CSCI 435: ALGORITHMS AND COMPLEXITY
9. LINEAR PROGRAMMING II

- *LP duality*
- *strong duality theorem*
- *applications*
- *ellipsoid algorithm*

LP duality

Primal problem.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

Goal. Find a lower bound on the optimal value.

Easy to do! Any feasible solution provides one.

Ex 1. $(A, B) = (34, 0) \Rightarrow z^* \ge 442$

Ex 2. $(A, B) = (0, 32) \Rightarrow z^* \ge 736$

Ex 3. $(A, B) = (7.5, 29.5) \Rightarrow z^* \ge 776$

Ex 4. $(A, B) = (12, 28) \Rightarrow z^* \ge 800$

LP duality

Primal problem.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

Goal. Find an upper bound on the optimal value.

Ex 1. Add 2 times the 1st inequality to the 2nd inequality:

$$\Rightarrow \quad z^* = \underbrace{13\,A + 23\,B}_{\text{objective function}} \le 14\,A + 34\,B \le 1120.$$

LP duality

Primal problem.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s.t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

Goal. Find an upper bound on the optimal value.

Ex 2. Multiply the 2nd inequality by 6:

$$\Rightarrow \quad z^* = 13A + 23B \le 24A + 24B \le 960.$$

LP duality

Primal problem.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s.t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

Goal. Find an upper bound on the optimal value.

Ex 3. Add the 1st inequality to 2 times the 2nd inequality:

$$\Rightarrow \quad z^* = 13A + 23B \le 13A + 23B \le 800.$$

Recall the lower bound. $(A, B) = (12, 28) \Rightarrow z^* \ge 800$

Combine the upper and lower bounds: $z^* = 800$.

LP duality

Primal problem.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s.t.} & 5A & + & 15B & \le & 480 \\ & & 4A & + & 4B & \le & 160 \\ & & 35A & + & 20B & \le & 1190 \\ & & A & , & B & \ge & 0 \end{array}$$

Idea. Add a nonnegative combination (C, H, M) of the constraints s.t.

$$\begin{array}{rl} 13A + 23B & \le \; (5C + 4H + 35M)\; A \; + \; (15C + 4H + 20M)\; B \\ & \le \; 480C + 160H + 1190M \end{array}$$

Dual problem. Find the best such upper bound.

$$\begin{array}{llrcrcrcr} \text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\ & \text{s.t.} & 5C & + & 4H & + & 35M & \ge & 13 \\ & & 15C & + & 4H & + & 20M & \ge & 23 \\ & & C & , & H & , & M & \ge & 0 \end{array}$$

LP duality: economic interpretation

Brewer. Find the optimal mix of beer and ale to maximize profits.

$$\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \le & 480 \\
 & & 4A & + & 4B & \le & 160 \\
 & & 35A & + & 20B & \le & 1190 \\
 & & A & , & B & \ge & 0
\end{array}$$

Entrepreneur. Buy individual resources from the brewer at minimum cost.

- C, H, M = unit price for corn, hops, and malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \ge & 13 \\
 & & 15C & + & 4H & + & 20M & \ge & 23 \\
 & & C & , & H & , & M & \ge & 0
\end{array}$$

Canonical form and double duals

Canonical form.

$$\begin{array}{llrcl}
\text{(P)} & \max c^T x & & & \\
 & \text{s. t.} & Ax & \le & b \\
 & & x & \ge & 0
\end{array}
\qquad
\begin{array}{llrcl}
\text{(D)} & \min y^T b & & & \\
 & \text{s. t.} & A^T y & \ge & c \\
 & & y & \ge & 0
\end{array}$$

Property. The dual of the dual is the primal.

Pf. Rewrite (D) as a maximization problem in canonical form, then take the dual.

$$\begin{array}{llrcl}
\text{(D')} & \max\ -y^T b & & & \\
 & \text{s. t.} & -A^T y & \le & -c \\
 & & y & \ge & 0
\end{array}
\qquad
\begin{array}{llrcl}
\text{(DD)} & \min\ -c^T z & & & \\
 & \text{s. t.} & -(A^T)^T z & \ge & -b \\
 & & z & \ge & 0
\end{array}$$

Taking duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a x = b_i$ $a x \le b_i$ $a x \ge b_i$	y_i unrestricted $y_i \ge 0$ $y_i \le 0$	variables
variables	$x_j \ge 0$ $x_j \le 0$ x_j unrestricted	$a^{\mathrm{T}} y \ge c_j$ $a^{\mathrm{T}} y \le c_j$ $a^{\mathrm{T}} y = c_j$	constraints

Pf. Rewrite the LP in standard form, then take the dual.

