Advanced Microeconomics (ECON 401)
Lecture 1
Preference Relations & Choice

Teng Wah Leo

Granted you have the basics pertaining to Microeconomics, regarding Consumer Theory, for further development of the theory, and its myriad of applications, we need now to revisit the underlying decision processes that underpin what you have understood about Indifference Curves and Utility Maximization.

The manner in which we will proceed is via two approaches. Consider first, the ideal situation when we can observe an agent's entire decision process, her likes and dislikes, and how it drives her choices. For us to accurately discuss the mechanism through which the agent makes her choice, we must then have a systematic, and universally understood language. We can do so by first imposing some fundamental assumptions, such as one of *rationality*. To be precise, we impose *Raionality Axioms*. It is these assumptions that will drive her ranking of objects of desire, or what is commonly termed *Preference Relations*. It is through these that the individual then makes her choices.

However, the world we exist in is not perfect, for instance, we normally do not see the complete decision process. Matter of fact, if we did, it would be a huge violation of their privacy. Of course, this privacy might be completely illusory today. The question then is whether we can learn about the preferences, and the process simply through observing the final choice itself? Well, we could if the choices are indeed a true reflection of their preference. But this is possible if there is some rule to her decision process, and if you thought *rationality*, you would be correct. It is with this assumption, that we can then track back, through how she would have made her decision, and consequently, her preferences.

1 Preference Relations & Choice Rules

1.1 Preference Relations

Let us start of with a set of potential choices that an individual agent could make, and denote it as $\mathcal{X}$. The idea of *preference relation* is used to describe how an individual would rank/rate one object compared to another, and is denoted as $\succsim$. More formally, we refer to $\succsim$ as a binary relation on all the objects/choices that the individual could make. Let's consider an example. Let both x_1 and $x_2 \in \mathcal{X}$. Then if the individual likes x_1 more, we have $x_1 \succsim x_2$, and we say that "x_1 is at least as good as x_2". You should then wonder if there are other more intense or less intense relations, if so, you would be correct. So that to complete

1. *Stict Preference Relation*, $\succ$, is then defined as,

$$x_1 \succ x_2 \quad \Leftrightarrow \quad x_1 \succsim x_2 \text{ but not } x_2 \succsim x_1 \tag{1}$$

2. *Indifference Relation*, $\sim$, is then defined as,

$$x_1 \sim x_2 \quad \Leftrightarrow \quad x_1 \succsim x_2 \;\&\; x_2 \succsim x_1 \tag{2}$$

In consequence, $\succsim$ is often refered to as *Weak Preference Relation*.

The three main assumptions that are typically adopted are,

1. *Completeness*: $\forall x_1, x_2 \in \mathcal{X}$, either $x_1 \succsim x_2$ or $x_2 \succsim x_1$ or both.

2. *Transitivity*: $\forall x_1, x_2, x_3 \in \mathcal{X}$, if $x_1 \succsim x_2$ and $x_2 \succsim x_3$, then $x_1 \succsim x_3$.

3. *Reflexive*: $\forall x \in \mathcal{X}$, $x \succsim x$.

However, for rationality, all we need are the first two assumptions. Formally,

Definition 1 *A Preference Relation, $\succsim$, is rational if it abides by Completeness and Transitivity.*

You might wonder why that is the case? Both these assumptions are not innocuous. Suppose you are not a fan of classical music, but are asked to rank between Johann Sebastian Bach, Giuseppe Tartini, and Felix Mendelssohn? Could you make a ranking off the cuff, without having ever heard any of their compositions? What *Completeness* thus imposes is that the individual has experienced the objects of desire, and knows how she feels about them, so that all her decisions are meditated choices. *Transitivity* is likewise a strong assumption as you may have easily seen. It basically prevents preferences from "cycling". To make it more concrete, suppose you are a classical music cognoscente, and you would rate the three composers in the following manner:

$$\begin{aligned} \text{Johann Sebastian Bach} &\succsim \text{Giuseppe Tartini} \\ \text{Giuseppe Tartini} &\succsim \text{Felix Mendelssohn} \end{aligned}$$

but when asked to rate between Johann Sebastian Bach, and Felix Mendelssohn, you vehemently insists that you think Felix Mendelssohn $\succsim$ Johann Sebastian Bach, which in turn would have meant that you should rank Felix Mendelssohn as preferred to Giuseppe Tartini.

It does seem immediately that these assumptions are fallible, and they may be so in two ways. Firstly, if you were to be offer choices that are *just perceptibly different*, an individual may not be able to discern between their difference, and yet when offered the extremes of this spectrum, she would reveal a *strict preference relation*. Secondly, the question has to do with the method we frame the question, or offer. Consider the same question on the above composers, but focused on for instance only the violin sonata's? Do you believe every classical music fan would maintain her preference ranking resolutely?

It should be noted that the fact that $\succsim$ is both *transitivite* and *complete* in turn has implications on $\succ$ and $\sim$.

Proposition 1 *If $\succsim$ reflects rationality, then,*

1. *$\succ$ is both irreflexive and transitive.*

2. *$\sim$ is reflexive, transitive and symmetric (if $x \sim y$ then $y \sim x$).*

Proof.

1. By *reflexivity* $\forall x \in \mathcal{X}$, $x \succsim x$. Suppose $\succ$ is *reflexive.* Then if $x \succ x$, we have $x \succsim x$ but not $x \precsim x$, which contradicts the first $\succsim$. Thus $\succ$ is *irreflexive.*

   Consider next, $\forall x, y, z \in \mathcal{X}$ such that $x \succ y$ and $y \succ z$. Since $x \succ y$, then $x \succsim y$ but not $x \precsim y$. Similarly $y \succ z$ implies $y \succsim z$ but not $y \precsim z$. Therefore, $x \succsim z$ but not $x \precsim z$, so that $x \succ z$, and $\succ$ is transitive.

2. For the second statement. It is clear that since if $x \sim x$ then $x \succsim x$ and $x \precsim x$, so that there is no contradiction in the implication, and $\sim$ is *reflexive*. Next $\forall x, y, z \in \mathcal{X}$, if $x \sim y$ and $y \sim z$, then each implies respectively that $x \succsim y$ and $x \precsim y$, and $y \succsim z$ and $y \precsim z$, so that $x \succsim z$ and $x \precsim z$, and we have $x \sim z$, and $\sim$ is *transitive*. The proof of *symmetry* is similar to that for *transitivity*.

