CSCI 435: Algorithms and Complexity
7. Amortized Analysis

- *binary counter*
- *multi-pop stack*
- *dynamic table*

Amortized analysis

Worst-case analysis. Determine the worst-case running time of a data structure operation as a function of the input size n.

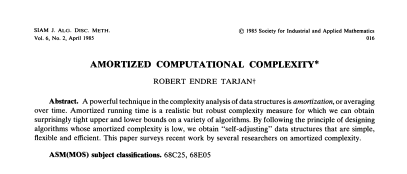
can be too pessimistic if the only way to encounter an expensive operation is when there were lots of previous cheap operations

Amortized analysis. Determine the worst-case running time of a sequence of n data structure operations as a function of the input size n.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of n push and pop operations takes $O(n)$ time in the worst case.

Amortized analysis: applications

- Splay trees.
- Dynamic tables.
- Fibonacci heaps.
- Garbage collection.
- Move-to-front list updating.
- Push–relabel algorithm for max flow.
- Path compression for disjoint-set union.
- Structural modifications to red–black trees.
- Security, databases, distributed computing, ...



SIAM J. ALG. DISC. METH.
Vol. 6, No. 2, April 1985

AMORTIZED COMPUTATIONAL COMPLEXITY*

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Abstract. A powerful technique in the complexity analysis of data structures is *amortization*, or averaging over time. Amortized running time is a realistic but robust complexity measure for which we can obtain surprisingly tight upper and lower bounds on a variety of algorithms. By following the principle of designing algorithms whose amortized complexity is low, we obtain "self-adjusting" data structures that are simple, flexible and efficient. This paper surveys recent work by several researchers on amortized complexity.

ASM(MOS) subject classifications. 68C25, 68E05

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Binary counter

Goal. Increment a k-bit binary counter $(\text{mod } 2^k)$.
Representation. $A[j] = j^{th}$ least significant bit of counter.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Cost model. Number of bits flipped.

Binary counter

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Representation. $A[j] = j^{th}$ least significant bit of counter.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	0
3	0	0	0	0	0	0	1	1
4	0	0	0	0	0	1	0	0
5	0	0	0	0	0	1	0	1
6	0	0	0	0	0	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	1	0	0	0
9	0	0	0	0	1	0	0	1
10	0	0	0	0	1	0	1	0
11	0	0	0	0	1	0	1	1
12	0	0	0	0	1	1	0	0
13	0	0	0	0	1	1	0	1
14	0	0	0	0	1	1	1	0
15	0	0	0	0	1	1	1	1
16	0	0	0	1	0	0	0	0

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n\,k)$ bits. ⟵ overly pessimistic upper bound

Pf. At most k bits flipped per increment. ▪

Aggregate method (brute force)

Aggregate method. Analyze the cost of a sequence of operations.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	Total cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Binary counter: aggregate method

Starting from the zero counter, in a sequence of n INCREMENT operations:

- Bit 0 flips n times.
- Bit 1 flips $\lfloor n / 2 \rfloor$ times.
- Bit 2 flips $\lfloor n / 4 \rfloor$ times.
- ...

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Bit j flips $\lfloor n / 2^j \rfloor$ times.
- The total number of bits flipped is $\sum_{j=0}^{k-1} \left\lfloor \frac{n}{2^j} \right\rfloor < n \sum_{j=0}^{\infty} \frac{1}{2^j}$

$$= 2n \quad \blacksquare$$

Remark. Theorem may be false if the initial counter is not zero.

Accounting method (banker's method)

Assign (potentially) different charges to each operation.

- D_i = data structure after i^{th} operation.
- c_i = actual cost of i^{th} operation. (can be more or less than actual cost)
- $\hat{c}_i$ = amortized cost of i^{th} operation = amount we charge operation i.
- $\hat{c}_i > c_i$: we store credits in data structure D_i to pay for future operations; $\hat{c}_i < c_i$: we consume credits in data structure D_i.
- Initial data structure D_0 starts with 0 credits.

Credit invariant. Total number of credits in the data structure ≥ 0.

$$\sum_{i=1} \hat{c}_i - \sum_{i=1} c_i \geq 0$$

← our job is to choose suitable amortized costs so that this invariant holds

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can be more or less than actual cost

Credit invariant. Total number of credits in the data structure ≥ 0.

$$\sum_{i=1} \hat{c}_i - \sum_{i=1} c_i \geq 0$$

Theorem. Starting from the initial data structure D_0, the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of n operations is $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$. ▪

credit invariant

Intuition. Measure running time in terms of credits (time = money).

