimination of double negation rely on the *law of excluded middle* (and, in fact, the three are all equivalent), but we won't get into further details here.[3]

For now, we will simply present the proof of the inference rule for proof by contradiction, with the $\neg\neg$ e rule on line 5 indicating the double negation elimination inference rule we will soon define:

1.	$\neg A \Rightarrow \bot$	premise
2.	$\neg A$	assumption
3.	$\bot$	$\Rightarrow$ e 1, 2
4.	$\neg\neg A$	$\neg$ i 2–3
5.	A	$\neg\neg$ e 4

[3]Everything mentioned here about the law of excluded middle applies only to the system of classical logic that we focus on in this course. There is another system called *intuitionistic logic* that rejects the validity of the law of excluded middle; classicists and intuitionists have engaged in bitter debates on the matter for the better part of the last century.

Example 9. Let's prove that the sequent $\neg(p \wedge q) \vdash \neg p \vee \neg q$ is valid. The proof, using our proof by contradiction inference rule, is as follows:

1.	$\neg(p \wedge q)$	premise
2.	$\neg(\neg p \vee \neg q)$	assumption
3.	$\neg p$	assumption
4.	$\neg p \vee \neg q$	$\vee$ i 3
5.	$\bot$	$\neg$ e 2, 4
6.	p	PBC 3–5
7.	$\neg q$	assumption
8.	$\neg p \vee \neg q$	$\vee$ i 7
9.	$\bot$	$\neg$ e 2, 8
10.	q	PBC 7–9
11.	$p \wedge q$	$\wedge$ i 6, 10
12.	$\bot$	$\neg$ e 1, 11
13.	$\neg p \vee \neg q$	PBC 2–12

This proof uses our inference rule multiple times, but each time the substructure of the proof is the same: starting with some assumption, we arrive at a contradiction, indicating to us that our assumption was incorrect. Within the first subproof formed by assuming the negation of the conclusion, we have two nested subproofs that each use proof by contradiction to obtain the propositions p and q by themselves. We then join these two propositions by introducing conjunction, only to arrive at another contradiction stemming from our premise. This ultimately leads to our desired conclusion.

2.5.3 Double Negation

From our definition of the negation connective, we know that doubly negating a truth value produces the original truth value. Therefore, double negation can be thought of as an identity operation; it doesn't affect the truth value of a formula, so we can insert or remove it anywhere we like.

The introduction and elimination inference rules for double negation, then, are as follows:

$$\frac{A}{\neg\neg A}\ \neg\neg\ \text{i} \qquad\qquad \frac{\neg\neg A}{A}\ \neg\neg\ \text{e}$$

Both the double negation introduction and double negation elimination inference rules can be obtained through an application of our other inference rules. The proof for the double negation introduction inference rule is as follows:

1.	A	premise
2.	$\neg A$	assumption
3.	$\bot$	$\neg$ e 1, 2
4.	$\neg\neg A$	$\neg$ i 2–3

Likewise, the proof for the double negation elimination inference rule (relying on our other derived inference rule for proof by contradiction) is as follows:

1.	$\neg\neg A$	premise
2.	$\neg A$	assumption
3.	$\bot$	$\neg$ e 2, 1
4.	A	PBC 2–3

Example 10. Let's prove that the sequent $(p \wedge \neg q) \Rightarrow r, \neg r, p \vdash q$ is valid. The proof, using our double negation inference rules, is as follows:

1.	$(p \wedge \neg q) \Rightarrow r$	premise
2.	$\neg r$	premise
3.	p	premise
4.	$\neg q$	assumption
5.	$p \wedge \neg q$	$\wedge$ i 3, 4
6.	r	$\Rightarrow$ e 1, 5
7.	$\bot$	$\neg$ e 2, 6
8.	$\neg\neg q$	$\neg$ i 4–7
9.	q	$\neg\neg$ e 8

After stating our three premises, we begin a subproof with the assumption $\neg q$ on line 4. The reason why we choose $\neg q$ as our assumption is so that we can combine it with p on line 3 to obtain $p \wedge \neg q$ on line 5, which appears in one of our premises. From here, we eliminate implication from our premise on line 1 to obtain r, and since $\neg r$ already appeared in the proof on line 2, we arrive at a contradiction. This means that the negation of our assumption $\neg q$ must hold, but since this is equivalent to $\neg\neg q$, we eliminate the double negation to obtain q as our conclusion.