■

1.2 Utility Functions

As you should be familiar by now, a way we quantify the realizations of an individuals preference relation, is through giving a systematic way of enumerating those choices. We are talking about the *utility function*. Formally,

Definition 2 *A function $u : \mathcal{X} \to \mathbb{R}$ is a utility function reflecting preference relation $\succsim$ if $\forall x, y \in \mathcal{X}$,*

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \tag{3}$$

Insofar as the utility function is an artifact we create, it is not unique. Indeed, any function that fulfils definition 2. Indeed, all we need is a strictly increasing function, and as a matter of fact, given any utility function, you can effect a transformation with a strictly increasing function, and yet maintian definition 2. For instance, let $f(.)$ be a strictly increasing function so that $\forall x_1, x_2 \in X$, if $x_1 \geq x_2$ then $f(x_1) \geq f(x_2)$. So that if $u(.)$ is reflective for the preference relation. Then, if $x \succsim y$, we have $u(x) \geq u(y) \Rightarrow f(u(x)) \geq f(u(y))$.

However, realize the following caveats. In and of itself, a preference relation $\succsim$ is just an ordering, or we say that it is *ordinal* in nature. In other words, the focus is on the relative ranking to the individual in question. So that our focus when we deal with utility functions is primarily on its ordinal element. *Cardinality* refers to for instance the distance between $u(x)$ and $u(y)$, which we are not concerned with. All this means that when we say that you can transform a utility function via a strictly increasing function, we are effectively highlihting that the essense we care most about is the *ordinal* nature, since cardinality changes with a transformation via an increasing function.

Further, since utility function maintains the ordering yielded by $\succsim$, the rationality assumption continues to hold, or impose itself on the utility function. Ask yourself if the utility function conforms with *transitivity* and *completeness*.

Proposition 2 *A utility function can represent a preference relation $\succsim$ only if the preference relation is rational.*

Proof. To perform the proof, we need to follow the statement in sequence, and show that if a utility function represents $\succsim$, then it must abide by completeness and transitivity, and in consequence, is rational.

1. Firstly, since $u(.)$ is a real valued function, $\forall x, y \in \mathcal{X} \subset \mathbb{R}^+$, we would have either $u(x) \geq u(y)$ or $u(x) \leq u(y)$. Since $u(.)$ represents $\succsim$, we would necessarily have either $x \succsim y$ or $x \precsim y$, so that $\succsim$ is complete.

2. Secondly, consider $x, y, z \in \mathcal{X} \subset \mathbb{R}^+$. Suppose we have $u(x) \geq u(y)$, and $u(y) \geq u(z)$ representing $x \succsim y$ and $y \succsim z$ respectively. Then the inequality sequence implies $u(x) \geq u(z)$, which implies that $x \succsim z$, and $\succsim$ is transitive.

Taken together, the representation by the utility function then implies that $\succsim$ is rational.
■

An important caveat to note here is that **not all rational preference relation can be represented by a utility function.** Can you think of an example? Another important point to note is that although the discussion thus far is couched in terms of a single good x, there is no reason that these ideas cannot be extended to a set of differing types of goods, in other words, a vector of goods $\mathbf{x}$.

1.3 Choice Rules

The alternative approach to examining decision making by individuals as noted earlier, is to treat the decisions made in the form of the choices as the starting point. But we need a more formal method to back track towards the individuals preferences so as to understand the entire process. To that end, we need a *choice structure* written as $(\mathcal{B}, C)$. From that we see that there are two elements to the structure.

1. $\mathcal{B}$ is an exhaustive family of nonempty subsets of all the possible choices an individual could make, $\mathcal{X}$. We can say this more succinctly as $B_i \in \mathcal{B}$ $i = \{1, 2, \ldots\}$, and $B \subset \mathcal{X}$. In more standard language, think of each element B_i as a possible budget set that you have learned in intermediate micro, so that $\mathcal{B}$ is a set of *budget sets.* So depending on the individuals budget, the set of objects/action changes. Further,

note that these subsets in $\mathcal{B}$ are not mututually exclusive. Although it is useful to relate this back to our standard budget constraint, these subsets are actually more general than that. These constraints can be institutional, physical, or social in nature. For example, even if you have the budget for a supercar, do you think it is an intelligent choice to have one in a city with congestions that can last for hours on end? Or even if you would love to have a pint, it may not be legal or socially acceptable within societies that frown upon its consumption.

2. $C(.)$ is a *choice rule*. This now refers to the decision aspect we observe. The choice rule determines the actual choice given the known budget, and this choice yields the observed action/object, and it is nonempty. In other words, $C(B) \subset B$. Think about to your intermediate Micro, when you were solving for the optimal choice, when the solution is the point of tangency between the budget line and the indifference curve, call that x^*. Well $C(B) = x^*$. However, in the general case here, $C(B)$ need not be a singleton.

Let have an example to make sure we understand this completely. Suppose $\mathcal{X} = \{a, b, c, d, e\}$, so that $\mathcal{B} = \{\{a, b\}, \{b, c, d\}, \{b, c, d, e\}\}$. A possible couplet in the choice structure could be $(\mathcal{B}, C_1(.))$. Without loss of generality, we could have $C_1(\{a, b\}) = b$, and $C_1(\{b, c, d\}) = b$, so that she would always choose b. However, at $C_1(\{b, c, d, e\}) = e$.

Like in much of what we have done in economics, you might realize that we need an assumption to link the observe choices, to the choice structure.

Definition 3 *A choice structure* $(\mathcal{B}, C(.))$ *satisfies the Weak Axiom of Revealed Preference if the following holds:*

> *If for* $B \in \mathcal{B}$ *where* $a, b \in B$ *an individual has* $a \in C(B)$*, then for* $B' \in \mathcal{B}$ *with* $a, b \in B'$*, we see* $b \in C(B')$*, then we must also have* $a \in C(B')$*.*

This is essentially saying that for two element that appears in two constraint sets, if we observe an element chosen in one, and another element chosen in the second, we can nonetheless still infer that the first element must have been still available, although it might not have been chosen. Look back now at the last example, and ask yourself if the *Weak Axiom of Revealed Preference* is met.

We can make a similar statement on the choice structure, but using the idea of preference relations. Since it is applied in an alternative fashion, we will denote it, for differentiation purposes, as $\succsim^*$.