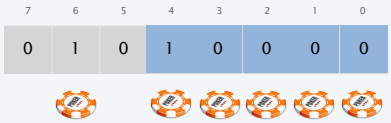
Binary counter: accounting method

Credits. One credit pays for a bit flip.
Invariant. Each 1 bit has one credit; each 0 bit has zero credits.

Accounting.

- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).
- Flip bit j from 1 to 0: pay for it with the 1 credit saved in bit j.

increment



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- Flip bit j from 0 to 1: charge 2 credits (use one and save one in bit j).
- Flip bit j from 1 to 0: pay for it with the 1 credit saved in bit j.

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

the rightmost 0 bit (unless counter overflows)

- Each INCREMENT operation flips at most one 0 bit to a 1 bit, so the amortized cost per INCREMENT ≤ 2.
- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Total actual cost of n operations $\leq$ sum of amortized costs $\leq 2n$. ▪

accounting method theorem

Potential method (physicist's method)

Potential function. $\Phi(D_i)$ maps each data structure D_i to a real number such that:

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each data structure D_i

Actual and amortized costs.

- c_i = actual cost of i^{th} operation.
- $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ = amortized cost of i^{th} operation.

our job is to choose a potential function so that the amortized cost of each operation is low

Potential method (physicist's method)

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Actual and amortized costs.

- c_i = actual cost of i^{th} operation.
- $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$ = amortized cost of i^{th} operation.

Theorem. Starting from the initial data structure D_0, the total actual cost of any sequence of n operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of operations is:

$$\begin{aligned} \sum_{i=1}^{n} \hat{c}_i &= \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\ &= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0) \\ &\geq \sum_{i=1}^{n} c_i \quad \blacksquare \end{aligned}$$

Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D.

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each data structure D_i

increment

7	6	5	4	3	2	1	0
0	1	0	1	0	0	0	0

Binary counter: potential method

Potential function. Let $\Phi(D)$ = number of 1 bits in the binary counter D.

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each data structure D_i

Theorem. Starting from the zero counter, a sequence of n INCREMENT operations flips $O(n)$ bits.

Pf.

- Suppose that the i^{th} INCREMENT operation flips t_i bits from 1 to 0.
- The actual cost is $c_i \leq t_i + 1$. ← operation flips at most one bit from 0 to 1 (no bits flipped to 1 when counter overflows)
- The amortized cost is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\leq c_i + 1 - t_i$$ ← potential decreases by 1 for t_i bits flipped from 1 to 0 and increases by 1 for the bit flipped from 0 to 1

$$\leq 2.$$

- Total actual cost of n operations $\leq$ sum of amortized costs $\leq 2n$. ▪

↑ potential method theorem

Famous potential functions

Fibonacci heaps. $\Phi(H) = 2\, trees(H) + 2\, marks(H)$

Splay trees. $\Phi(T) = \sum_{x \in T} \lfloor \log_2 size(x) \rfloor$

Move-to-front. $\Phi(L) = 2\, inversions(L, L^*)$

Preflow–push. $\Phi(f) = \sum_{v\,:\,excess(v) > 0} height(v)$

Red–black trees. $\Phi(T) = \sum_{x \in T} w(x)$

$$w(x) = \begin{cases} 0 & \text{if } x \text{ is red} \\ 1 & \text{if } x \text{ is black and has no red children} \\ 0 & \text{if } x \text{ is black and has one red child} \\ 2 & \text{if } x \text{ is black and has two red children} \end{cases}$$

CSCI 435: ALGORITHMS AND COMPLEXITY
7. AMORTIZED ANALYSIS

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- *dynamic table*

Multipop stack

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
- POP(S): remove and return the most recently added element.
- MULTI-POP(S, k): remove the most recently added k elements.

```
MULTI-POP(S, k)
FOR i = 1 TO k
    POP(S).
```

Exceptions. We assume POP throws an exception if stack is empty.

Multipop stack

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
- POP(S): remove and return the most recently added element.
- MULTI-POP(S, k): remove the most recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n^2)$ time.

Pf.

- Use a singly linked list.
- POP and PUSH take $O(1)$ time each.
- MULTI-POP takes $O(n)$ time. ▪

overly pessimistic upper bound

Multipop stack: aggregate method

Goal. Support operations on a set of elements:

- PUSH(S, x): add element x to stack S.
- POP(S): remove and return the most-recently added element.
- MULTI-POP(S, k): remove the most-recently added k elements.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.

- An element is popped at most once for each time that it is pushed.
- There are $\leq n$ PUSH operations.
- Thus, there are $\leq n$ POP operations (including those made within MULTI-POP). ▪

better upper bound

Multipop stack: accounting method

Credits. 1 credit pays for either a PUSH or a POP.
Invariant. Every element on the stack has 1 credit.

