CSCI 435: Algorithms and Complexity
8. Linear Programming I

- *a refreshing example*
- *standard form*
- *fundamental questions*
- *geometry*
- *linear algebra*
- *simplex algorithm*

Linear programming

Linear programming. Optimize a linear function subject to linear inequalities.

$$
\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
& \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\
& & x_j & \ge & 0 & 1 \le j \le n
\end{array}
$$

$$
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
& \text{s.t.} & Ax & = & b \\
& & x & \ge & 0
\end{array}
$$

Linear programming

Linear programming. Optimize a linear function subject to linear inequalities.

Generalizes: $Ax = b$, two-person zero-sum games, shortest path, max flow, assignment problem, matching, multicommodity flow, MST, min weighted arborescence, …

What is the significance?

- Design poly-time algorithms.
- Design approximation algorithms.
- Solve NP-hard problems using branch-and-cut.

Ranked among the most important scientific advances of the 20th century!

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Reference: *The Allocation of Resources by Linear Programming*, Scientific American, by Bob Bland

Brewery problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

How can the brewer maximize profits?

- Devote all resources to ale: 34 barrels of ale ⇒ $442
- Devote all resources to beer: 32 barrels of beer ⇒ $736
- 7.5 barrels of ale, 29.5 barrels of beer ⇒ $776
- 12 barrels of ale, 28 barrels of beer ⇒ $800

Brewery problem

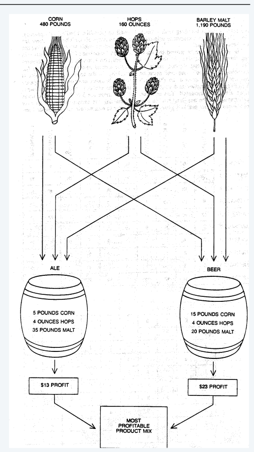
objective function

Ale Beer

$$
\begin{array}{lrcrcrl}
\max & 13A & + & 23B & & & \text{Profit} \\
\text{s. t.} & 5A & + & 15B & \le & 480 & \text{Corn} \\
 & 4A & + & 4B & \le & 160 & \text{Hops} \\
 & 35A & + & 20B & \le & 1190 & \text{Malt} \\
 & A & , & B & \ge & 0 &
\end{array}
$$

constraint

decision variable



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Standard form of a linear program

Standard form.

- Input: real numbers a_{ij}, c_j, b_i.
- Output: real numbers x_j.
- n = number of decision variables, m = number of constraints.
- Maximize the linear objective function subject to linear inequalities.

$$
\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\
 & & x_j & \ge & 0 & 1 \le j \le n
\end{array}
$$

$$
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
 & \text{s.t.} & Ax & = & b \\
 & & x & \ge & 0
\end{array}
$$

"Linear". No x^2, $x\,y$, $\arccos(x)$, etc.
"Programming". Planning.
(term predates computer programming)

Brewery problem: converting to standard form

Original input.

$$
\begin{array}{lrcrcr}
\max & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \le & 480 \\
 & 4A & + & 4B & \le & 160 \\
 & 35A & + & 20B & \le & 1190 \\
 & A & , & B & \ge & 0
\end{array}
$$

Standard form.

- Add a slack variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{lrcrcccccccr}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_H & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_M & = & 1190 \\
 & A & , & B & , & S_C & , & S_H & , & S_M & \ge & 0
\end{array}
$$

Equivalent forms

Converting to standard form.

$$
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
 & \text{s. t.} & Ax & = & b \\
 & & x & \geq & 0
\end{array}
$$

Less than to equality. $x + 2y - 3z \leq 17 \Rightarrow x + 2y - 3z + s = 17, s \geq 0$
Greater than to equality. $x + 2y - 3z \geq 17 \Rightarrow x + 2y - 3z - s = 17, s \geq 0$
Min to max. $\min\ x + 2y - 3z \Rightarrow \max\ -x - 2y + 3z$
Unrestricted to nonnegative. x unrestricted $\Rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$

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Fundamental questions

Linear programming. For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, does there exist $x \in \mathbb{R}^n$ such that $Ax = b,\ x \geq 0,\ c^T x \geq \alpha$?