Definition 4 *For a Choice Structure $(\mathcal{B}, C(.))$, the revealed preference relation $\succsim^*$ is,*

$$a \succsim^* b \Leftrightarrow \text{for } B \in \mathcal{B} \text{ such that } a, b \in B, \text{ we have } a \in C(B).$$

This is intuitively saying that for any set of objects/actions, when we observe an element chosen, when other alternative are availble/feasible, i.e. is in the same constraint set, then the decision maker has reveal her preference for the chosen object/action.

1.4 The Relationship

What you might be thinking now is that all these does make a lot of sense, and might even venture that the relationship between the concept of *preference relation* and *choice rules* are obvious. But is it really. It might be helpful to write out the implications of the connection you are trying to draw here.

1. Given that we have endowed preference relations with rationality, do you think it is necessarily the case that when faced with a constraint set, a decision maker's choice would necessarily generate a choice structure that abides with the weak axiom of revealed preference?

2. On the other hand, is it necessarily the case that a decision maker whose behavior is indeed captured by the choice structure, and reflects the weak axiom of revealed preference, would in turn allow us to see her preferences as being consistent with both the attendant conditions of rationality?

As you should have realized by now, you need to be precise with what you are saying, and be able to proof what you are saying is true. Fortunately for you, the answer to both of the above as in the positive. Let's be formal about this.

Proposition 3 *The Choice Structure generated by $\succsim$, $(\mathcal{B}, C^*(, \succsim))$ will satisty the weak axiom of revealed preference.*

Proof. First let $B \in \mathcal{B}$, and without loss of generality, let $a, b \in B$, and let $a \in C^*(B, \succsim)$. This thus means that $a \succsim b$. Next consider another constraint set where both a and b are elements of, $a, b \in B'$, but $b \in C^*(B', \succsim)$. So $\forall d \in B'$, we have revealed to us $b \succsim d$. By transitivity, $a \succsim d$, $\Rightarrow a \in C^*(B', \succsim)$, and the $(\mathcal{B}, C^*(.))$ satisfies the weak axiom of revealed preference. ■

Now for the other question. Before that, we need another definition before beginning.

Definition 5 *Given $(\mathcal{B}, C(.))$, $\succsim$ rationalizes $C(.)$ in relation to $\mathcal{B}$ if,*

$$C(B) \;=\; C^*(B, \succsim)$$

$\forall B \in \mathcal{B}$.

What the definition is saying is that $\succsim$ rationalizes the choice rule $C(.)$ on $\mathcal{B}$ if the optimal choices generated by $\succsim$ coincides with those of $C(.)$ $\forall B \in \mathcal{B}$. You might also wish to ask if there are other $\succsim$ that might generate the same choice structure? What do you think?

Proposition 4 *For $(\mathcal{B}, C(.))$ such that,*

1. *the weak axiom of revealed preference is satisfied,*

2. *$\mathcal{B}$ includes all subsets of X of up to three elements*

then there is a rational $\succsim$ that rationalizes $C(.)$ relative to $\mathcal{B}$ in the sense of $C(B) = C^*(B, \succsim)$ $\forall B \in \mathcal{B}$, and this $\succsim$ is the only preference relation that can do so.

The proof of the proposition will not be discussed due to its length. It should nonetheless be noted that this preference is a very specialized and strong one, since it requires the constraint set to be exhaustive. In this case, consideration of decisions from either direction is completely equivalent. The primary genesis of the caveat is that this theory works when we are talking about specific types of constraints, such as a budget constraint. When the constraint has a broader definition, the weak axiom of revealed preference might not be sufficient in those cases, since the possibilities would not be included in $\mathcal{B}$.

2 Utility Maximization

2.1 Assumptions & the Utility Function

As mentioned, although our discussions thus far have considered the objects of desire as scalar in nature, it is easily generalized to multiple objects, in which case, we can than describe a vector of objects of desire,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix}$$

as opposed to x.

Before we examine the decision process of the individual as used typically in microeconomics, as you would well remember, we need to define the space of all possible consumption goods, and then the constraint on the set. You can conceive of this set $X \subset \mathbb{R}^K$. However, since we are talking about consumption, we will focus as usual on the non-negative choices, and define the consumption set as:

$$X = \mathbb{R}_K^+ + \{0\} = \{\mathbf{x} \in \mathbb{R}_K^+ : x_k \geq 0, k = 1, \ldots, K\}$$

In the discussion, some of the following features will necessarily be adopted as assumptions as they become apparent.

1. *Weak Monotonicity*: If $\mathbf{x} \geq \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y}$.

2. *Strong Monotonicity*: If $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y} \Rightarrow \mathbf{x} \succ \mathbf{y}$.

3. *Local Nonsatiation*: For any $\mathbf{x} \in X \subset \mathbb{R}_K^+$, and $\epsilon > 0$, there will be some vector $\mathbf{y} \in X$ such that $|\mathbf{x} - \mathbf{y}| < \epsilon$ (Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$) such that $\mathbf{y} \succ \mathbf{x}$.

4. *Convexity*: For $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in X$ where $\mathbf{x} \succsim \mathbf{z}$ and $\mathbf{y} \succsim \mathbf{z}$, $\Rightarrow \alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \succsim \mathbf{z}$ $\forall$ $\alpha \in [0, 1]$.

5. *Strict Convexity*: For $\mathbf{x} \neq \mathbf{y}$, and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in X$, if $\mathbf{x} \succsim \mathbf{z}$ and $\mathbf{y} \succsim \mathbf{z}$, $\Rightarrow \alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \succ \mathbf{z}$ $\forall$ $\alpha \in [0, 1]$.

6. *Continuity*: $\forall \mathbf{x}, \mathbf{y} \in X$, the sets $\{\mathbf{x} : \mathbf{x} \succsim \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \precsim \mathbf{y}\}$ are closed sets $\Rightarrow$ $\{\mathbf{x} : \mathbf{x} \succ \mathbf{y}\}$ and $\{\mathbf{x} : \mathbf{x} \prec \mathbf{y}\}$ are open sets.

These assumptions should be clear in themselves. The first essentially says that "at least as much of everything is at least as good", while the stricter relation says that "at least as much of everything, and strictly more of some, is strictly better". On the aggregate, it says the objects of desire are just that, desirable! The third assumption would be familiar to some of you. It basically rules out "thick" indifference curves, so that an individual will always be able to find something they $\succ$. The last two assumptions basically ensures that if two sets of objects are in a set, a convex combination will still be in the set. To see how these assumptions could be utilized, consider the following theorem:

Theorem 5 ***Existence of a utility function:*** *Suppose preferences are complete, reflexive, transitive, continuous, and strongly monotonic. Then there exists a continuous utility function* $u : \mathbb{R}_K^+ \to \mathbb{R}$ *which represents those preferences.*

Proof. Let $\mathbf{e} = [1, \ldots, 1]' \in \mathbb{R}_K^+$. Without loss of generality, let $\mathbf{x} \in X$. We need to show that there is number $u(\mathbf{x})$ such that $\mathbf{x} \sim u(\mathbf{x})\mathbf{e}$.