Accounting.
- PUSH(S, x): charge 2 credits.
  - Use 1 credit to pay for pushing x now.
  - Store 1 credit to pay for popping x at some point in the future.
- POP(S): charge 0 credits.
- MULTIPOP(S, k): charge 0 credits.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf.
- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per operation ≤ 2.
- Total actual cost of n operations $\leq$ sum of amortized costs $\leq 2n$. ▪

accounting method theorem

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.
- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 1: push]
- Suppose that the i^{th} operation is a PUSH.
- The actual cost is $c_i = 1$.
- The amortized cost is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$.

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 2: pop]
- Suppose that the i^{th} operation is a POP.
- The actual cost is $c_i = 1$.
- The amortized cost is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$.

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [Case 3: multi-pop]

- Suppose that the i^{th} operation is a MULTI-POP of k objects.
- The actual cost is $c_i = k$.
- The amortized cost is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$.

Multipop stack: potential method

Potential function. Let $\Phi(D)$ = number of elements currently on the stack.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each D_i.

Theorem. Starting from an empty stack, any intermixed sequence of n PUSH, POP, and MULTI-POP operations takes $O(n)$ time.

Pf. [putting everything together]

- The amortized cost is $\hat{c}_i \leq 2$. ⟵ 2 for push; 0 for pop; 0 for multi-pop
- The sum of amortized costs $\hat{c}_i$ of the n operations is $\leq 2n$.
- The total actual cost $\leq$ sum of amortized cost $\leq 2n$. ▪

↑ potential method theorem

CSCI 435: ALGORITHMS AND COMPLEXITY
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- *multi-pop stack*
- *dynamic table*

Dynamic table

Goal. Store items in a table (e.g., for hash table, binary heap).

- Two operations: INSERT and DELETE.
  - too many items inserted ⇒ expand table.
  - too many items deleted ⇒ contract table.
- Requirement: if table contains m items, then space $\in \Theta(m)$.

Theorem. Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n^2)$ time.

overly pessimistic upper bound

Pf. Each INSERT or DELETE takes $O(n)$ time. ▪

Dynamic table: insert only

- When inserting into an empty table, allocate a table of capacity 1.
- When inserting into a full table, allocate a new table of twice the capacity and copy all items.
- Insert item into table.

insert	old capacity	new capacity	insert cost	copy cost
1	0	1	1	–
2	1	2	1	1
3	2	4	1	2
4	4	4	1	–
5	4	8	1	4
6	8	8	1	–
7	8	8	1	–
8	8	8	1	–
9	8	16	1	8
⋮	⋮	⋮	⋮	⋮

Cost model. Number of items written (due to insertion or copy).

Dynamic table: insert only (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let c_i denote the cost of the i^{th} insertion.

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Starting from an empty table, the cost of a sequence of n INSERT operations is:

$$\begin{aligned} \sum_{i=1}^{n} c_i &\le n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j \\ &< n + 2n \\ &= 3n \quad \blacksquare \end{aligned}$$

Dynamic table demo: insert only (accounting method)

Insert. Charge 3 credits (use 1 to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

insert N

capacity = 16

A	B	C	D	E	F	G	H	I	J	K	L	M			

Dynamic table: insert only (accounting method)

Insert. Charge 3 credits (use 1 to insert; save 2 with new item).

Invariant. 2 credits with each item in right half of table; none in left half.

Pf. [by induction]

slight cheat if table capacity = 1
(can charge only 2 credits for first insert)

- Each newly inserted item gets 2 credits.
- When table doubles from k to $2k$, $k / 2$ items in the table have 2 credits.
  - these k credits pay for the work needed to copy the k items
  - now, all k items are in left half of table (and have 0 credits)

Theorem. [via accounting method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf.

- Invariant $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per INSERT $= 3$.
- Total actual cost of n operations $\leq$ sum of amortized cost $\leq 3n$. ▪

accounting method theorem

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i) - capacity(D_i)$.

number of elements

capacity of array

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i ← immediately after doubling $capacity(D_i) = 2\ size(D_i)$

1	2	3	4	5	6		

size = 6
capacity = 8
Φ = 4

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements / capacity of array

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i

Case 0. [first insertion]

- Actual cost $c_1 = 1$.
- $\Phi(D_1) - \Phi(D_0) = (2\ size(D_1)\ -\ capacity(D_1))\ -\ (2\ size(D_0)\ -\ capacity(D_0))$
  $= 1.$
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_1) - \Phi(D_0))$
  $= 1 + 1$
  $= 2.$

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements / capacity of array

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i

Case 1. [no array expansion] $capacity(D_i) = capacity(D_{i-1})$.

