CSCI 435: Algorithms and Complexity
3. Approximation Algorithms

- load balancing
- center selection
- weighted vertex cover: pricing method
- weighted vertex cover: LP rounding
- knapsack problem

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-complete problem. What should I do?
A. Theory says you're unlikely to find a poly-time algorithm.

Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve specific instances of the problem.

This lecture's focus. Solving problems as close to optimally as possible.

ρ-approximation algorithm.

- Runs in polynomial time.
- Solves arbitrary instances of the problem.
- Finds a solution that is within some ratio ρ of the optimum.

Challenge. Need to prove that a solution's value is close to optimum, without even knowing what the optimum value is.

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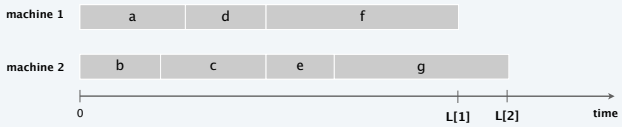
Load balancing

Input. m identical machines; $n \geq m$ jobs, each job j has a processing time t_j.
- Job j must run contiguously on one machine.
- Each machine can process at most one job at a time.

Def. Let $S[i]$ be the subset of jobs assigned to machine i.
The load of a machine i is $L[i] = \sum_{j \in S[i]} t_j$.

Def. The makespan is the maximum load on any machine $L = \max_i L[i]$.

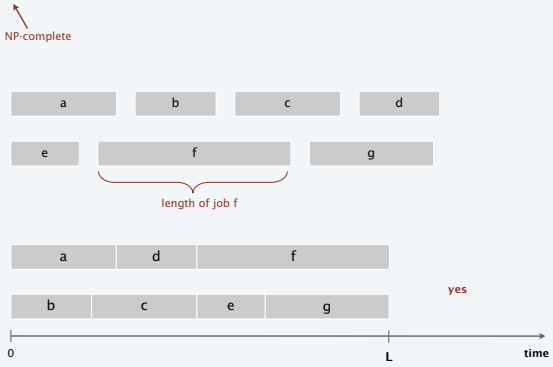
Load balancing. Assign each job to a machine to minimize makespan.



Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if we have just $m = 2$ machines.

Pf. PARTITION $\leq_P$ LOAD-BALANCE.



Load balancing: list scheduling

List-scheduling algorithm.
- Consider n jobs in some fixed order.
- Assign each job j to the machine i whose load is smallest so far.

LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO m
- $L[i] \leftarrow 0$. ← load on machine i
- $S[i] \leftarrow \emptyset$. ← jobs assigned to machine i

FOR $j = 1$ TO n
- $i \leftarrow \text{argmin}_k L[k]$. ← machine i has smallest load
- $S[i] \leftarrow S[i] \cup \{j\}$. ← assign job j to machine i
- $L[i] \leftarrow L[i] + t_j$. ← update load of machine i

RETURN $S[1], S[2], \ldots, S[m]$.

Implementation. $O(n \log m)$ using a priority queue for loads $L[k]$.

Load balancing: list scheduling analysis

Theorem. [Graham 1966] The greedy algorithm is a 2-approximation.

- First worst-case analysis of an approximation algorithm.
- Need to compare the resulting solution to the optimal makespan L^*.

Lemma 1. For all k, the optimal makespan is $L^* \geq t_k$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. The optimal makespan is $L^* \geq \frac{1}{m} \sum_k t_k$.

Pf.

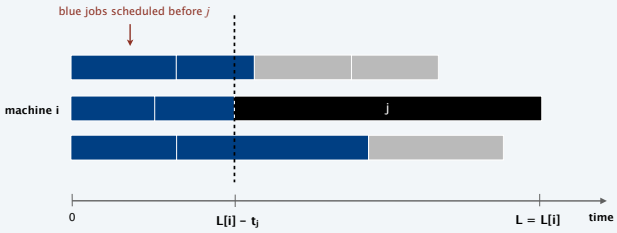
- The total processing time is $\Sigma_k t_k$.
- One of m machines must do at least $1 / m$ fraction of total work. ▪

Load balancing: list scheduling analysis

Theorem. [Graham 1966] The greedy algorithm is a 2-approximation.