LP duality: example

Primal.

$$\begin{array}{lrcrcrcl}
\min & x_1 & - & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & -x_1 & & & + & 4x_3 & = & 5\\
 & 2x_1 & + & 3x_2 & - & 5x_3 & \geq & 6\\
 & & & 7x_2 & & & \leq & 8\\
 & x_1 & & & & & \geq & 0\\
 & & & x_2 & & & & \text{free}\\
 & & & & & x_3 & \leq & 0.
\end{array}$$

Primal program has three constraints

Dual.

$$\begin{array}{lrcrcrcl}
\max & 5y_1 & + & 6y_2 & + & 8y_3 & & \\
\text{s.t.} & -y_1 & + & 2y_2 & & & \leq & 1\\
 & & & 3y_2 & + & 7y_3 & = & -2\\
 & 4y_1 & - & 5y_2 & & & \geq & 3\\
 & y_1 & & & & & & \text{free}\\
 & & & y_2 & & & \geq & 0\\
 & & & & & y_3 & \leq & 0.
\end{array}$$

Dual program has three variables

LP duality: example

Primal.

$$\begin{array}{lrcrcrcl}
\min & x_1 & - & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & -x_1 & & & + & 4x_3 & = & 5\\
 & 2x_1 & + & 3x_2 & - & 5x_3 & \geq & 6\\
 & & & 7x_2 & & & \leq & 8\\
 & x_1 & & & & & \geq & 0\\
 & & & x_2 & & & & \text{free}\\
 & & & & & x_3 & \leq & 0.
\end{array}$$

First constraint is =

Second constraint is ≥

Third constraint is ≤

Dual.

$$\begin{array}{lrcrcrcl}
\max & 5y_1 & + & 6y_2 & + & 8y_3 & & \\
\text{s.t.} & -y_1 & + & 2y_2 & & & \leq & 1\\
 & & & 3y_2 & + & 7y_3 & = & -2\\
 & 4y_1 & - & 5y_2 & & & \geq & 3\\
 & y_1 & & & & & & \text{free}\\
 & & & y_2 & & & \geq & 0\\
 & & & & & y_3 & \leq & 0.
\end{array}$$

Corresponding variable is free

Corresponding variable is non-neg

Corresponding variable is non-pos

LP duality: example

Primal.

$$\begin{array}{lrcrcrcl}
\min & x_1 & - & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & -x_1 & & & + & 4x_3 & = & 5\\
 & 2x_1 & + & 3x_2 & - & 5x_3 & \geq & 6\\
 & & & 7x_2 & & & \leq & 8\\
 & x_1 & & & & & \geq & 0\\
 & & & x_2 & & & & \text{free}\\
 & & & & & x_3 & \leq & 0.
\end{array}$$

Primal program has three variables

Dual.

$$\begin{array}{lrcrcrcl}
\max & 5y_1 & + & 6y_2 & + & 8y_3 & & \\
\text{s.t.} & -y_1 & + & 2y_2 & & & \leq & 1\\
 & & & 3y_2 & + & 7y_3 & = & -2\\
 & 4y_1 & - & 5y_2 & & & \geq & 3\\
 & y_1 & & & & & & \text{free}\\
 & & & y_2 & & & \geq & 0\\
 & & & & & y_3 & \leq & 0.
\end{array}$$

Dual program has three constraints

LP duality: example

Primal.

$$\begin{array}{lrcrcrcl}
\min & x_1 & - & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & -x_1 & & & + & 4x_3 & = & 5 \\
 & 2x_1 & + & 3x_2 & - & 5x_3 & \geq & 6 \\
 & & & 7x_2 & & & \leq & 8 \\
 & x_1 & & & & & \geq & 0 \\
 & & & x_2 & & & & \text{free} \\
 & & & & & x_3 & \leq & 0.
\end{array}$$