2.6 Soundness and Completeness

You may have noticed at this point that, unlike with semantic tableaux, we haven't presented an algorithm to construct a natural deduction proof for a given sequent. This is because, unlike with semantic tableaux, the formulas that may appear in a natural deduction proof are not limited to the set of subformulas of the original formula. This complicates the situation quite a bit, since we're unable to use what we have (the formula) to determine conclusively what we need (the next step in a proof). As we mentioned at the beginning of this lecture, the structure of a natural deduction proof is very similar to how a human might prove a statement, and computers lack the human insight that allows us to view a proof holistically.

That being said, not all is lost. While we may not have an algorithm to *construct* a proof for a given sequent, it is possible to develop heuristics that *search for* the proof of a sequent. Tools using such heuristics are called *automated theorem provers*. We can also develop tools called *proof assistants* that take as input a completed proof and verify that each step of the proof is legal.

Keeping our focus on natural deduction, though, we have one big goal remaining: we must prove that natural deduction is both sound and complete. To achieve this, we will relate the notions of a sequent ($\vdash$) and of logical consequence ($\vDash$) to show that the two are, in a sense, the same; that is, one holds if and only if the other holds.

Theorem 11 (Soundness and completeness of natural deduction)**.** *Let A_1, A_2, …, A_n, and B be formulas. Then $A_1, A_2, \ldots, A_n \vdash B$ is a valid sequent if and only if $A_1, A_2, \ldots, A_n \vDash B$ holds.*

This theorem asserts in one direction that if a sequent is valid, then the conclusion of the sequent is a logical consequence of the set of premises. In the other direction, the theorem states that for any formula B that is a logical consequence of some set of formulas $A_1, A_2, \ldots, A_n$, there is a valid sequent corresponding to that formula.

2.6.1 Proving Soundness

A proof of the soundness of natural deduction demonstrates that, given a proof of a sequent $A_1, A_2, \ldots, A_n \vdash B$, there is no possible interpretation where every proposition A_1 through A_n is true while B is false. In other words, a valid sequent abides by our semantic rules for propositional logic. Thus, the basis of our proof will be to show that, in all interpretations $\mathscr{I}$ where $v_{\mathscr{I}}(A_1) = \cdots = v_{\mathscr{I}}(A_n) = \text{T}$, we have that $v_{\mathscr{I}}(B) = \text{T}$ as well.

Proof of Soundness. Suppose that $A_1, A_2, \dots, A_n \vdash B$ is a valid sequent, and consider the proof of this sequent. We will prove soundness by way of induction on the length k of the proof, where the length is given by the number of lines in the proof.

For the base case ($k = 1$), our proof must be of the form

$$1. \quad A_1 \quad \text{premise}$$

which corresponds to a sequent of the form $A_1 \vdash A_1$. We know that this is the only possible sequent with a proof of length 1, since all natural deduction rules contribute more than one line to a proof. Clearly, if $v_{\mathscr{I}}(A_1) = \mathrm{T}$ in the premise, then $v_{\mathscr{I}}(A_1) = \mathrm{T}$ in the conclusion. Therefore, $A_1 \vDash A_1$.