Pf. Consider the load $L[i]$ of the bottleneck machine i. ← machine that ends up with highest load

- Let j be the last job scheduled on machine i.
- When job j was assigned to machine i, i had the smallest load.
  Its load before the assignment was $L[i] - t_j$; hence, $L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.



Load balancing: list scheduling analysis

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- Let j be the last job scheduled on machine i.
- When job j was assigned to machine i, i had the smallest load.
  Its load before the assignment was $L[i] - t_j$; hence, $L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.
- Sum inequalities over all k and divide by m:

$$\begin{aligned} L[i] - t_j &\leq \frac{1}{m} \sum_k L[k] \\ &= \frac{1}{m} \sum_k t_k \\ &\leq L^*. \end{aligned}$$

Lemma 2 →

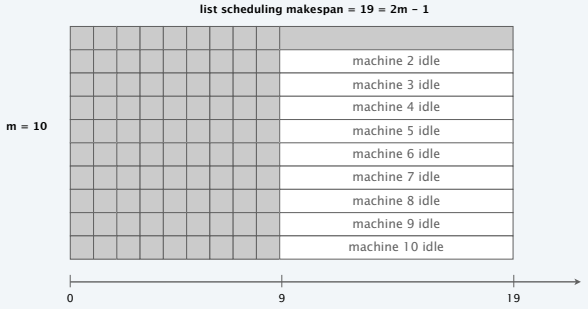
- Now, $L = L[i] = \underbrace{(L[i] - t_j)}_{\leq L^*} + \underbrace{t_j}_{\leq L^*} \leq 2L^*$ ▪

above inequality ↑ Lemma 1 ↑

Load balancing: list scheduling analysis

Q. Is our analysis tight?
A. Essentially, yes.

Ex: m machines, first $m(m-1)$ jobs have length 1, last job has length m.



Load balancing: list scheduling analysis

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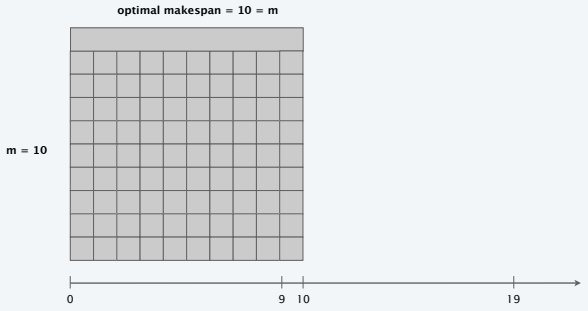

Load balancing: LPT rule

Longest processing time (LPT). Sort n jobs in decreasing order of processing times, then run the list scheduling algorithm.

LPT-LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

SORT jobs and renumber so that $t_1 \geq t_2 \geq \ldots \geq t_n$.

FOR $i = 1$ TO m
  $L[i] \leftarrow 0$. ← load on machine i
  $S[i] \leftarrow \emptyset$. ← jobs assigned to machine i

FOR $j = 1$ TO n
  $i \leftarrow \operatorname{argmin}_k L[k]$. ← machine i has smallest load
  $S[i] \leftarrow S[i] \cup \{j\}$. ← assign job j to machine i
  $L[i] \leftarrow L[i] + t_j$. ← update load of machine i

RETURN $S[1], S[2], \ldots, S[m]$.

Load balancing: LPT rule

Observation. If bottleneck machine i has only 1 job, then this is optimal.

Pf. Any solution must schedule that job. ▪

Lemma 3. If there are more than m jobs, then $L^* \geq 2t_{m+1}$.

Pf.

- Consider the processing times of the first $m+1$ jobs $t_1 \geq t_2 \geq \ldots \geq t_{m+1}$.
- Each job takes at least t_{m+1} time.
- There are $m+1$ jobs and m machines, so by the pigeonhole principle, at least one machine gets two jobs. ▪

Theorem. The LPT rule is a 3/2-approximation algorithm.