Q. Is LP in **NP**?
Q. Is LP in co-**NP**?
Q. Is LP in **P**?

Input size.
- n = number of variables.
- m = number of constraints.
- L = number of bits to encode input.

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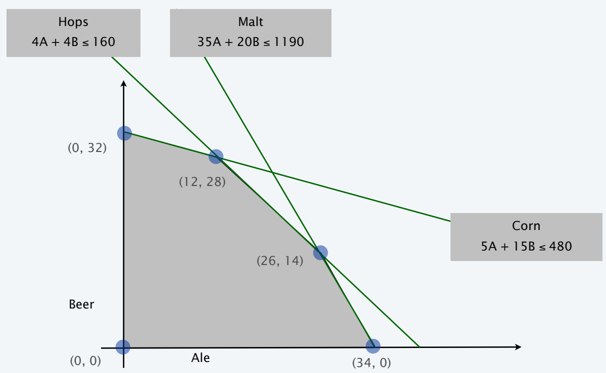
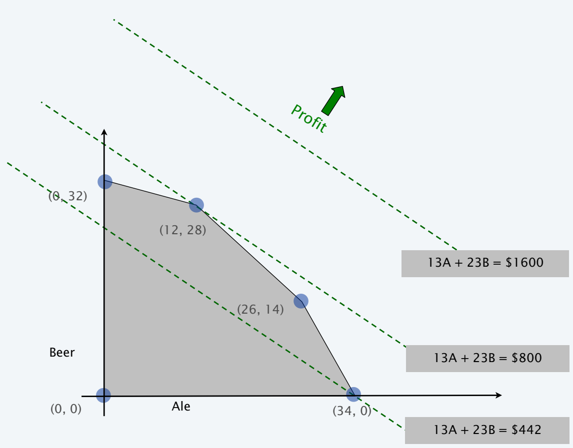
Brewery problem: feasible region

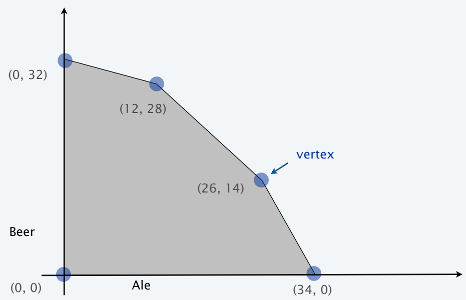


Brewery problem: objective function



Brewery problem: geometry

Observation. Regardless of the objective function coefficients, an optimal solution occurs at a vertex.



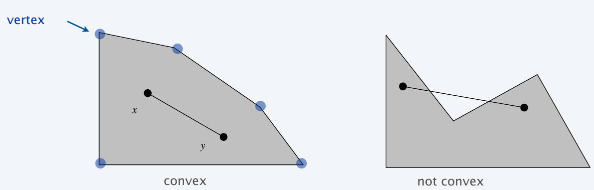
Convexity

Convex set. If two points x and y are in the set, then so is $\lambda x + (1-\lambda) y$ for $0 \le \lambda \le 1$.

convex combination

not a vertex iff $\exists\, d \neq 0$ s.t. $x \pm d$ in set

Vertex. A point x in the set that can't be written as a strict convex combination of two distinct points in the set.



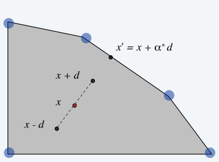
Observation. The LP feasible region is a convex set.

Purification

Theorem. If there exists an optimal solution to (P), then there exists a solution that is a vertex.

$$\text{(P)} \quad \max \; c^T x \quad \text{s.t.} \; Ax = b, \; x \ge 0$$

Intuition. If x is not a vertex, move in a non-decreasing direction until you reach a boundary. Repeat.



Purification

Theorem. If there exists an optimal solution to (P), then there exists a solution that is a vertex.

Pf.