← First variable is non-neg

← Second variable is free

← Third variable is non-pos

Dual.

$$\begin{array}{lrcrcrcl}
\max & 5y_1 & + & 6y_2 & + & 8y_3 & & \\
\text{s.t.} & -y_1 & + & 2y_2 & & & \leq & 1 \\
 & & & 3y_2 & + & 7y_3 & = & -2 \\
 & 4y_1 & - & 5y_2 & & & \geq & 3 \\
 & y_1 & & & & & & \text{free} \\
 & & & y_2 & & & \geq & 0 \\
 & & & & & y_3 & \leq & 0.
\end{array}$$

← Corresponding constraint is $\leq$

← Corresponding constraint is $=$

← Corresponding constraint is $\geq$

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LP weak duality

Theorem. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, if (P) and (D) are nonempty, then max ≤ min.

$$\begin{array}{llrcl}
\text{(P)} & \max c^T x & & & \\
 & \text{s. t.} & Ax & \leq & b \\
 & & x & \geq & 0
\end{array}
\qquad
\begin{array}{llrcl}
\text{(D)} & \min y^T b & & & \\
 & \text{s. t.} & A^T y & \geq & c \\
 & & y & \geq & 0
\end{array}$$

Pf. Suppose $x \in \mathbb{R}^m$ is feasible for (P) and $y \in \mathbb{R}^n$ is feasible for (D).

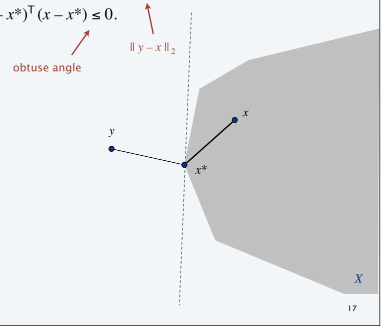
- $y \geq 0,\ A x \leq b \quad \Rightarrow \quad y^T A x \leq y^T b$
- $x \geq 0,\ A^T y \geq c \quad \Rightarrow \quad y^T A x \geq c^T x$
- Combine: $c^T x \leq y^T A x \leq y^T b$. ▪

Question. Can we turn the inequality $\underset{\text{weak duality}}{\leq}$ into an equality $\underset{\text{strong duality}}{=}$?

Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then $\min \{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \mathbb{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$, we have $(y - x^*)^T (x - x^*) \leq 0$.



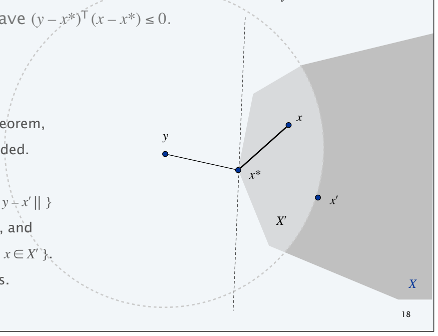
$\| y - x \|_2$

obtuse angle

Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then $\min \{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \mathbb{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$, we have $(y - x^*)^T (x - x^*) \leq 0$.



Pf (part 1).

- Define $f(x) = \| y - x \|$.
- Want to apply Weierstrass' theorem, but X is not necessarily bounded.
- $X \neq \emptyset \;\Rightarrow$ there exists $x' \in X$.
- Define $X' = \{ x \in X : \| y - x \| \leq \| y - x' \| \}$ so that X' is closed, bounded, and $\min \{ f(x) : x \in X \} = \min \{ f(x) : x \in X' \}$.
- By Weierstrass, the min exists.

Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then $\min \{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \mathbb{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$, we have $(y - x^*)^T (x - x^*) \leq 0$.

Pf (part 2).