For the inductive case, suppose that $A_1, A_2, \dots, A_n \vdash B$ is a valid sequent with a proof of length k. Further suppose that the statement we wish to prove is true for all values less than k. Our proof must be of the form

$$\begin{array}{lll} 1. & A_1 & \text{premise} \\ 2. & A_2 & \text{premise} \\ & \vdots & \\ n. & A_n & \text{premise} \\ & \vdots & \\ k. & B & \text{(justification)} \end{array}$$

There are two “missing” components in this proof: the intermediate lines represented by the dots, and the justification used to obtain B on line k. While the intermediate lines are covered by our inductive hypothesis, we must establish ourselves that no matter what justification was used to obtain line k of the proof, the property of logical consequence holds.

To establish this, we perform a case-based analysis involving each of our inference rules. This is, unfortunately, somewhat tedious, so we will only consider one example case in this proof. The remaining cases are omitted but similar.

Example Case 1: $\wedge$ i. Suppose the justification on line k used the inference rule $\wedge$ i k_1, k_2. Because of this, we know that $B = B_1 \wedge B_2$ for some subformulas B_1 and B_2 appearing earlier in the proof on lines k_1 and k_2, respectively. Since $k_1 < k$ and $k_2 < k$, there exist sequents $A_1, A_2, \dots, A_n \vdash B_1$ and $A_1, A_2, \dots, A_n \vdash B_2$ both having proofs with length less than k. By our inductive hypothesis, we know that both $A_1, A_2, \dots, A_n \vDash B_1$ and $A_1, A_2, \dots, A_n \vDash B_2$ hold. Following our semantic rules for propositional logic, we can conclude that $A_1, A_2, \dots, A_n \vDash B_1 \wedge B_2$ also holds as desired.

After showing that logical consequence holds for all inference rules, we conclude that the overall claim holds by the principle of mathematical induction. □

2.6.2 Proving Completeness

The next step, proving the completeness of natural deduction, demonstrates that if B is a logical consequence of the formulas A_1 through A_n, then there is a valid sequent $A_1, A_2, \dots, A_n \vdash B$ establishing this.

Proof of Completeness. Our proof will proceed in three steps:

1. Show that $\vDash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots))$ holds;
2. Show that $\vdash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots))$ is valid; and
3. Show that $A_1, A_2, \dots, A_n \vdash B$ is valid.

Step 1. Suppose that $A_1, A_2, \dots, A_n \vDash B$ holds. We wish to show that the formula $A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots))$ is true for all interpretations $\mathscr{I}$; that is, the formula is a tautology.

Since the formula consists of a series of implications, it evaluates to false only if each premise A_1 through A_n evaluates to true and B evaluates to false. However, if this is the case, then our assumption that $A_1, A_2, \ldots, A_n \vDash B$ holds is incorrect. Therefore, $\vDash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B)\cdots))$ must hold.

Step 2. Suppose that $\vDash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B)\cdots))$ holds. We wish to show that $\vdash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B)\cdots))$ is valid.

Given that our formula contains n distinct propositional variables, by our assumption we know that the formula evaluates to true in all 2^n lines of its corresponding truth table. To construct a proof for the corresponding theorem, we will translate each line of the truth table into a sequent and combine these 2^n sequents into a proof for the overall theorem. For this, we require the following proposition:

Proposition 12. *Let A be a formula containing n distinct propositional variables $p_1, p_2, \ldots, p_n$. Consider some line ℓ of the truth table for A. For all $1 \leq i \leq n$, take $\hat{p}_i = p_i$ if p_i in line ℓ is true, and take $\hat{p}_i = \neg p_i$ if p_i in line ℓ is false. Then*

1. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash A$ is provable if the entry for A in line ℓ is true; and*
2. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg A$ is provable if the entry for A in line ℓ is false.*

This proposition can be proved using structural induction, though we omit the lengthy proof here.

We apply the proposition to the formula $\vDash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B)\cdots))$. Since we know that the formula is a tautology by the previous step, the formula evaluates to true in all 2^n lines of its truth table. By our proposition, we have 2^n different proofs of the sequent $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash A$, depending on the values taken by $\hat{p}_i$.