Let $G = \{t \in \mathbb{R} : t\mathbf{e} \succsim \mathbf{x}\}$ and $L = \{t \in \mathbb{R} : \mathbf{x} \succsim t\mathbf{e}\}$. By strong monotonicity, G is nonempty. Likewise, since $t \in \mathbb{R}$ so that zero is in the set, L is nonempty. By continuity, both G and L are both closed sets. Since the real line $\mathbb{R}$ is continuous and connected, there must exist a number t_x such that $t_x\mathbf{e} \sim \mathbf{x}$.

Let

$$\begin{aligned} u(\mathbf{x}) &= t_x \qquad \text{where } t_x\mathbf{e} \sim \mathbf{x} \\ u(\mathbf{y}) &= t_y \qquad \text{where } t_y\mathbf{e} \sim \mathbf{y} \end{aligned}$$

Then without loss of generality, let $t_x > t_y$, so that by strong monotonicity $t_x\mathbf{e} \succ t_y\mathbf{e}$. In turn, transitivity implies that,

$$\mathbf{x} \sim t_x\mathbf{e} \ \succ \ t_y\mathbf{e} \sim \mathbf{y}$$

The same reason runs in the reverse for $\mathbf{x} \prec \mathbf{y}$. The remainder of the proof is to show that $u(.)$ is continuous. ■

2.2 Consumer Behavior

With the method of modelling individual preferences determined, we can now try to understand how preferences are optimized. We know the with preferences alone, an individual may consume till she is completely surfeit of want, and desire. However, that is highly unlikely, since we are constrained numerous constraints, including budgetary, social, and legal ones. We will as usual focus only on the budgetary constraint, but always keeping in mind that the ideas needs to be further generalized.

Let the wealth/income that an individual faces be denoted as m, a scalar number. Let there be K desirable goods/actions, with quantities $\mathbf{x} = [x_1, \ldots, x_k]'$, and their attendant prices $\mathbf{p} = [p_1, \ldots, p_K]'$. Then we can denote the *Walrasian* budget set as,

$$B = \{\mathbf{x} \in X \subset \mathbb{R}^K_+ : \mathbf{p}'\mathbf{x} \leq m\}$$

We can then frame the question of preference optimization in a manner you should be familiar now with,

$$\begin{aligned} & \max u(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{p}'\mathbf{x} \leq m \\ & \mathbf{x} \in X \end{aligned}$$

One of the first issues you might be wondering, at least at the abstract level, whether a solution exists to this problem. For such constrained problems, we generally need the objective function to be continuous, and that the constraint itself to be closed and bounded. Since, by assumption, the objective is continous, we are half way there. It turns out that if $p_i > 0$, $i = 1, \ldots, K$, and $m \geq 0$, the constraint would be closed and bounded. The issue arises only when some $p_j = 0$, $j = 1, \ldots, K$. In that case, if the object is very desirable, we might have the case where the individual might desire an infinite number of the product. So to keep our discussion within the realm of the normal, and reasonable, we will ignore those cases, so that our budget set would be closed and bounded, and a solution would exist.

You might also observe that in the very general setup, the optimal choice is actually independent of the choice of utility function. Since my optimizing an individual's own preference, all we need is for the optimizing choice to be such that $\mathbf{x}^* \succsim \mathbf{x}$ for all $\mathbf{x} \in X$ and $\mathbf{p}'\mathbf{x} \leq m$. This means then that any utility function that achieves this, can be used.

Finally, observe that the budget set is homogeneous of degree one, in the sense that if prices and income/wealth increases by the same factor, the budget set will be left unchanged.

Some of you may wonder about the necessity of keeping the budget constraint as an inequality, and you would be right to question. Intuitively, why would the individual consume within the budget set, if her objective is to optimize her preference. However, in this setup, we need to make use of the assumptions to achieve that result. How then would we show the the budget constraint should be an equality. To show this, let us suppose that the optimizing choice is actually inside the budget set, so that $\mathbf{p}'\mathbf{x}^* < m$. But since it is strictly less than the wealth/income m, there must be other bundles of $\mathbf{x} \in X$ are are feasible, and less than m. At the same time, by local nonsatiation, there must exist another combination $\mathbf{x}^{**}$ such that $\mathbf{x}^{**} \succ \mathbf{x}^*$, and yet $\mathbf{p}'\mathbf{x}^{**} \leq m$, which thus means that $\mathbf{x}^*$ could not have been optimizing the preferences, subject to the budget constraint, which means then in the final analysis that $\mathbf{p}'\mathbf{x} = m$. Thus the previous constrained problem can be written as,

$$\begin{aligned} v(\mathbf{p}, m) &= \max u(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{p}'\mathbf{x} = m \end{aligned}$$

Where $v(\mathbf{p}, m)$ is the utility at the optimal choice of $\mathbf{x}(\mathbf{p}, m) \equiv \mathbf{x}^*$. We call this utility $v(\mathbf{p}, m)$ the *Indirect Utility Function*. Observe that the function $\mathbf{x}(\mathbf{p}, m)$ is in effect a demand function, but more accurately, is a *Demanded Bundle Function* that tells us the optimal choice of the bundle $\mathbf{x}(\mathbf{p}, m)$ for each vector of prices $\mathbf{p}$, and income m. It is interesting to note that if the price vector and income changes by the same factor, then demand should remain unchanged, so that $\mathbf{x}(\mathbf{p}, m) = \mathbf{x}(\alpha\mathbf{p}, \alpha m)$, where $\alpha = \mathbb{R}$, which in turn implies that the demanded bundle function is homogeneous of degree zero.