- x^* min distance $\Rightarrow \| y - x^* \|^2 \leq \| y - x \|^2$ for all $x \in X$.
- By convexity: if $x \in X$, then $x^* + \varepsilon\,(x - x^*) \in X$ for all $0 < \varepsilon < 1$.
- $\| y - x^* \|^2 \leq \| y - x^* - \varepsilon(x - x^*) \|^2$
  $= \| y - x^* \|^2 + \varepsilon^2 \| (x - x^*) \|^2 - 2\,\varepsilon\,(y - x^*)^T (x - x^*)$
- Thus, $(y - x^*)^T (x - x^*) \leq \tfrac{1}{2}\,\varepsilon\, \| (x - x^*) \|^2$.
- Letting $\varepsilon \to 0^+$, we obtain the desired result. ▪

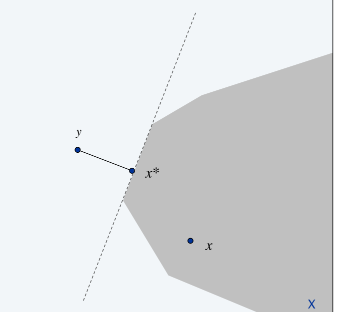
Separating hyperplane theorem

Theorem. Let $X \subset \mathbb{R}^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists a hyperplane $H = \{ x \in \mathbb{R}^m : a^T x = \alpha \}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X.

$a^T x \geq \alpha$ for all $x \in X$
$a^T y < \alpha$

Pf.

- Let x^* be the closest point in X to y.
- By the projection lemma,
  $(y - x^*)^T (x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = x^* - y \neq 0$ and $\alpha = a^T x^*$.
- If $x \in X$, then $a^T(x - x^*) \geq 0$;
  thus, $a^T x \geq a^T x^* = \alpha$.
- Also, $a^T y = a^T (x^* - a) = \alpha - ||a||^2 < \alpha$. ▪



$H = \{ x \in \mathbb{R}^m : a^T x = \alpha \}$

Farkas' lemma

Theorem. [Farkas, 1902] For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, exactly one of the following two systems holds:

$$\text{(I)} \quad \exists\, x \in \mathbb{R}^n \quad \begin{array}{rrcl} \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$$\text{(II)} \quad \exists\, y \in \mathbb{R}^m \quad \begin{array}{rrcl} \text{s.t.} & A^T y & \geq & 0 \\ & y^T b & < & 0 \end{array}$$

Pf. [not both] Suppose x satisfies (I) and y satisfies (II).
Then $0 > y^T b = y^T A x \geq 0$, which is a contradiction. ▪

Pf. [at least one] Suppose (I) is infeasible. We will show that (II) is feasible.

- Consider $S = \{ A x : x \geq 0 \}$ so that S is closed and convex, and let $b \notin S$.
- Let $y \in \mathbb{R}^m, \alpha \in \mathbb{R}$ be a hyperplane that separates b from S:
  $y^T b < \alpha$, $y^T s \geq \alpha$ for all $s \in S$.
- $0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$.
- $y^T A x \geq \alpha$ for all $x \geq 0 \Rightarrow y^T A \geq 0$, since x can be arbitrarily large. ▪

LP strong duality

Theorem. [Gale–Kuhn–Tucker, 1951; Dantzig–von Neumann, 1947]
For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, if (P) and (D) are nonempty, then max = min.

$$\text{(P)} \quad \max\, c^T x \quad \begin{array}{rrcl} \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

$$\text{(D)} \quad \min\, y^T b \quad \begin{array}{rrcl} \text{s.t.} & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

Pf. [max ≤ min] Weak LP duality.
Pf. [min ≤ max] Suppose that max < α. We show that min < α.

$$\text{(I)} \quad \exists\, x \in \mathbb{R}^n \quad \begin{array}{rrcl} \text{s.t.} & Ax & \leq & b \\ & -c^T x & \leq & -\alpha \\ & x & \geq & 0 \end{array}$$

$$\text{(II)} \quad \exists\, y \in \mathbb{R}^m,\ z \in \mathbb{R} \quad \begin{array}{rrcl} \text{s.t.} & A^T y - c\, z & \geq & 0 \\ & y^T b - \alpha\, z & < & 0 \\ & y,\ z & \geq & 0 \end{array}$$

- By the definition of α, (I) infeasible ⇒ (II) feasible, by Farkas' lemma.

LP strong duality

Pf. (cont'd)

$$\text{(II)} \quad \exists\, y \in \mathbb{R}^m,\ z \in \mathbb{R} \quad \text{s.t.} \quad \begin{aligned} A^T y - cz &\geq 0 \\ y^T b - \alpha z &< 0 \\ y, z &\geq 0 \end{aligned}$$

Let y, z be a solution to (II).