We prove the overall theorem by appealing to each of the 2^n proofs we have available to us. For each such proof, we have a set of premises on the left-hand side of the sequent corresponding to the values taken by $\hat{p}_i$: each propositional variable is either a literal or its negation. To remove these premises from the left-hand side, we use the *law of excluded middle*: for all r, the formula $r \vee \neg r$ holds. Applying the law of excluded middle to all propositional variables, we obtain a total of 2^n subproofs at the deepest layer of our proof. We then repeatedly apply the inference rule $\vee$ e until all of our premises are removed from the left-hand sides.

Example of Step 2 Application: Suppose that $\neg q, p \Rightarrow q \vDash \neg p$. We want to construct a proof for the theorem $\vdash \neg q \Rightarrow ((p \Rightarrow q) \Rightarrow \neg p)$, which we denote by ϕ. Applying our proposition to the corresponding truth table, we get the following set of sequents:

p	q	ϕ	
T	T	T	$p, q \vdash \phi$
T	F	T	$p, \neg q \vdash \phi$
F	T	T	$\neg p, q \vdash \phi$
F	F	T	$\neg p, \neg q \vdash \phi$

The proof for the theorem $\vdash \neg q \Rightarrow ((p \Rightarrow q) \Rightarrow \neg p)$ is as follows:

```
p ∨ ¬p                                              LEM
┌─────────────────────────────┐┌─────────────────────────────┐
│ p                   assump. ││ ¬p                  assump. │
│ q ∨ ¬q                  LEM ││ q ∨ ¬q                  LEM │
│ ┌────────────┐┌────────────┐││ ┌────────────┐┌────────────┐│
│ │ q  assump. ││ ¬q assump. │││ │ q  assump. ││ ¬q assump. ││
│ │ ⋮          ││ ⋮          │││ │ ⋮          ││ ⋮          ││
│ │ φ          ││ φ          │││ │ φ          ││ φ          ││
│ └────────────┘└────────────┘││ └────────────┘└────────────┘│
│ φ                       ∨ e ││ φ                       ∨ e │
└─────────────────────────────┘└─────────────────────────────┘
φ                                                   ∨ e
```

Step 3. Consider the proof for the theorem $\vdash A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B)\cdots))$ from the previous step, and add each of A_1 through A_n as premises to this proof. Then, starting with A_1 and continuing in sequence to A_n, apply the inference rule $\Rightarrow$e a total of n times. Doing so, we arrive at the conclusion B, and the process serves as a proof for the sequent $A_1, A_2, \ldots, A_n \vdash B$, thus proving that the sequent is valid. □

	Introduction	Elimination
Conjunction ($\wedge$)	$\dfrac{A_1 \quad A_2}{A_1 \wedge A_2} \wedge \text{ i}$	$\dfrac{A_1 \wedge A_2}{A_1} \wedge \text{ e}_1 \quad \dfrac{A_1 \wedge A_2}{A_2} \wedge \text{ e}_2$
Disjunction ($\vee$)	$\dfrac{A_1}{A_1 \vee A_2} \vee \text{ i}_1 \quad \dfrac{A_2}{A_1 \vee A_2} \vee \text{ i}_2$	$\dfrac{A_1 \vee A_2 \quad \boxed{\begin{array}{c} A_1 \\ \vdots \\ B \end{array}} \quad \boxed{\begin{array}{c} A_2 \\ \vdots \\ B \end{array}}}{B} \vee \text{ e}$
Implication ($\Rightarrow$)	$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{B} \Rightarrow \text{ i}$	$\dfrac{A \quad A \Rightarrow B}{B} \Rightarrow \text{ e}$
Negation ($\neg$)	$\dfrac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A} \neg \text{ i}$	$\dfrac{A \quad \neg A}{\bot} \neg \text{ e}$
Contradiction ($\bot$)	(no introduction inference rule)	$\dfrac{\bot}{A} \bot \text{ e}$

Table 1: Summary of natural deduction inference rules