Pf. [similar to proof for list scheduling]

- Consider the load $L[i]$ of bottleneck machine i. ← assuming machine i has at least 2 jobs, we have $j \geq m+1$
- Let j be the last job scheduled on machine i.

$$L = L[i] = \underbrace{(L[i] - t_j)}_{\leq L^*} + \underbrace{t_j}_{\leq \frac{1}{2} L^*} \leq \frac{3}{2} L^* \text{ ▪}$$

as before → ← Lemma 3 (since $t_{m+1} \geq t_j$)

Load balancing: LPT rule

Q. Is our 3/2 analysis tight?

A. No.

Theorem. [Graham 1969] LPT rule is a 4/3-approximation.

Pf. More sophisticated analysis of same algorithm.

Q. Is Graham's 4/3 analysis tight?

A. Essentially, yes.

Ex.

- m machines; $n = 2m+1$ jobs.
- 2 jobs of length $m, m+1, \ldots, 2m-1$ and one more job of length m.
- Then, $L / L^* = (4m-1) / (3m)$.

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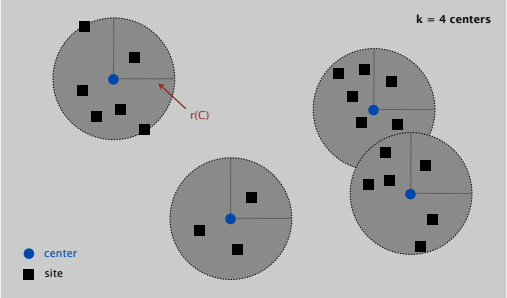
Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select a set of k centers C so that the maximum distance $r(C)$ from a site to the nearest center is minimized.



Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select a set of k centers C so that the maximum distance $r(C)$ from a site to the nearest center is minimized.

Notation.

- $dist(x, y)$ = distance between sites x and y.
- $dist(s_i, C) = \min_{c \in C} dist(s_i, c)$ = distance from s_i to closest center.
- $r(C) = \max_i dist(s_i, C)$ = smallest covering radius.

Goal. Find set of centers C that minimizes $r(C)$, subject to $|C| = k$.

Distance function properties.

- $dist(x, x) = 0$ [identity]
- $dist(x, y) = dist(y, x)$ [symmetry]
- $dist(x, y) \leq dist(x, z) + dist(z, y)$ [triangle inequality]

Center selection: a false start

Greedy algorithm. Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

Remark: This can be arbitrarily bad!

Center selection: greedy algorithm

Better greedy algorithm. Repeatedly choose the next center to be the site farthest from any existing centers.

GREEDY-CENTER-SELECTION $(k, n, s_1, s_2, \ldots, s_n)$

$C \leftarrow \varnothing$.

REPEAT k times

  Select a site s_i with maximum distance $dist(s_i, C)$.

  $C \leftarrow C \cup s_i$.

RETURN C.

site farthest from any center

Property. Upon termination, all centers in C are pairwise at least $r(C)$ apart.

Pf. By construction of algorithm.

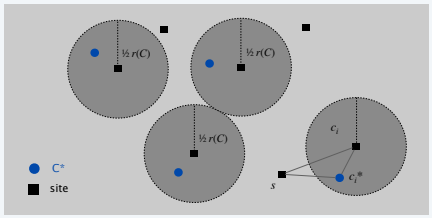
Center selection: analysis of greedy algorithm

Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Pf. [by contradiction] Assume $r(C^*) < \frac{1}{2} r(C)$.

- For each site $c_i \in C$, consider ball of radius $\frac{1}{2} r(C)$ around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^*.
- Consider any site s and its closest center $c_i^* \in C^*$.
- $dist(s, C) \le dist(s, c_i) \le dist(s, c_i^*) + dist(c_i^*, c_i) \le 2r(C^*)$.

  Δ-inequality; ≤ r(C*) since c_i^* is closest center
- Thus, $r(C) \le 2r(C^*)$. ▪



Center selection: analysis of greedy algorithm

Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Theorem. The greedy algorithm is a 2-approximation for the center selection problem.

Remark. The greedy algorithm always places centers at sites, but is still within a factor of 2 of the best solution that is allowed to place centers anywhere.

e.g., points in the plane

Question. Is there hope of a 3/2-approximation? 4/3?