Case 1. $[z = 0]$

- Then, $\{\, y \in \mathbb{R}^m : A^T y \geq 0,\ y^T b < 0,\ y \geq 0 \,\}$ is feasible.
- Farkas' lemma $\Rightarrow$ $\{\, x \in \mathbb{R}^n : Ax \leq b, x \geq 0 \,\}$ is infeasible.
- Contradiction, since by our assumption, (P) is nonempty.

Case 2. $[z > 0]$

- Scale y, z so that y satisfies (II) and $z = 1$.
- Resulting y is feasible to (D) and $y^T b < \alpha$. ▪

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LP duality: economic interpretation

Brewer. Find the optimal mix of beer and ale to maximize profits.

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$A^* = 12$
$B^* = 28$
$OPT = 800$

Entrepreneur. Buy individual resources from the brewer at minimum cost.

$$\text{(D)} \quad \begin{array}{lrcrcrcr} \min & 480C & + & 160H & + & 1190M & & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \geq & 13 \\ & 15C & + & 4H & + & 20M & \geq & 23 \\ & C & , & H & , & M & \geq & 0 \end{array}$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

LP duality. Market clears.

LP duality: sensitivity analysis

Q. How much should the brewer be willing to pay (marginal price) for additional supplies of scarce resources?

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s.t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & & 15C & + & 4H & + & 20M & \geq & 23 \\
 & & C & , & H & , & M & \geq & 0
\end{array}
$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

A. Corn \$1, hops \$2, malt \$0.

Q. Suppose a new product, "light beer", is proposed. It requires 2 corn, 5 hops, and 24 malt. How much profit must be obtained from light beer to justify diverting resources from the production of beer and ale?

A. At least 2 (\$1) + 5 (\$2) + 24 (\$0) = \$12 / barrel.

not light beer;
American beer

Fundamental questions

Linear programming. For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, does there exist $x \in \mathbb{R}^n$ such that $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Q. Is LP in **NP** ∩ co-**NP**?
A. Yes!

- We already showed that LP is in **NP**.
- If the LP is infeasible, then apply Farkas' lemma to get a certificate of infeasibility:

$$
\begin{array}{llrcl}
\text{(II)} & \exists\, y \in \mathbb{R}^m,\ z \in \mathbb{R} & & & \\
 & \text{s.t.} & A^T y & \geq & 0 \\
 & & y^T b - \alpha z & < & 0 \\
 & & z & \geq & 0
\end{array}
$$

or equivalently,
$y^T b - \alpha z = -1$

- This proves that LP is in co-**NP**.

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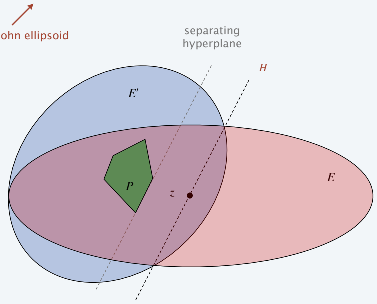
Geometric divide-and-conquer

To find a point in P:

- Maintain an ellipsoid E containing P.
- If the center of the ellipsoid, z, is in P, stop;
  otherwise, find the hyperplane separating z from P.
- Find the smallest ellipsoid E' containing half the ellipsoid E.

and consider the corresponding half-ellipsoid $½\, E = E \cap H$

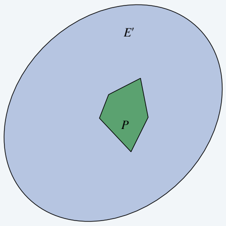
Lowner–John ellipsoid



Geometric divide-and-conquer

To find a point in P:

- Maintain an ellipsoid E containing P.
- If the center of the ellipsoid, z, is in P, stop;
  otherwise, find the hyperplane separating z from P.
- Find the smallest ellipsoid E' containing half the ellipsoid E.
- Repeat.



Ellipsoid algorithm

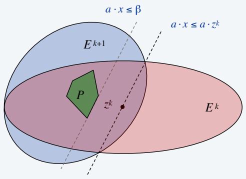
Goal. Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, find $x \in \mathbb{R}^n$ such that $\underbrace{Ax \le b}_{P}$.

Ellipsoid algorithm.