- Suppose x is an optimal solution that is not a vertex.
- There exists a direction $d \neq 0$ such that $x \pm d \in P$.
- $A d = 0$ because $A(x \pm d) = b$.
- Assume $c^T d \leq 0$ (by taking either d or $-d$).
- Consider $x + \lambda d, \ \lambda > 0$:

Case 1. [there exists j such that $d_j < 0$]

- Increase λ to λ^* until the first new component of $x + \lambda d$ hits 0.
- $x + \lambda^* d$ is feasible, since $A(x + \lambda^* d) = Ax = b$ and $x + \lambda^* y \geq 0$.
- $x + \lambda^* d$ has one more zero component than x.
- $c^T x' = c^T (x + \lambda^* d) = c^T x + \lambda^* c^T d \leq c^T x$.

$d_k = 0$ whenever $x_k = 0$ because $x \pm d \in P$

Purification

Theorem. If there exists an optimal solution to (P), then there exists a solution that is a vertex.

Pf.

- Suppose x is an optimal solution that is not a vertex.
- There exists a direction $d \neq 0$ such that $x \pm d \in P$.
- $A d = 0$ because $A(x \pm d) = b$.
- Assume $c^T d \leq 0$ (by taking either d or $-d$).
- Consider $x + \lambda d, \ \lambda > 0$:

Case 2. [$d_j \geq 0$ for all j]

- $x + \lambda d$ is feasible for all $\lambda \geq 0$, since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$.
- As $\lambda \to \infty, \ c^T(x + \lambda d) \to \infty$ because $c^T d < 0$. ▪

if $c^T d = 0$, choose d so that case 1 applies

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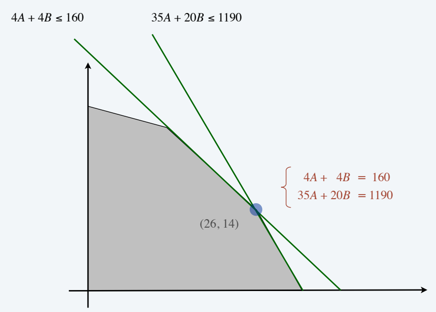
Feasible solutions

Intuition. A vertex in $\mathbb{R}^m$ is uniquely specified by m linearly independent equations.



Basic feasible solution

Theorem. Let $P = \{ x : Ax = b, x \ge 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then, x is a vertex iff A_B has linearly independent columns.

Notation. Let B = set of column indices.
Define A_B to be the subset of columns of A indexed by B.

Ex. $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$

$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{1, 3\}, \quad A_B = \begin{bmatrix} 2 & 3 \\ 7 & 2 \\ 0 & 0 \end{bmatrix}$$

Basic feasible solution

Theorem. Let $P = \{ x : Ax = b, x \ge 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then, x is a vertex iff A_B has linearly independent columns.

Pf. [$\Rightarrow$]

- Assume x is not a vertex.
- There exists a direction $d \ne 0$ such that $x \pm d \in P$.
- $A\,d = 0$ because $A(x \pm d) = b$.
- Define $B' = \{ j : d_j \ne 0 \}$.
- $A_{B'}$ has linearly dependent columns since $d \ne 0$.
- Moreover, $d_j = 0$ whenever $x_j = 0$ because $x \pm d \ge 0$.
- Thus $B' \subseteq B$, so $A_{B'}$ is a submatrix of A_B.
- Therefore, A_B has linearly dependent columns.

Basic feasible solution

Theorem. Let $P = \{ x : Ax = b, x \geq 0 \}$. For $x \in P$, define $B = \{ j : x_j > 0 \}$. Then, x is a vertex iff A_B has linearly independent columns.