- Actual cost $c_i = 1$.
- $\Phi(D_i) - \Phi(D_{i-1}) = (2\ size(D_i)\ -\ capacity(D_i))\ -\ (2\ size(D_{i-1})\ -\ capacity(D_{i-1}))$
  $= 2.$
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  $= 1 + 2$
  $= 3.$

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements / capacity of array

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i

Case 2. [array expansion] $capacity(D_i) = 2\ capacity(D_{i-1})$.

- Actual cost $c_i = 1 + capacity(D_{i-1})$.
- $\Phi(D_i) - \Phi(D_{i-1}) = (2\ size(D_i)\ -\ capacity(D_i))\ -\ (2\ size(D_{i-1})\ -\ capacity(D_{i-1}))$
  $= 2 - capacity(D_i) + capacity(D_{i-1})$
  $= 2 - capacity(D_{i-1}).$
- Amortized cost $\hat{c}_i = c_i + (\Phi(D_i) - \Phi(D_{i-1}))$
  $= 1 + capacity(D_{i-1}) + (2 - capacity(D_{i-1}))$
  $= 3.$

Dynamic table: insert only (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any sequence of n INSERT operations takes $O(n)$ time.

Pf. Let $\Phi(D_i) = 2\ size(D_i)\ -\ capacity(D_i)$.

number of elements ↑ $size(D_i)$ — capacity of array ↑ $capacity(D_i)$

- $\Phi(D_0) = 0$
- $\Phi(D_i) \geq 0$ for each D_i

[putting everything together]

- Amortized cost per operation $\hat{c}_i \leq 3$.
- Total actual cost of n operations $\leq$ sum of amortized cost $\leq 3n$. ▪

↑ potential method theorem

Dynamic table: doubling and halving

Thrashing.

- INSERT: when inserting into a full table, double the capacity.
- DELETE: when deleting from a table that is ½-full, halve the capacity.

Efficient solution.

- When inserting into an empty table, initialize the table size to 1; when deleting from a table of size 1, free the table.
- INSERT: when inserting into a full table, double the capacity.
- DELETE: when deleting from a table that is ¼-full, halve the capacity.

Memory usage. A dynamic table uses $\Theta(n)$ memory to store n items.

Pf. Table is always between 25% and 100% full. ▪

Dynamic table: insert and delete (aggregate method)

Theorem. [via aggregate method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

Pf.

- In between resizing events, each INSERT and DELETE takes $O(1)$ time.
- Consider the total amount of work between two resizing events.
  - Just after the table is doubled to size m, it contains $m/2$ items.
  - Just after the table is halved to size m, it contains $m/2$ items.
  - Just before the next resizing, it contains either $m/4$ or m items.
  - After resizing to m, we must perform $\Omega(m)$ operations before we resize again (either $\geq m/2$ insertions or $\geq m/4$ deletions).
- Resizing a table of size m requires $O(m)$ time. ▪

Dynamic table demo: insert and delete (accounting method)

Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).
Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

Invariant 1. 2 credits with each item in right half of table.
Invariant 2. 1 credit with each empty slot in left half of table.

delete M

capacity = 16

Dynamic table: insert and delete (accounting method)

Insert. Charge 3 credits (1 to insert; save 2 with item if in right half).
Delete. Charge 2 credits (1 to delete; save 1 in empty slot if in left half).

discard any existing or extra credits

Invariant 1. 2 credits with each item in right half of table. ← to pay for expansion
Invariant 2. 1 credit with each empty slot in left half of table. ← to pay for contraction

Theorem. [via accounting method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

Pf.

- Invariants $\Rightarrow$ number of credits in data structure ≥ 0.
- Amortized cost per operation ≤ 3.
- Total actual cost of n operations $\leq$ sum of amortized cost $\leq 3n$. ▪

accounting method theorem

Dynamic table: insert and delete (potential method)

Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of n INSERT and DELETE operations takes $O(n)$ time.

Pf sketch.

- Let $\alpha(D_i) = size(D_i) / capacity(D_i)$.
- Define $\Phi(D_i) = \begin{cases} 2\,size(D_i) - capacity(D_i) & \text{if } \alpha(D_i) \geq 1/2 \\ \frac{1}{2}\,capacity(D_i) - size(D_i) & \text{if } \alpha(D_i) < 1/2 \end{cases}$
- $\Phi(D_0) = 0, \Phi(D_i) \geq 0$. [a potential function]
- When $\alpha(D_i) = 1/2, \Phi(D_i) = 0$. [zero potential after resizing]
- When $\alpha(D_i) = 1, \Phi(D_i) = size(D_i)$. [can pay for expansion]
- When $\alpha(D_i) = 1/4, \Phi(D_i) = size(D_i)$. [can pay for contraction]

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