So the next big question is to ask, how can we solve the constrained problem? Fortunately, this is where your expertise in calculus come to bear once again. The problem can be written as a Lagrangian,

$$\mathcal{L} = u(\mathbf{x}) - \lambda(\mathbf{p}'\mathbf{x} - m)$$

This would give us the following set of first order conditions.

$$\frac{\partial u(\mathbf{x})}{\partial x_i} - \lambda p_i = 0 \text{ for } i = 1, \ldots, K$$

which in turn can be reexpressed as,

$$\frac{\frac{\partial u(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial u(\mathbf{x}^*)}{\partial x_j}} = \frac{p_i}{p_j} \quad \text{for } i, j = 1, \ldots, K$$

which is the equilibrium condition you are most familiar with. As a reminder, the left hand side is known as your *Marginal Rate of Substitution*, while your right hand side is the *Economic Rate of Substitution* (since it reflects the relative prices). And in your Intermediate classes, this merely highlights that at equilibrium occurs at the point of tangency between the indifference curve and the budget hyperplane.

This ideas could be generalized even more succinctly using matrix algrebra. Let $\mathbf{dx}$ be a perturbation to $\mathbf{x}^*$, the optimal choice. Then in order to examine optimality, we can perturb the optimal choice in the budget set, and represent it as,

$$\mathbf{p}(\mathbf{x}^* \pm \mathbf{dx}) = 0$$

But we know at equilibrium, $\mathbf{px} = m$, which in turn means that $\mathbf{pdx} = 0$, which implies that the $\mathbf{dx}$ vector must be orthogonal to the price vector $\mathbf{p}$, or more precisely, the *normal vector* (equivalent to the gradient of a line). Or frame in terms of what you understand in basic calculus, this gives the first order conditions that you are familiar with.

We also need to similarly examine the change on the indifference curve/utility function. Here denote the perturbation to the utility function evaluated at the optimal choice $\mathbf{D}u(\mathbf{x}^*)$. Then since any perturbation cannot effect any change to the utility, we have,

$$\mathbf{D}u(\mathbf{x}^*)\mathbf{dx} = 0$$

where

$$\mathbf{D}u(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial u(\mathbf{x}^*)}{\partial x_1} \\ \frac{\partial u(\mathbf{x}^*)}{\partial x_2} \\ \vdots \\ \frac{\partial u(\mathbf{x}^*)}{\partial x_K} \end{bmatrix}$$

is the *gradient vector* or simply as the *Jacobian*. Since both the above, and the orthogonality condition must hold, this means that $\mathbf{D}u(\mathbf{x}^*)$ must be proportional to $\mathbf{p}$, which is in effect, the equilibrium condition.

Nonetheless, the above exposition is incomplete, since we have yet to examine if the choices have indeed maximized the individual's wellbeing.This is achieved through examining the second order condition, which can be written as,

$$\mathbf{h}'\mathbf{D}^2u(\mathbf{x}^*)\mathbf{h} \leq 0$$

where for all $\mathbf{h}$, we have $\mathbf{ph} = 0$, and where

$$\mathbf{D}^2 u(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial^2 \mathbf{x}^*}{\partial x_1^2} & \frac{\partial^2 \mathbf{x}^*}{\partial x_1 x_2} & \cdots & \frac{\partial^2 \mathbf{x}^*}{\partial x_1 x_K} \\ \frac{\partial^2 \mathbf{x}^*}{\partial x_2 x_1} & \frac{\partial^2 \mathbf{x}^*}{\partial x_2^2} & \cdots & \frac{\partial^2 \mathbf{x}^*}{\partial x_2 x_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathbf{x}^*}{\partial x_K x_1} & \frac{\partial^2 \mathbf{x}^*}{\partial x_K x_2} & \cdots & \frac{\partial^2 \mathbf{x}^*}{\partial x_K^2} \end{bmatrix}$$

and is known as a *Hessian Matrix.* What we need to verify is that the Hessian be negative semidefinite, or simply to be *locally quasiconcave.*

Definition 6 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is **quasiconcave** if the upper contour sets of the function are convex sets. Formally, sets of the form $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x} \geq a)\}$ are convex $\forall a$*

In turns,

Definition 7 *A function $f(\mathbf{x})$ is **quasiconvex** if $-f(\mathbf{x})$ is quasiconcave.*

This essentially means that the *upper contour set* be unanimously above the budget hyperplane. Technically, all this means is that you need the leading principle minors of the hessian matrix to alternate in sign. Compared to what you have learned in prior microeconomic classes, all this means is this is equivalent to verifying that the second order derivative be non-positive.

2.3 Indirect Utility

As noted prior, the indirect utility function $v(\mathbf{p}, m)$ gives the value of the utility evaluated at $\mathbf{x}^*(\mathbf{p}, m)$. It has some interesting properties we can examine:

1. $v(\mathbf{p}, m)$ is nonincreasing in $\mathbf{p} \Rightarrow$ for $\mathbf{p} \geq \mathbf{p}'$, $v(\mathbf{p}, m) \leq v(\mathbf{p}', m)$. As well, $v(\mathbf{p}, m)$ is nondecreasing in m.

2. $v(\mathbf{p}, m)$ is homogeneous of degree 0 in $(\mathbf{p}, m)$.

3. $v(\mathbf{p}, m)$ is quasiconvex in $\mathbf{p} \Rightarrow \{\mathbf{p} : v(\mathbf{p}, m) \leq k\}$ is a convex set for all k.

4. $v(\mathbf{p}, m)$ is continuous at all $\mathbf{p} \gg 0$, $m > 0$.

Proof.

1. Let $B = \{\mathbf{x} : \mathbf{px} \leq m\}$ and $B' = \{\mathbf{x} : \mathbf{p}'\mathbf{x} \leq m\}$ for $\mathbf{p}' \geq \mathbf{p}$. Then $B' \subset B$. Which means that the contour set over B is at least as large as that over B' and the property follows. The same arguement holds for m.

2. The arguement here is as follows: $v(\alpha\mathbf{p}, \alpha m) = v(\mathbf{p}, m) = \alpha^0 v(\mathbf{p}, m)$, and the property follows.