Dominating set reduces to center selection

Theorem. Unless **P** = **NP**, there is no ρ-approximation for the center selection problem for any ρ < 2.

Pf. We can use a $(2-\varepsilon)$-approximation algorithm for CENTER-SELECTION to solve DOMINATING-SET in poly-time.

- Let $G = (V, E)$, k be an instance of DOMINATING-SET.
- Construct an instance G' of CENTER-SELECTION with sites V and distances:
  - $dist(u, v) = 1$ if $(u, v) \in E$
  - $dist(u, v) = 2$ if $(u, v) \notin E$
- Note that G' satisfies the triangle inequality.
- G has a dominating set of size k iff there exist k centers C^* with $r(C^*) = 1$.
- Thus, if G has a dominating set of size k, a $(2-\varepsilon)$-approximation algorithm for CENTER-SELECTION would find a solution C^* with $r(C^*) = 1$ since it cannot use any edge of distance 2. ▪

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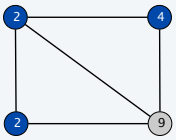
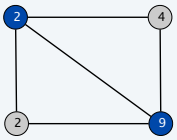
Weighted vertex cover

Vertex cover. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in E has at least one end in S.

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4 = 8



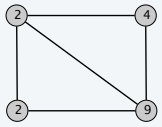
weight = 2 + 9 = 11

Pricing method

Pricing method. Each edge must be covered by some vertex.
Edge $e = (i, j)$ pays a price $p_e \geq 0$ to use both vertices i and j.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i: \sum_{e=(i,j)} p_e \leq w_i$$



Fairness lemma. For any vertex cover S and any fair prices p_e, $\sum_e p_e \leq w(S)$.

Pf. $\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S)$. ▪

each edge e covered by at least one node in S

sum fairness inequalities for each node in S

Pricing method

Pricing method. Set prices and find vertex cover simultaneously.

WEIGHTED-VERTEX-COVER (G, w)

$S \leftarrow \emptyset$.

FOREACH $e \in E$

$p_e \leftarrow 0$.

$\sum_{e=(i,j)} p_e = w_i$

WHILE (there exists an edge (i, j) such that neither i nor j is tight)

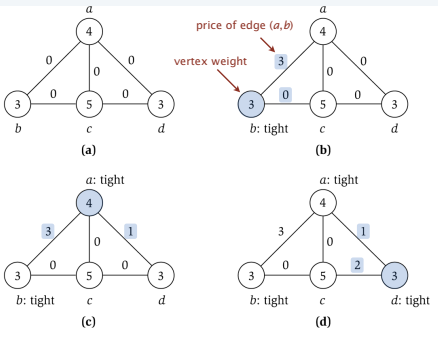
Select such an edge $e = (i, j)$.

Increase p_e as much as possible until i or j is tight.

$S \leftarrow$ set of all tight nodes.

RETURN S.

Pricing method: example



Pricing method: analysis

Theorem. Pricing method is a 2-approximation for WEIGHTED-VERTEX-COVER.

Pf.

- The algorithm terminates, since at least one new node becomes tight after each iteration of the while loop.
- Let S = set of all tight nodes upon termination of algorithm.
  S is a vertex cover: if some edge (i, j) is uncovered, then neither i nor j is tight.
  But then the while loop would not terminate.
- Let S^* be an optimal vertex cover. We show that $w(S) \le 2\, w(S^*)$.

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \le \sum_{i \in V} \sum_{e=(i,j)} p_e = 2 \sum_{e \in E} p_e \le 2w(S^*). \quad \blacksquare$$

↑ all nodes in S are tight; ↑ $S \subseteq V$, prices ≥ 0; ↑ each edge counted twice; ↑ fairness lemma

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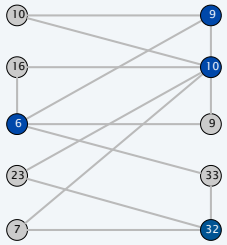
Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \ge 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.



total weight = 6 + 9 + 10 + 32 = 57

Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.

Integer linear programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i.

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

  Vertex covers are in 1–1 correspondence with 0/1 assignments: $S = \{ i \in V : x_i = 1 \}$.