Pf. [⇐]

- Assume A_B has linearly dependent columns.
- There exists $d \neq 0$ such that $A_B \, d = 0$.
- Extend d to $\mathbb{R}^n$ by adding 0 components.
- Now, $A \, d = 0$ and $d_j = 0$ whenever $x_j = 0$.
- For sufficiently small λ, $x \pm \lambda d \in P \Rightarrow x$ is not a vertex. ▪

Basic feasible solution

Theorem. Given $P = \{ x : Ax = b, x \geq 0 \}$, x is a vertex iff there exists $B \subseteq \{ 1, \ldots, n \}$ such that $|B| = m$ and:

- A_B is nonsingular.
- $x_B = A_B^{-1} \, b \geq 0$.
- $x_N = 0$.

basic feasible solution

Pf. Augment A_B with linearly independent columns (if needed). ▪

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 7 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 16 \\ 0 \end{bmatrix}$$

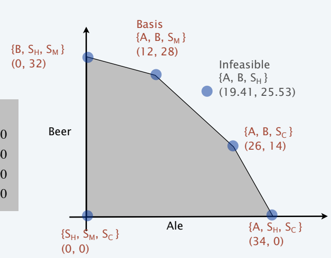
$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \{ 1, 3, 4 \}, \quad A_B = \begin{bmatrix} 2 & 3 & 0 \\ 7 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Assumption. $A \in \mathbb{R}^{m \times n}$ has full row rank.

Basic feasible solution: example

Basic feasible solutions.

$$\begin{array}{lrcrcccccl} \max & 13A & + & 23B & & & & & & \\ \text{s.t.} & 5A & + & 15B & + & S_C & & & = & 480 \\ & 4A & + & 4B & & & + & S_H & = & 160 \\ & 35A & + & 20B & & & & + \; S_M & = & 1190 \\ & A & , & B & , & S_C & , & S_H \; , \; S_M & \geq & 0 \end{array}$$



Fundamental questions

Linear programming. For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, does there exist $x \in \mathbb{R}^n$ such that $Ax = b,\ x \geq 0,\ c^T x \geq \alpha$?

Q. Is LP in **NP**?
A. Yes!

- Number of vertices $\leq C(n, m) = \binom{n}{m} \leq n^m$.
- Cramer's rule $\Rightarrow$ we can check a vertex in poly-time.

Cramer's rule. For $B \in \mathbb{R}^{n \times n}$ invertible, $b \in \mathbb{R}^n$, the solution to $Bx = b$ is given by:

$$x_i = \frac{\det(B_i)}{\det(B)}$$

replace ith column of B with b

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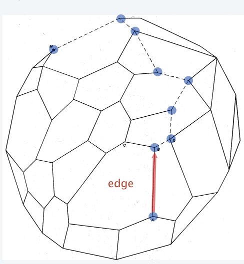
Simplex algorithm: intuition

Simplex algorithm. [Dantzig, 1947] Move from a basic feasible solution to an adjacent one, without decreasing the objective function.

replace one basic variable with another



Greediness. A basic feasible solution is optimal iff no adjacent one is better.
Challenge. Number of basic feasible solutions can be exponential!

Simplex algorithm: initialization

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
13A & + & 23B & & & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & & & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Simplex algorithm: pivot 1

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
13A & + & 23B & & & & & & & - & Z & = & 0 \\
\hline
5A & + & \boxed{15B} & + & S_C & & & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & & & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Substitute: $B = 1/15\,(480 - 5A - S_C)$

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Simplex algorithm: pivot 1

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Q. Why pivot on column 2 (or 1)?
A. Each unit increase in B increases the objective value by \$23.

Q. Why pivot on row 2?
A. Preserves feasibility by ensuring that RHS ≥ 0.

min ratio rule: min { 480/15, 160/4, 1190/20 }

Simplex algorithm: pivot 2

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
\tfrac{16}{3}A \;-\; \tfrac{23}{15}S_C \;-\; Z = -736 \\
\hline
\tfrac{1}{3}A + B + \tfrac{1}{15}S_C = 32 \\
\tfrac{8}{3}A - \tfrac{4}{15}S_C + S_H = 32 \\
\tfrac{85}{3}A - \tfrac{4}{3}S_C + S_M = 550 \\
A,\; B,\; S_C,\; S_H,\; S_M \ge 0
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Substitute: $A = 3/8\ (32 + 4/15\ S_C - S_H)$