3. Let $\mathbf{p}$ and $\mathbf{p}'$ be such that $v(\mathbf{p}) \leq k$ and $v(\mathbf{p}') \leq k\ \forall k$. Define $\mathbf{p}'' = \alpha\mathbf{p} + (1-\alpha)\mathbf{p}'$. Define the respective three budget sets as,

$$\begin{aligned} B &= \{\mathbf{x} : \mathbf{px} \leq m\} \\ B' &= \{\mathbf{x} : \mathbf{p}'\mathbf{x} \leq m\} \\ B'' &= \{\mathbf{x} : \mathbf{p}''\mathbf{x} \leq m\} \end{aligned}$$

   Given the definition of quasiconcavity, and quasiconvexity, we need to show that $-v(\mathbf{p}, m)$ is quasiconcave, which implies that $-v(\mathbf{p}, m) \geq k$ is a convex set. Stated another way, we thus need to show that $v(\mathbf{p}, m)$ is quasiconcave, and that $v(\mathbf{p}, m) \leq k$ is a convex set. This can be achieved by showing that the budget set generated by a convex combination of $\mathbf{p}$ and $\mathbf{p}'$ would still be in the union of their respective budget sets. Specifically, we need to show that $B'' \subset B \cup B'$. Suppose that is not true, so that we have $\alpha\mathbf{px} + (1-\alpha)\mathbf{p}'\mathbf{x} \leq m$, but $\mathbf{px} > m$ and $\mathbf{p}'\mathbf{x} > m$. But this would imply that $\alpha\mathbf{pxx} > \alpha m$ and $(1-\alpha)\mathbf{p}'\mathbf{x} > (1-\alpha)m$, which in turn implies that $\alpha\mathbf{px} + (1-\alpha)\mathbf{p}'\mathbf{x} > m$, and we have a contradiction.

   This thus means that,

$$\begin{aligned} v(\mathbf{p}'', m) &= \max u(\mathbf{x}) \text{ ,such that } \mathbf{x} \in B'' \\ &\leq \max u(\mathbf{x}) \text{ ,such that } \mathbf{x} \in B \cup B' \\ &\leq k \text{ ,since } v(\mathbf{p}, m) \leq k \ \&\ v(\mathbf{p}', m) \leq k \end{aligned}$$

and the property follows. ■

You may well ask what is the significance of the indirect utility, and how does it differ from the usual indifference curve generated? Note firstly that should we depict this diagrammatically, the indifference curve should be drawn with the prices on the vertices. Secondly, it should still look like a rectangular hyperbola. Thirdly, the set above the indifference curve is now known as the *lower contour set*, since the*price indifference* curve is nonincreasing in the price vector, so that utility is increasing as we move towards the origin.

Another important insight is that we can now depict the indirect utility also as the relationship between the utility and income. And since the indirect utility is strictly increasing in m, for a given utility, we can always solve for the "optimal" income that would give us that level of utility in terms of the price vector $\mathbf{p}$. The function that helps us draw the link between income and utility is the *expenditure function*, $e(\mathbf{p}, u)$.

This then suggest an alternative view to the utility maximization problem, written as a expenditure minimization problem instead:

$$\begin{aligned} e(\mathbf{p}, u) &= \min \mathbf{px} \\ \text{subject to } u(\mathbf{x}) &\geq u \end{aligned}$$

This thus instead gives the minimum cost of attaining the minimal level of desired utility/felicity.

The properties of the expenditure function are:

1. $e(\mathbf{p}, u)$ is nondecreasing in $\mathbf{p}$.

2. $e(\mathbf{p}, u)$ is homogeneous of degree 1 in $\mathbf{p}$.

3. $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$.

4. $e(\mathbf{p}, u)$ is continuous in $\mathbf{p}$, for $p \gg 0$.

5. If $h(\mathbf{p}, u)$ is the expenditure minimizing vector of goods necessary to achieve u at prices $\mathbf{p}$, $\Rightarrow h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p},u)}{\partial p_i}$ for $i = 1, \ldots, K$, for $p_i > 0$.

Proof.

1. Let $\mathbf{x}$ and $\mathbf{x}'$ be expenditure minimizing vector of goods associated with $\mathbf{p}$ and $\mathbf{p}'$ respectively. Without loss of generality, let $\mathbf{p}' \geq \mathbf{p}$. Then $\mathbf{px} \leq \mathbf{px}'$ by expenditure minimization, and $\mathbf{px}' \leq \mathbf{p}'\mathbf{x}'$, so that $\mathbf{px} \leq \mathbf{p}'\mathbf{x}'$, and $e(\mathbf{p}, u) \leq e(\mathbf{p}', u)$.

2. What the second property is saying can be summarized as,

$$e(\alpha\mathbf{p}, u) = \alpha e(\mathbf{p}, u)$$

which is saying that if prices are changed by the same factor α, the optimal choice remains the same. Put another way, if $\mathbf{x}$ optimizes at $\mathbf{p}$, it would optimize at $\alpha\mathbf{p}$.

Suppose not, so that at $\alpha\mathbf{p}$, the optimizing vector is $\mathbf{x}'$, so that $\alpha\mathbf{px} > \alpha\mathbf{px}'$, $\Rightarrow$ that $\mathbf{px} > \mathbf{px}'$. But that would mean that $\mathbf{x}$ was not expenditure minimizing in the first place, so that $\mathbf{x}$ must be expenditure minimizing in both prices. This thus means that changing prices by α with no impact on relative prices simply changes expenditure by the factor α. In other words,

$$e(\alpha\mathbf{p}, u) = \alpha\mathbf{px} = \alpha e(\mathbf{p}, u)$$

and the property follows.

3. To show that $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$, we need to show that

$$e(\mathbf{p}'', u) \geq \alpha e(\mathbf{p}', u) + (1-\alpha)e(\mathbf{p}, u)$$

where $\mathbf{p}'' = \alpha\mathbf{p}' + (1-\alpha)\mathbf{p}$, for $\alpha \in [0, 1]$. In other words, the usual definition for a concave function. Denote the optimal choice under $\mathbf{p}$ and $\mathbf{p}'$ be $\mathbf{x}$ and $\mathbf{x}'$ respectively. This means that the optimizing choice under $\mathbf{p}''$ would be $\mathbf{x}''$, so that

$$e(\mathbf{p}'', u) = \mathbf{p}''\mathbf{x}'' = \alpha\mathbf{p}'\mathbf{x}'' + (1-\alpha)\mathbf{px}''$$

But we know that

$$\begin{aligned} e(\mathbf{p}', u) = \mathbf{p}'\mathbf{x}' &\leq \mathbf{p}'\mathbf{x}'' \\ e(\mathbf{p}, u) = \mathbf{px} &\leq \mathbf{px}'' \\ \Rightarrow e(\mathbf{p}'', u) &= \alpha\mathbf{p}'\mathbf{x}'' + (1-\alpha)\mathbf{px}'' \\ &\geq \alpha\mathbf{p}'\mathbf{x}' + (1-\alpha)\mathbf{px} \\ &= \alpha e(\mathbf{p}', u) + (1-\alpha)e(\mathbf{p}, u) \end{aligned}$$

and the property follows.