- Objective function: minimize $\sum_i w_i x_i$.

- For every edge (i, j), we must take either vertex i or j (or both): $x_i + x_j \geq 1$.

Weighted vertex cover: ILP formulation

Integer linear programming formulation.

$$\begin{array}{lllll} (ILP) & \min & \sum_{i \in V} w_i x_i & & \\ & \text{s.t.} & x_i + x_j & \geq & 1 \qquad (i,j) \in E \\ & & x_i & \in & \{0, 1\} \qquad i \in V \end{array}$$

Observation. If x^* is an optimal solution to *ILP*, then $S = \{ i \in V : x_i^* = 1 \}$ is a min-weight vertex cover.

Integer linear programming

Given integers a_{ij}, b_i, and c_j, find integers x_j that satisfy:

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & Ax & \geq \; b \\ & x & \geq \; 0 \\ & x & \text{integral} \end{array} \qquad \begin{array}{lllll} \min & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \\ & x_j & & \text{integral} & 1 \leq j \leq n \end{array}$$

Observation. Formulating VERTEX-COVER as an integer linear program proves that INTEGER-PROGRAMMING is an **NP**-hard optimization problem.

~~Integer~~ linear programming

Given integers a_{ij}, b_i, and c_j, find real numbers x_j that satisfy:

$$\begin{array}{lrl} \min & c^T x & \\ \text{s.t.} & Ax & \geq b \\ & x & \geq 0 \end{array}$$

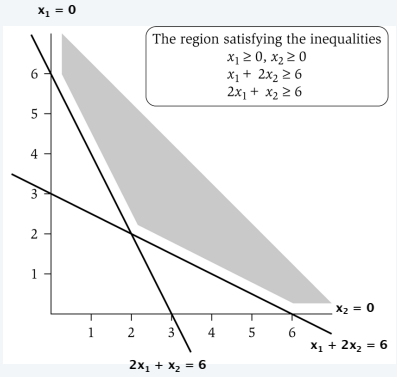
$$\begin{array}{lrll} \min & \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \geq b_i & 1 \leq i \leq m \\ & x_j & \geq 0 & 1 \leq j \leq n \end{array}$$

Linear. No x^2, xy, $\arccos(x)$, $x(1-x)$, etc.

Simplex algorithm. [Dantzig 1947] Can solve LP in practice.
Ellipsoid algorithm. [Khachiyan 1979] Can solve LP in poly-time.
Interior point algorithms. [Karmarkar 1984, Renegar 1988, ...]
Can solve LP both in practice and in poly-time.

LP feasible region

LP geometry in 2D.



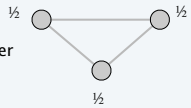
Weighted vertex cover: LP relaxation

Linear programming relaxation.

$$\begin{array}{lrll} (LP) & \min & \sum_{i \in V} w_i x_i & \\ & \text{s.t.} & x_i + x_j & \geq 1 \quad (i,j) \in E \\ & & x_i & \geq 0 \quad i \in V \end{array}$$

Observation. Optimal value of LP $\leq$ optimal value of ILP.
Pf. LP has fewer constraints.

Note. LP solution x^* may not correspond to a vertex cover (even if all weights are 1).



Q. How can solving LP help us find a low-weight vertex cover?
A. Solve LP and round fractional values in x^*.

Weighted vertex cover: LP rounding algorithm

Lemma. If x^* is an optimal solution to LP, then $S = \{ i \in V : x_i^* \geq \frac{1}{2} \}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq \frac{1}{2}$ or $x_j^* \geq \frac{1}{2}$ (or both) $\Rightarrow$ (i, j) covered.

Pf. [S has desired weight]

- Let S^* be an optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \;\geq\; \sum_{i \in S} w_i x_i^* \;\geq\; \frac{1}{2} \sum_{i \in S} w_i$$

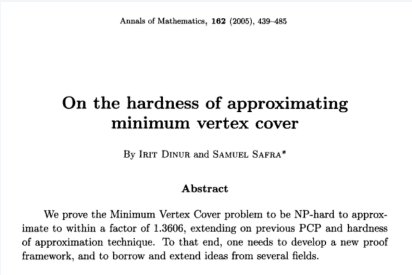
LP is a relaxation; $x_i^* \geq \frac{1}{2}$

Theorem. The rounding algorithm is a 2-approximation algorithm.