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
- S_C - 2\,S_H - Z = -800 \\
\hline
B + \tfrac{1}{10}S_C + \tfrac{1}{8}S_H = 28 \\
A - \tfrac{1}{10}S_C + \tfrac{3}{8}S_H = 12 \\
- \tfrac{25}{6}S_C - \tfrac{85}{8}S_H + S_M = 110 \\
A,\; B,\; S_C,\; S_H,\; S_M \ge 0
\end{array}
$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex algorithm: optimality

Q. When to stop pivoting?
A. When all coefficients in the top row are nonpositive.

Q. Why is the resulting solution optimal?
A. Any feasible solution satisfies the system of equations in the tableaux.

- In particular: $Z = 800 - S_C - 2\,S_H$, $S_C \ge 0$, $S_H \ge 0$.
- Thus, the optimal objective value $Z^* \le 800$.
- Current basic feasible solution has value 800 ⇒ optimal.

$$
\begin{array}{l}
\text{max } Z \text{ subject to} \\
- S_C - 2\,S_H - Z = -800 \\
\hline
B + \tfrac{1}{10}S_C + \tfrac{1}{8}S_H = 28 \\
A - \tfrac{1}{10}S_C + \tfrac{3}{8}S_H = 12 \\
- \tfrac{25}{6}S_C - \tfrac{85}{8}S_H + S_M = 110 \\
A,\; B,\; S_C,\; S_H,\; S_M \ge 0
\end{array}
$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex tableaux: matrix form

Initial simplex tableaux.

$$
\begin{array}{rcl}
c_B^T x_B + c_N^T x_N & = & Z \\
A_B x_B + A_N x_N & = & b \\
x_B \;,\; x_N & \ge & 0
\end{array}
$$

Simplex tableaux corresponding to basis B.

$$
\begin{array}{rcl}
(c_N^T - c_B^T A_B^{-1} A_N)\, x_N & = & Z - c_B^T A_B^{-1} b \\
I\, x_B + A_B^{-1} A_N\, x_N & = & A_B^{-1} b \\
x_B \;,\; x_N & \ge & 0
\end{array}
$$

← subtract $c_B{}^T A_B{}^{-1}$ times constraints

← multiply by $A_B{}^{-1}$

basic feasible solution	optimal basis
$x_B = A_B{}^{-1} b \ge 0$, $x_N = 0$	$c_N{}^T - c_B{}^T A_B{}^{-1} A_N \le 0$

Corner cases

We're missing some details for corner cases of the simplex algorithm.

Q. What happens if the min ratio test fails?
Q. How do we find the initial basis?
Q. How to guarantee termination?

Unboundedness

Q. What happens if the min ratio test fails?

all coefficients in entering column are nonpositive

$$
\begin{array}{l}
\max Z \text{ subject to} \\
\begin{array}{ccccccccccccc}
 & & & & & + & 2x_4 & + & 20x_5 & - & Z & = & 2 \\
x_1 & & & & & - & 4x_4 & - & 8x_5 & & & = & 3 \\
 & & x_2 & & & + & 5x_4 & - & 12x_5 & & & = & 4 \\
 & & & & x_3 & & & & & & & = & 5 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & & & \ge & 0
\end{array}
\end{array}
$$

A. Unbounded objective function.

$$Z = 2 + 20x_5 \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 + 8x_5 \\ 4 + 12x_5 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Initial basis

Q. How do we find the initial basis?

$$
\begin{array}{llrcl}
\text{(P)} & \max & c^T x & & \\
 & \text{s.t.} & Ax & = & b \\
 & & x & \ge & 0
\end{array}
$$

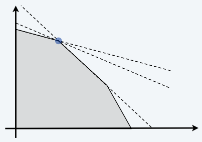
A. Solve (P′), starting from the basis consisting of all the z_i variables.

$$
\begin{array}{llrcl}
\text{(P}'\text{)} & \max & \sum_{i=1}^{m} z_i & & \\
 & \text{s.t.} & Ax + Iz & = & b \\
 & & x, \quad z & \ge & 0
\end{array}
$$

- Case 1: $\min > 0 \Rightarrow$ (P) is infeasible.
- Case 2: $\min = 0$, basis has no z_i variables $\Rightarrow$ okay to start Phase II.
- Case 3: $\min = 0$, basis has z_i variables. Pivot z_i variables out of basis. If successful, start Phase II; else, remove linear dependent rows.

