Extending the Lyndon-Schützenberger Theorem

SOGMSC 2016

Taylor J. Smith

Joint work with Jeffrey Shallit

David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario, Canada

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Table of Contents

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Table of Contents

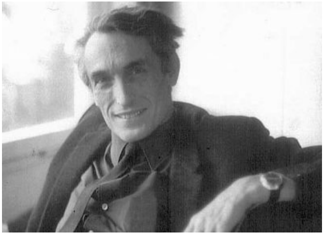
Background

- Combinatorics on words is the branch of mathematics that focuses on combinations, sequences, and patterns of symbols.
- By investigating properties of words, we get many useful applications (e.g. pattern matching).
- Most of the time, we only consider words in one dimension.
- What happens to some of these properties if we introduce a second dimension?

Background

- The **Lyndon-Schützenberger theorem** defines a set of conditions for when the concatenation of two words x and y commutes; that is, when $xy = yx$.
- This theorem is one of the most well-known results in the field of combinatorics on words. (For a proof, see the paper by Lyndon and Schützenberger.)

R. Lyndon



M.-P. Schützenberger

Table of Contents

1D Preliminaries

- A **symbol** is a primitive, atomic unit, such as a letter. A nonempty set of symbols is called an **alphabet**.
- A **word** (or string) is a combination of symbols.
- A **language** is a set of words.
- Given a word w, we can take the n-th **power** of w, written w^n, by repeating w a total of n times.

1D Preliminaries

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Example

Given the alphabet $\Sigma = \{\mathtt{a}, \mathtt{c}, \mathtt{t}\}$, we can form the word $w = \mathtt{cat}$ and the language $L = \{\mathtt{at},\ \mathtt{act},\ \mathtt{cat},\ \mathtt{tac}\}$.

1D Lyndon-Schützenberger Theorem

Theorem (1D Lyndon-Schützenberger Theorem)

Let $x, y \in \Sigma^+$. Then the following three conditions are equivalent:

1. $xy = yx$;
2. *There exist $z \in \Sigma^+$ and integers $k, l > 0$ such that $x = z^k$ and $y = z^l$;*
3. *There exist integers $i, j > 0$ such that $x^i = y^j$.*

1D Lyndon-Schützenberger Theorem

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Theorem (1D Lyndon-Schützenberger Theorem)

Let $x, y \in \Sigma^+$. Then the following ***five*** *conditions are equivalent:*

1. $xy = yx$;
2. *There exist $z \in \Sigma^+$ and integers $k, l > 0$ such that $x = z^k$ and $y = z^l$;*
3. *There exist integers $i, j > 0$ such that $x^i = y^j$;*
4. ***There exist integers $r, s > 0$ such that $x^r y^s = y^s x^r$;***
5. $\boldsymbol{x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset}$.

3. There exist integers $i, j > 0$ such that $x^i = y^j$.

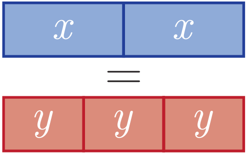
$\Downarrow$

4. There exist integers $r, s > 0$ such that $x^r y^s = y^s x^r$.

Proof.

If $x^i = y^j$, then comparing prefixes and suffixes reveals that $x^i y^j = y^j x^i$.

Take $r = i$ and $s = j$ to get $x^r y^s = y^s x^r$. $\square$



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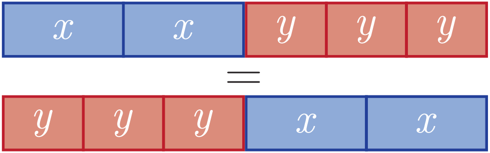


1D Lyndon-Schützenberger Theorem

4. There exist integers $r, s > 0$ such that $x^r y^s = y^s x^r$.

$\Downarrow$

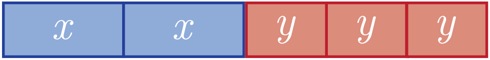
5. $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

Proof.

Let $z = x^r y^s$. Then $z \in x\{x, y\}^*$.

By condition 4, we know that $z = y^s x^r$, so $z \in y\{x, y\}^*$.

Therefore, $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$. $\square$

$z =$ 

1D Lyndon-Schützenberger Theorem

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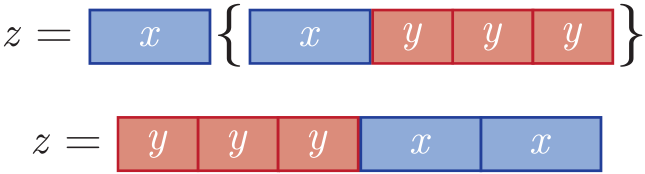

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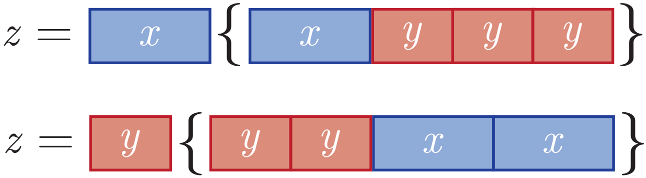

5. $x\{x, y\}^* \cap y\{x, y\}^* \neq \emptyset$.