4. We will relegate the discussion regarding continuity elsewhere.

5. To show that we can obtain the Hicksian demand from examining the expenditure function, let's define the expenditure minimizing vector be $\mathbf{h}^*$ that provides the desired level of utility u given price vector $\mathbf{p}^*$. Next define a function in terms of price vector $\mathbf{p}$,

$$f(\mathbf{p}) = e(\mathbf{p}, u) - \mathbf{ph}^*$$

In essence, we are asking with the creation of such a function what the optimal price vector ought to be to minimize the difference between the expenditure function $e(.,.)$, and the amount of income of expended. Observe that since $e(\mathbf{p}, u)$ is the minimized expenditure, $f(\mathbf{p})$ is nonpositive. Further, at $\mathbf{p}^*$, $f(\mathbf{p}^*) = 0$, which is the maximum value attainable, and since we know $e(\mathbf{p}, u)$ is concave in $\mathbf{p}$ as shown above, we have,

$$\frac{\partial f(\mathbf{p}^*)}{\partial p_i} = \frac{\partial e(\mathbf{p}^*, u)}{\partial p_i} - h_i^* = 0$$

$\forall\ i = 1, \ldots, K$, and the property follows.

■

As might have been highlighted to you, the solution you obtained in maximizing the utility of the agent, is what is commonly known as the *Walrasian/Marshallian Demand*. On the other hand, in minimizing the expenditure on goods, we obtain in turn what is known as the *Hicksian Demand*. It should be clear that since both solutions are essentially solving the same problem, the should be the same, and they in did are. The question is how do you then obtain one from the other?

First, before we get ahead of ourselves, let us understand what a *Hicksian Demand* is. It is typically denoted as $h(\mathbf{p}, u)$, and it reveals the optimal vectors that achieves the target level of utility. Observe the difference to the *Walrasian/Marshallian Demand*, $x(\mathbf{p}, m)$, where the target is dependent in stead on income, m. In deed, the Hicksian demand is typically also known as the *Compensated Demand* since it is "constructed" by varying prices and income so that utility is maintained. Think of the income as being adjusted for price changes, so that utility remains constant. It is indeed an interesting construct, but it must be noted that there is a good reason why we focus on the Marshallian demand, as opposed to the Hicksian. It's to do with the former being observable, at least its equilbrium realizations, while the Hicksian is not.

2.4 Important Identities

We will see how all the above functions are related to each other now. Rewriting the two types of problems,

$$\begin{aligned} v(\mathbf{p}, m^*) &= \max u(\mathbf{x}) \\ \text{subject to } \mathbf{p}'\mathbf{x} &\leq m \end{aligned}$$

and

$$
\begin{aligned}
e(\mathbf{p}, u^*) &= \min \mathbf{p}'\mathbf{x} \\
\text{subject to } u(\mathbf{x}) &\geq u^*
\end{aligned}
$$

Since as we observed, the solution in both this problems should yield the same solution, we have,

1. $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m \Rightarrow$ the minimum expenditure to achieve the the optimal utility $v(\mathbf{p}, m)$ is m.

2. $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u \Rightarrow$ the maximum utility achieved from income $e(\mathbf{p}, u)$ is u.

3. $x_i(\mathbf{p}, m) \equiv h_i(\mathbf{p}, v(\mathbf{p}, m)) \Rightarrow$ The Marshallian demand at m must be the same as the Hicksian demand at $v(\mathbf{p}, m)$.

4. $h_i(\mathbf{p}, u) \equiv x_i(\mathbf{p}, m) \Rightarrow$ The Hicksiam demand at u is the same as the Marshallian demand at $e(\mathbf{p}, u)$.

This gives rise to an interesting and useful identity, *Roy's Identity.*

Proposition 6 ***Roy's Identity:*** *Let* $\mathbf{x}(\mathbf{p}, m)$ *be the Marshallian demand function. Then,*

$$
x_i(\mathbf{p}, m) = -\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_i}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}}
$$

for $i = 1, \ldots, k$. *This is true as long as* $u(.)$ *is continuous, increasing and concave/quasiconcave in* $\mathbf{x}$*, and* $p_i > 0$*, and* $m > 0$*.*

Proof. From identity 3, we have,

$$
\mathbf{x}(\mathbf{p}^*, m^*) = \mathbf{h}(\mathbf{p}^*, u^*)
$$

so that given income $e(\mathbf{p}, u^*)$, the highest utility achievable is u^*, that is,

$$
v(\mathbf{p}^*, e(\mathbf{p}^*, u^*)) = u^*
$$

Differentiating the entire equation with respect to p_i for $i = 1, \ldots, k$,

$$
0 = \frac{\partial e(\mathbf{p}^*, m^*)}{\partial p_i} + \frac{\partial v(\mathbf{p}^*, m^*)}{\partial m} \frac{\partial e(\mathbf{p}^*, u^*)}{\partial p_i}
$$

So that combining identity 3, we have,

$$
x_i(\mathbf{p}^*, m^*) \equiv h_i(\mathbf{p}^*, u^*) \equiv \frac{\partial e(\mathbf{p}^*, u^*)}{\partial p_i} \equiv -\frac{\frac{\partial v(\mathbf{p}^*, m^*)}{\partial p_i}}{\frac{\partial v(\mathbf{p}^*, m^*)}{\partial m}}
$$

$\forall\ i = 1, \ldots, k$, and the result follows. ■

3 Duality & Welfare Evaluation

3.1 Slutsky Equation

We are now ready to perform some comparative statics, which as in the intermediate courses, we are going to examine the effects that changes in the parameters of prices and income has on the agent's demand choices. In the process, we will more directly discern the relationship between both the *Walrasian* and *Hicksian* demands. More precisely, we will see this through the *Slutsky equation*, which shows that although the *Hicksian* demand is not observable, its derivative, however can be found using the derivatives of the *Walrasian* demand.