Termination: degeneracy

Degeneracy. New basis, same vertex.



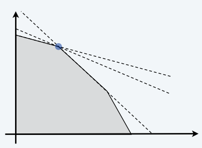
Degenerate pivot. Min ratio = 0.

max Z subject to

$$
\begin{array}{lllllllllll}
 & & & \frac{3}{4}x_4 & - & 20x_5 & + & \frac{1}{2}x_6 & - & 6x_7 & - Z = 0 \\
\hline
x_1 & & & + \frac{1}{4}x_4 & - & 8x_5 & - & x_6 & + & 9x_7 & = 0 \\
 & x_2 & & + \frac{1}{2}x_4 & - & 12x_5 & - & \frac{1}{2}x_6 & + & 3x_7 & = 0 \\
 & & x_3 & & & & + & x_6 & & & = 1 \\
x_1, & x_2, & x_3, & x_4, & & x_5, & & x_6, & & x_7 & \geq 0
\end{array}
$$

Termination: cycling

Degeneracy. New basis, same vertex.



Cycling. Infinite loop by cycling through different bases that all correspond to the same vertex.

Anti-cycling rules.

- Bland's rule: choose the eligible variable with the smallest index.
- Random rule: choose the eligible variable uniformly at random.
- Lexicographic rule: perturb constraints to ensure nondegeneracy.

Termination: lexicographic rule

Intuition. No degeneracy ⇒ no cycling.

Perturbed problem.

much much greater, say $\varepsilon_i = \delta^i$ for small δ

$$
\begin{array}{llll}
(P') & \max & c^T x & \\
 & \text{s.t.} & Ax = b + \varepsilon & \\
 & & x \geq 0 &
\end{array}
\quad \text{where } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \text{ such that } \varepsilon_1 \succ \varepsilon_2 \succ \cdots \succ \varepsilon_n
$$

Lexicographic rule. Apply a perturbation virtually by manipulating ε symbolically:

$$17 + 5\varepsilon_1 + 11\varepsilon_2 + 8\varepsilon_3 \leq 17 + 5\varepsilon_1 + 14\varepsilon_2 + 3\varepsilon_3$$

Termination: lexicographic rule

Intuition. No degeneracy $\Rightarrow$ no cycling.

Perturbed problem.

much much greater, say $\varepsilon_i = \delta^i$ for small δ

$$(P') \quad \max \ c^T x \quad \text{s.t.} \quad Ax = b + \varepsilon, \quad x \geq 0 \qquad \text{where } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \text{ such that } \varepsilon_1 \succ \varepsilon_2 \succ \cdots \succ \varepsilon_n$$

Claim. In the perturbed problem, $x_B = A_B^{-1}(b + \varepsilon)$ is always nonzero.

Pf. The j^{th} component of x_B is a (nonzero) linear combination of the components of $b + \varepsilon$

$\Rightarrow$ contains at least one of the ε_i terms.

which can't cancel

Corollary. No cycling!

Simplex algorithm: practice

Remarkable property. In practice, the simplex algorithm typically terminates after at most $2(m + n)$ pivots.

but no polynomial pivot rule known

Issues.

- Avoid stalling.
- Choose the pivot.
- Maintain sparsity.
- Ensure numerical stability.
- Preprocess to eliminate variables and constraints.

Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints!

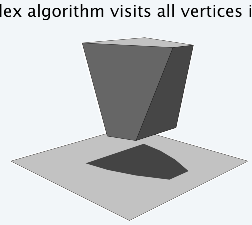
Simplex algorithm: practice

Remarkable property. In practice, the simplex algorithm typically terminates after at most $2(m + n)$ pivots.

but no polynomial pivot rule known

Worst-case performance. In the worst case, the simplex algorithm requires exponential time.

Klee–Minty cube. A "squashed" hypercube that has 2^D vertices in D dimensions. The simplex algorithm visits all vertices in the worst case.



Reference: Wikipedia