- Let E_0 be an ellipsoid containing P.
- Set $k = 0$.
- While the center z^k of the ellipsoid E^k is not in P :
  - Find a constraint, say $a \cdot x \le \beta$, that is violated by z^k
  - Let E^{k+1} be the min-volume ellipsoid containing $\underbrace{E^k \cap \{\, x : a \cdot x \le a \cdot z^k \}}_{\text{half-ellipsoid } ½\, E}$
  - Increment $k = k + 1$.

enumerate constraints

easy to compute



Shrinking lemma

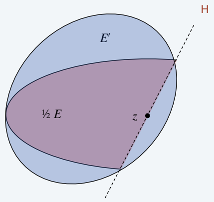
Ellipsoid. Given $D \in \mathbb{R}^{n \times n}$ positive definite and $z \in \mathbb{R}^n$, then

$$E \;=\; \{\; x \in \mathbb{R}^n : (x-z)^T D^{-1}(x-z) \;\leq\; 1 \;\}$$

is an ellipsoid centered on z with $\text{vol}(E) = \sqrt{\det(D)} \;\times\; vol(B(0,\; 1))$.

unit sphere

Key lemma. Every half-ellipsoid ½ E is contained in an ellipsoid E' with $\text{vol}(E') \,/\, \text{vol}(E) \leq e^{-1/(2n+1)}$.



Shrinking lemma: unit sphere

Special case. E = unit sphere, $H = \{ x : x_1 \geq 0 \}$.

$$E \;=\; \{\; x : \sum_{i=1}^{n} (x_i)^2 \;\leq\; 1 \;\}$$

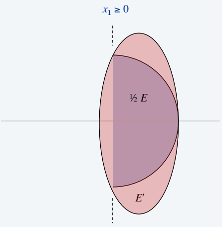
$$E' \;=\; \{\; x : \left(\tfrac{n+1}{n}\right)^2 (x_1 \;-\tfrac{1}{n+1})^2 \;+\tfrac{n^2-1}{n^2} \;\sum_{i=2}^{n} (x_i)^2 \;\leq\; 1 \;\}$$

Claim. E' is an ellipsoid containing ½ $E = E \cap H$.

Pf. If $x \in$ ½ E, then

$$\begin{aligned}
&\left(\frac{n+1}{n}\right)^2 \left(x_1 \;-\; \frac{1}{n+1}\right)^2 \;+\; \frac{n^2-1}{n^2} \;\sum_{i=2}^{n} x_i^{\,2} \\
&\quad= \;\frac{n^2+2n+1}{n^2} x_1^{\,2} \;-\; \left(\frac{n+1}{n}\right)^2 \frac{2x_1}{n+1} \;+\; \frac{1}{n^2} + \;\frac{n^2-1}{n^2} \;\sum_{i=2}^{n} x_i^{\,2} \\
&\quad= \;\frac{2n+2}{n^2} x_1^{\,2} \;-\; \frac{2n+2}{n^2} x_1 \;+\; \frac{1}{n^2} + \;\frac{n^2-1}{n^2} \;\sum_{i=1}^{n} x_i^{\,2} \\
&\quad= \;\frac{2n+2}{n^2} x_1(x_1-1) \;+\; \frac{1}{n^2} + \;\frac{n^2-1}{n^2} \;\sum_{i=1}^{n} x_i^{\,2} \\
&\quad\leq \;0 + \;\frac{1}{n^2} + \;\frac{n^2-1}{n^2} \\
&\quad= \;1
\end{aligned}$$

$0 \leq x_1 \leq 1$ $\quad$ $\sum x_i^2 \leq 1$



Shrinking lemma: unit sphere

Special case. E = unit sphere, $H = \{ x : x_1 \geq 0 \}$.

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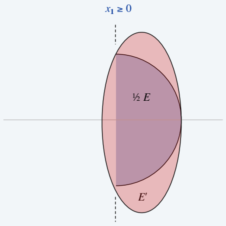
Claim. E' is an ellipsoid containing ½ $E = E \cap H$.