$\Downarrow$

1. $xy = yx$.

Proof.

By induction on $|xy|$.

- Both the base case ($|xy| = 2$) and the case where $|x| = |y|$ are trivial.

1D Lyndon-Schützenberger Theorem

5. $x\{x,y\}^* \cap y\{x,y\}^* \neq \emptyset$.

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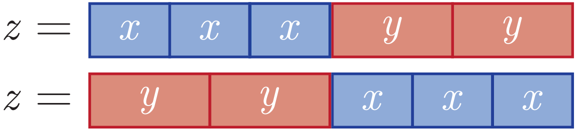
Proof.

By induction on $|xy|$.

- Without loss of generality, assume $|x| < |y|$.
  Let z be as before. Since $z \in x\{x,y\}^*$ and $z \in y\{x,y\}^*$ by condition 5, we know x is a proper prefix of y.



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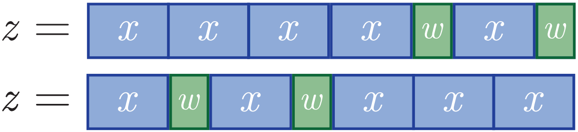
- Let $y = xw$. Then z has the prefixes xx and xw, so $x^{-1}z \in x\{x, w\}^*$ and $x^{-1}z \in w\{x, w\}^*$. Thus, $x\{x, w\}^* \cap w\{x, w\}^* \neq \emptyset$.
  By induction, condition 1 holds for x and w, so $xw = wx$ and therefore $yx = (xw)x = x(wx) = xy$. □



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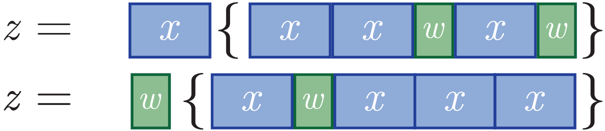

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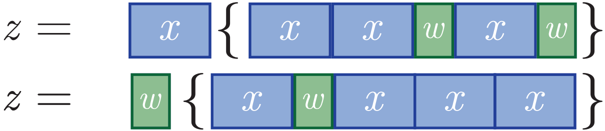

Table of Contents

2D Preliminaries

- We consider a **two-dimensional word** to be an array of symbols from some alphabet Σ.
- $\Sigma^{m \times n}$ is the set of all $m \times n$ rectangular arrays M of elements chosen from Σ.
- If $M \in \Sigma^{m \times n}$, then $M^{p \times q}$ is the $pm \times qn$ rectangular array constructed by repeating M in p rows and q columns.

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Example

If $M = \begin{bmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{bmatrix}$, then $M^{2\times 3} = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} & \mathtt{c} & \mathtt{d} & \mathtt{c} & \mathtt{d} \\ \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} & \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} & \mathtt{c} & \mathtt{d} & \mathtt{c} & \mathtt{d} \end{bmatrix}$.

2D Preliminaries

- We can **concatenate** two arrays A and B, but we must insist on a matching of dimension.
- If A is $m \times n_1$ and B is $m \times n_2$, then $A \varobar B$ is the $m \times (n_1 + n_2)$ array obtained by placing B to the right of A.
- If A is $m_1 \times n$ and B is $m_2 \times n$, then $A \ominus B$ is the $(m_1 + m_2) \times n$ array obtained by placing B beneath A.

2D Preliminaries

- We can **concatenate** two arrays A and B, but we must insist on a matching of dimension.
- If A is $m \times n_1$ and B is $m \times n_2$, then $A \oplus B$ is the $m \times (n_1 + n_2)$ array obtained by placing B to the right of A.
- If A is $m_1 \times n$ and B is $m_2 \times n$, then $A \ominus B$ is the $(m_1 + m_2) \times n$ array obtained by placing B beneath A.

Example

If $A_1 = \begin{bmatrix} \mathtt{a} & \mathtt{b} \end{bmatrix}$ and $B_1 = \begin{bmatrix} \mathtt{c} & \mathtt{d} \end{bmatrix}$, then $A_1 \ominus B_1 = \begin{bmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{bmatrix}$.

Example

If $A_2 = \begin{bmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{d} & \mathtt{e} \end{bmatrix}$ and $B_2 = \begin{bmatrix} \mathtt{c} \\ \mathtt{f} \end{bmatrix}$, then $A_2 \oplus B_2 = \begin{bmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} \\ \mathtt{d} & \mathtt{e} & \mathtt{f} \end{bmatrix}$.