Pf. Above lemma + the fact that LP can be solved in poly-time.

Weighted vertex cover inapproximability

Theorem. [Dinur–Safra 2004] If **P** ≠ **NP**, then there exists no ρ-approximation algorithm for WEIGHTED-VERTEX-COVER for any $\rho < 1.3606$ (even if all weights are 1).



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On the hardness of approximating minimum vertex cover

By IRIT DINUR and SAMUEL SAFRA*

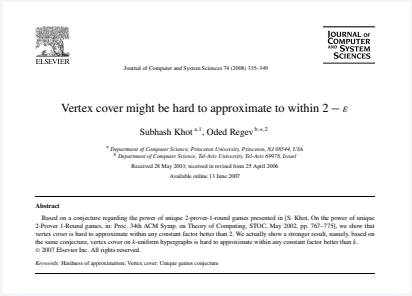
Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and to borrow and extend ideas from several fields.

Open research problem. Close the gap.

Weighted vertex cover inapproximability

Theorem. [Kohot–Regev 2008] If the Unique Games Conjecture is true, then there exists no $2 - \varepsilon$ approximation algorithm for WEIGHTED-VERTEX-COVER for any $\varepsilon > 0$.



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Vertex cover might be hard to approximate to within $2 - \varepsilon$

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Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [S. Khot, On the power of unique 2-Prover 1-Round games, in: Proc. 34th ACM Symp. on Theory of Computing, STOC, May 2002, pp. 767–775], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Keywords: Hardness of approximation; Vertex cover; Unique games conjecture

Open research problem. Prove the Unique Games Conjecture.

CSCI 435: Algorithms and Complexity
3. Approximation Algorithms

- *load balancing*
- *center selection*
- *weighted vertex cover: pricing method*
- *weighted vertex cover: LP rounding*
- *knapsack problem*

Polynomial-time approximation schemes

PTAS. $(1+\varepsilon)$-approximation algorithm for any constant $\varepsilon > 0$.

- Load balancing. [Hochbaum–Shmoys 1987]
- Euclidean TSP. [Arora, Mitchell 1996]
- Many more!

Consequence. PTAS produces an arbitrarily high quality solution, but at the expense of trading off accuracy for time.

Knapsack problem

Knapsack problem.

- Given n objects and a knapsack.
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← assume $w_i \le W$ for each i
- Knapsack has weight limit W.
- Goal: fill knapsack so as to maximize total value.

Ex: $\{ 3, 4 \}$ has value 40.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

original instance (W = 11)

Knapsack problem is NP-complete

KNAPSACK. Given a set X, weights $w_i \geq 0$, values $v_i \geq 0$, a weight limit W, and a target value V, is there a subset $S \subseteq X$ such that:

$$\sum_{i \in S} w_i \leq W$$
$$\sum_{i \in S} v_i \geq V$$

SUBSET-SUM. Given a set X, values $u_i \geq 0$, and an integer U, is there a subset $S \subseteq X$ whose elements sum to exactly U?

Theorem. SUBSET-SUM $\leq_P$ KNAPSACK.

Pf. Given an instance $(u_1, \ldots, u_n, U)$ of SUBSET-SUM, create the following KNAPSACK instance:

$$v_i = w_i = u_i \qquad \sum_{i \in S} u_i \leq U$$
$$V = W = U \qquad \sum_{i \in S} u_i \geq U$$

Knapsack problem: dynamic programming

Def. $OPT(i, w)$ = max value subset of items $1, \ldots, i$ with weight limit w.

Case 1. OPT does not select item i.
- OPT selects best of $1, \ldots, i-1$ using up to weight limit w.

Case 2. OPT selects item i.
- New weight limit $= w - w_i$.
- OPT selects best of $1, \ldots, i-1$ using up to weight limit $w - w_i$.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{ OPT(i-1, w),\ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Theorem. The algorithm computes the optimal value in $O(n\,W)$ time.
- Not polynomial in input size.
- Polynomial time if weights are small integers.