Pf (cont'd). The volume of the ellipsoid is proportional to the side lengths:

$$\begin{aligned}
\frac{vol(E')}{vol(E)} \;&=\; \left(\tfrac{n^2}{n^2-\,1}\right)^{\frac{n-1}{2}} \;\left(\tfrac{n}{n\;\;+1}\right) \\
&=\; \left(1 \;+\; \tfrac{1}{n^2-\,1}\right)^{\frac{n-1}{2}} \;\left(1 \;-\; \tfrac{1}{n\;\;+1}\right) \\
&\leq\; e^{\frac{1}{n^2-1}\;\frac{n-1}{2}} \quad e^{\frac{-1}{n+1}} \\
&=\; e^{\frac{-1}{2(n+1)}}
\end{aligned}$$

$1 + x \leq e^x$



Shrinking lemma

Shrinking lemma. The min-volume ellipsoid containing the half-ellipsoid $\frac{1}{2} E = E \cap \{ x : a \cdot x \le a \cdot z \}$ is defined by:

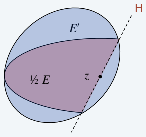
$$z' = z - \frac{1}{n+1} \frac{Da}{\sqrt{a^T D a}} , \quad D' = \frac{n^2}{n^2 - 1} \left(D - \frac{2}{n+1} \frac{D a a^T D}{a^T D a} \right)$$

$$E' = \{ x \in \mathbb{R}^n : (x - z')^T (D')^{-1} (x - z') \le 1 \}$$

Moreover, $\mathrm{vol}(E') / \mathrm{vol}(E) < e^{-1/(2n+1)}$.

Pf sketch.

- We proved the special case where E = unit sphere, $H = \{ x : x_1 \ge 0 \}$.
- Ellipsoids are affine transformations of unit spheres.
- Volume ratios are preserved under affine transformations.



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Moreover, $\mathrm{vol}(E') / \mathrm{vol}(E) < e^{-1/(2n+1)}$.

Corollary. The ellipsoid algorithm terminates after at most $2(n+1) \ln (\mathrm{vol}(E_0) / \mathrm{vol}(P))$ steps.

Ellipsoid algorithm

Theorem. LP is in **P**.

Pf sketch.

- Use the shrinking lemma.
- Set the initial ellipsoid E_0 so that $\mathrm{vol}(E_0) \le 2^{cnL}$.
- Perturb $Ax \le b$ to $Ax \le b + \varepsilon \Rightarrow$ either P is empty or $\mathrm{vol}(P) \ge 2^{-cnL}$.
- Use the bit complexity model (to deal with square roots).
- Purify to a vertex solution.

Caveat. This is a theoretical result. Do not implement.

$O(mn^3 L)$ arithmetic ops on numbers of size $O(L)$,
where L = number of bits to encode input

New York Times article

Nov. 27, 1979.

Leonid Khachiyan

An Approach to Difficult Problems

Mathematicians disagree as to the ultimate practical value of Leonid Khachiyan's new technique, but concur that in any case it is an important theoretical accomplishment.

Mr. Khachiyan's method is believed to offer an approach for the linear programming of computers to solve so-called "traveling salesman" problems. Such problems are among the most intractable in mathematics. They involve, for instance, finding the shortest route by which a salesman could visit a number of cities without his path touching the same city twice.

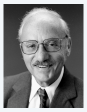
Each time a new city is added to the route, the problem becomes very much more complex. Very large numbers of variables must be calculated from large numbers of equations using a system of linear programming. At a certain point, the compexity becomes so great that a computer would require billions of years to find a solution.

In the past, "traveling salesmen" problems, including the efficient scheduling of airline crews or hospital nursing staffs, have been solved on computers using the "simplex method" invented by George B. Dantzig of Stanford University.

As a rule, the simplex method works well, but it offers no guarantee that after a certain number of computer steps it will always find an answer. Mr. Khachiyan's approach offers a way of telling right from the start whether or not a problem will be soluble in a given number of steps.

Two mathematicians conducting research at Stanford already have applied the Khachiyan method to develop a program for a pocket calculator, which has solved problems that would not have been possible with a pocket calculator using the simplex method.

Mathematically, the Khachiyan approach uses equations to create imaginary ellipsoids that encapsulate the answer, unlike the simplex method, in which the answer is represented by the intersections of the sides of polyhedrons. As the ellipsoids are made smaller and smaller, the answer is known with greater precision. MALCOLM W. BROWNE



George Dantzig