2D Lyndon-Schützenberger Theorem

Theorem (2D Lyndon-Schützenberger Theorem)

Let A and B be nonempty arrays. Then the following three conditions are equivalent:

1. *There exist positive integers p_1, p_2, q_1, q_2 such that* $A^{p_1 \times q_1} = B^{p_2 \times q_2}$*;*
2. *There exist a nonempty array C and positive integers r_1, r_2, s_1, s_2 such that* $A = C^{r_1 \times s_1}$ *and* $B = C^{r_2 \times s_2}$*;*
3. *There exist positive integers t_1, t_2, u_1, u_2 such that* $A^{t_1 \times t_2} \circ B^{u_1 \times u_2} = B^{u_1 \times u_2} \circ A^{t_1 \times t_2}$ *where $\circ$ can be $\varobar$ or $\ominus$.*

Remark

- Conditions 1, 2, and 3 in the 2D version correspond to conditions 3, 2, and 4, respectively, in the 1D version.
- Here, we prove $2 \Rightarrow 1$ and $2 \Rightarrow 3$. (Other directions omitted.)

2. There exist a nonempty array C and positive integers r_1, r_2, s_1, s_2 such that $A = C^{r_1 \times s_1}$ and $B = C^{r_2 \times s_2}$.

$\Downarrow$

1. There exist positive integers p_1, p_2, q_1, q_2 such that $A^{p_1 \times q_1} = B^{p_2 \times q_2}$.

Proof.

Let $p_1 = r_2$, $p_2 = r_1$, $q_1 = s_2$, and $q_2 = s_1$. Then

$$\begin{aligned} A^{p_1 \times q_1} &= (C^{r_1 \times s_1})^{p_1 \times q_1} \\ &= C^{p_1 r_1 \times q_1 s_1} \\ &= C^{r_2 p_2 \times s_2 q_2} \\ &= (C^{r_2 \times s_2})^{p_2 \times q_2} \\ &= B^{p_2 \times q_2}. \end{aligned}$$

□

2. There exist a nonempty array C and positive integers r_1, r_2, s_1, s_2 such that $A = C^{r_1 \times s_1}$ and $B = C^{r_2 \times s_2}$.

$\Downarrow$

3. There exist positive integers t_1, t_2, u_1, u_2 such that $A^{t_1 \times t_2} \circ B^{u_1 \times u_2} = B^{u_1 \times u_2} \circ A^{t_1 \times t_2}$ where $\circ$ can be $\varobar$ or $\ominus$.

Proof.

Assume the operation is $\varobar$. (The proof is similar for $\ominus$.) Let $t_1 = r_2$, $t_2 = r_1$, $u_1 = s_2$, and $u_2 = s_1$. Then

$$\begin{aligned} A^{t_1 \times u_1} \varobar B^{t_2 \times u_2} &= (C^{r_1 \times s_1})^{t_1 \times u_1} \varobar (C^{r_2 \times s_2})^{t_2 \times u_2} \\ &= C^{r_1 t_1 \times s_1 u_1} \varobar C^{r_2 t_2 \times s_2 u_2} \\ &\vdots \\ &= C^{r_2 t_2 \times s_2 u_2} \varobar C^{r_1 t_1 \times s_1 u_1} \\ &= (C^{r_2 \times s_2})^{t_2 \times u_2} \varobar (C^{r_1 \times s_1})^{t_1 \times u_1} \\ &= B^{t_2 \times u_2} \varobar A^{t_1 \times u_1}. \end{aligned}$$

□

2D Lyndon-Schützenberger Theorem

- What does this theorem give us?
  - Conditions to check for and to match patterns in large arrays.
  - An algorithm to determine the primitivity or periodicity of a given array.
  - An algorithm to find the "primitive root" of a given array.
- For details, see our paper!

$A^{m \times n} =$

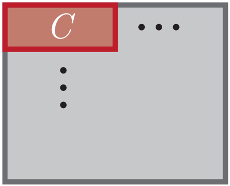

Table of Contents

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Conclusions

- The one-dimensional version of the Lyndon-Schützenberger theorem admits two new equivalent conditions.
- There exists an analogous two-dimensional version of the Lyndon-Schützenberger theorem that is remarkably similar to the one-dimensional case.
- This two-dimensional version of the theorem allows us to investigate many useful properties of two-dimensional words.

Future Work

- Is there a two-dimensional analogue to conditions 1 and 5 of the one-dimensional Lyndon-Schützenberger theorem?
- Do higher-dimensional analogues (e.g. 3D) of this theorem exist? What information can we gain from these analogues?

References

[1] R. C. Lyndon and M.-P. Schützenberger. The equation $a^M = b^N c^P$ in a free group. *Mich. Math. J.*, 9(4):289–298, 1962.

[2] J. Shallit and T. J. Smith. Periodicity in rectangular arrays. arXiv:1602.06915.