Knapsack problem: dynamic programming revisited

Def. $OPT(i, v)$ = min weight of a knapsack for which we can obtain a solution of value $\geq v$ using a subset of items $1, \ldots, i$.

Note. Optimal value is the largest value v such that $OPT(n, v) \leq W$.

Case 1. OPT does not select item i.
- OPT selects best of $1, \ldots, i-1$ that achieves value $\geq v$.

Case 2. OPT selects item i.
- Consumes weight w_i, need to achieve value $\geq v - v_i$.
- OPT selects best of $1, \ldots, i-1$ that achieves value $\geq v - v_i$.

$$OPT(i, v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } v > 0 \\ \min\{ OPT(i-1, v),\ w_i + OPT(i-1, v - v_i) \} & \text{otherwise} \end{cases}$$

Knapsack problem: dynamic programming revisited

Theorem. The algorithm computes the optimal value in $O(n^2 v_{max})$ time, where v_{max} is the maximum of any value.

Pf.

- The optimal value is $V^* \leq n v_{max}$.
- There is one subproblem for each item and for each value $v \leq V^*$.
- We take $O(1)$ time per subproblem. ▪

Remarks.

- Still not polynomial in input size!
- Polynomial time if values are small integers.

Knapsack problem: polynomial-time approximation scheme

Intuition for approximation algorithm.

- Round all values up to lie within smaller range.
- Run second DP algorithm on rounded/scaled instance.
- Return optimal items from rounded instance.

item	value	weight
1	934221	1
2	5956342	2
3	17810013	5
4	21217800	6
5	27343199	7

original instance (W = 11)

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

rounded instance (W = 11)

Knapsack problem: polynomial-time approximation scheme

Round up all values:

- $0 < \varepsilon \leq 1$ = precision parameter.
- v_{max} = largest value in original instance.
- θ = scaling factor = $\varepsilon v_{max} / 2n$.

$$\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \quad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with $\bar{v}$ are equivalent to optimal solutions to problem with $\hat{v}$.

Intuition. $\bar{v}$ is close to v so the optimal solution using $\bar{v}$ is nearly optimal; $\hat{v}$ is small and integral so the DP algorithm is fast.

Knapsack problem: polynomial-time approximation scheme

Theorem. If S is the solution found by the rounding algorithm and S^* is any other feasible solution, then $(1+\epsilon)\sum_{i \in S} v_i \geq \sum_{i \in S^*} v_i$.

Pf. Let S^* be any feasible solution satisfying the weight constraint.

$$\begin{aligned}
\sum_{i \in S^*} v_i &\leq \sum_{i \in S^*} \bar{v}_i && \text{always round up} \\
&\leq \sum_{i \in S} \bar{v}_i && \text{solve rounded instance optimally} \\
&\leq \sum_{i \in S} (v_i + \theta) && \text{never round up by more than } \theta \\
&\leq \sum_{i \in S} v_i + n\theta && |S| \leq n \\
&= \sum_{i \in S} v_i + \tfrac{1}{2}\,\epsilon\, v_{\max} && \theta = \varepsilon\, v_{\max} / 2n \\
&\leq (1+\epsilon)\sum_{i \in S} v_i && v_{\max} \leq 2 \textstyle\sum_{i \in S} v_i
\end{aligned}$$

subset containing only the item of largest value

choosing $S^* = \{ \max \}$

$$v_{\max} \leq \sum_{i \in S} v_i + \tfrac{1}{2}\,\epsilon\, v_{\max} \leq \sum_{i \in S} v_i + \tfrac{1}{2}\, v_{\max}$$

thus

$$v_{\max} \leq 2\sum_{i \in S} v_i$$

Knapsack problem: polynomial-time approximation scheme

Theorem. For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1+\varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

Pf.

- We have already proved the accuracy bound.
- DP algorithm running time is $O(n^2 \hat{v}_{\max})$, where $\hat{v}_{\max} = \left\lceil \frac{v_{\max}}{\theta} \right\rceil = \left\lceil \frac{2n}{\epsilon} \right\rceil$