Theorem 7 *The Slutsky equation is,*

$$\frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} = \frac{\partial h_j(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_i} - \frac{\partial x_j(\mathbf{p}, m)}{\partial m} x_i(\mathbf{p}, m)$$

Proof. Let $\mathbf{x}^* \equiv \mathbf{x}^*(\mathbf{p}^*, m^*)$, and $u^* = u(\mathbf{x}^*)$. And we already know that,

$$\mathbf{h}(\mathbf{p}, u^*) \equiv \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u^*))$$

Thus differentiating this identity with respect to p_i, and evaluating the derivative at $\mathbf{p}^*$, we have,

$$\frac{\partial h_j(\mathbf{p}^*, u^*)}{\partial p_i} = \frac{\partial x_j(\mathbf{p}^*, m^*)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}^*, m^*)}{\partial m} \frac{\partial e(\mathbf{p}^*, u^*)}{\partial p_i} \tag{4}$$

The left hand side says we are examining how a price change would affect the Hicksian demand. The right hand side in turn tells us that the former change is composed of two elements. The first element is that of the change in the Walrasian demand holding income constant at m^*. The second tells us the change in demand when income changes, multiplied by a factor of the change in income to maintain utility at a fixed level of u^*. In addition, we already know from prior that this last term in the last element is just $x_i^*(\mathbf{p}^*, m^*)$, so that we have,

$$\frac{\partial h_j(\mathbf{p}^*, u^*)}{\partial p_i} = \frac{\partial x_j(\mathbf{p}^*, m^*)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}^*, m^*)}{\partial m} x_i^*(\mathbf{p}^*, m^*)$$

and the equation follows. ■ Writing the Slutsky equation, you would notice that this is essentially the decomposition of price change into the substitution, and income effects.

The Slutsky equation can be written in matrix form, and in turn, we may observe certain features.

$$\begin{aligned} \Delta_p \mathbf{x}(\mathbf{p}, m) &= \Delta_p \mathbf{h}(\mathbf{p}, u) - \Delta_m \mathbf{x}(\mathbf{p}, m)\mathbf{x} \end{aligned}$$

$$\begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial p_1} & \cdots & \frac{\partial x_k}{\partial p_k} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_k} \end{bmatrix} - \begin{bmatrix} \frac{\partial x_1}{\partial m} \\ \vdots \\ \frac{\partial x_k}{\partial m} \end{bmatrix} \begin{bmatrix} x_1 & \ldots & x_k \end{bmatrix}$$

With this technicalities out of the way, we can discuss some interesting properties.

1. The matrix of substitution effects, $\frac{\partial h_j(\mathbf{p},u)}{\partial p_i}$ is negative semidefinite since,

$$\begin{aligned} \frac{\partial h_j(\mathbf{p}, u)}{\partial p_i} &= \frac{\partial \frac{\partial e(\mathbf{p},u)}{\partial p_j}}{\partial p_i} \\ &= \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i \partial p_j} \end{aligned}$$

   and the concavity of the expenditure function.

2. The matrix of substitution effects is symmetric since,

$$\frac{\partial h_j(\mathbf{p}, u)}{\partial p_i} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i \partial p_j} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_j \partial p_i} = \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j}$$

3. The compensated own price effect is always nonpositive,

$$\frac{\partial h_i(\mathbf{p}, u)}{\partial p_i} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i^2} \leq 0$$

   which follows from the negative semidefinite substitution matrix, since a negative semidefinite matrix will always have nonpositive diagonal elements.

4. Because of point 1 to 3, the substitution matrix of equation 4 is symmetric and negative semidefinite.

3.2 Duality of Consumption

We will examine now how we can obtain the direct utility from the indirect utility function. Following the common practice, we will normalize income to 1. First note that the normalized indirect utility problem can be written as,

$$\begin{aligned} v(\mathbf{p}) &= \max_{\mathbf{x}} u(\mathbf{x}) \\ \text{subject to } \mathbf{px} &= 1 \end{aligned}$$

Then the duality basically says that we can always obtain the direct utility to solving,

$$\begin{aligned} u(\mathbf{x}) &= \min_{\mathbf{p}} v(\mathbf{p}) \\ \text{subject to } \mathbf{px} &= 1 \end{aligned}$$

You would be right to wonder how did the maximization problem become a minimization? To see the intuition, let $\mathbf{x}$ be the optimal vector at $\mathbf{p}$, so that $v(\mathbf{p}) = u(\mathbf{x})$. However, let $\mathbf{p}'$ be another price vector that satisfies the budget constraint in the sense that $\mathbf{p}'\mathbf{x} = 1$. This would simply be another budget line that passes through $\mathbf{x}$. Since we know that although it is still affordable, it would not be chosen, because the relative prices has changed, necessarily, the utility obtained at $\mathbf{p}'$ must be at least as much as at $u(\mathbf{x})$. This thus translates to $v(\mathbf{p}') \geq u(\mathbf{x}) = v(\mathbf{p})$. This in turn says that the minimum indirect utility over all possible $\mathbf{p}$ that satisfies the budget constraint would give us $u(\mathbf{x})$!

3.3 Revealed Preference

To recapitulate what we have done thus far, which is essentially taking preference as the primitive concept, and in the process deriving the restrictions that utility maximization imposes on the observed demand functions, as represented within the Slutsky equation in the form of the substitution effect being necessarily symmetric, and negative semidefinite. It should be noted that although, conceptually, the demand is observable, it is not the same as observing the actuall demand function, since what we observe are actually equilibrium choices given the market equilibrium, and we know equilibrium price changes may be generated by other concerns, which causes shifts in demand. The question then is whether we can not use preferences as the primitive concept, and yet return to the idea that individuals are utility maximizing?

We had already gone through this idea of revealed preference prior. We will now examine this in additional detail. Before that we need additional notation. Firstly, if $\mathbf{p}'\mathbf{x}' \geq \mathbf{p}'\mathbf{x}$, then it must be that $u(\mathbf{x}') \geq u(\mathbf{x})$, which just says that since $\mathbf{x}$ was within the budget set, the fact it was not chosen means that $\mathbf{x}'$ must yield at least as high a level of welfare. We will define this situation as $\mathbf{x}'$ *directly revealed preferred to* $\mathbf{x}$, and denote it as $\mathbf{x}'R^D\mathbf{x}$. This thus means that if we observe $\mathbf{x}'R^D\mathbf{x}$, we may infer that $u(\mathbf{x}') \geq u(\mathbf{x})$. If instead we have $\mathbf{p}'\mathbf{x}' \geq \mathbf{p}'\mathbf{x}$, then we may conclude that $\mathbf{x}'$ is *strictly directly revealed preferred to* $\mathbf{x}$, and this can be written succinctly as $\mathbf{x}'P^Dx$. If following a sequence of directly revealed preference, we observe $\mathbf{x}'$ revealed preferred to $\mathbf{x}$ via transitivity, we denote it as $\mathbf{x}'R\mathbf{x}$. If we assume our observations are generated by utility maximization, then $\mathbf{x}'R\mathbf{x}$ implies that $u(\mathbf{x}') \geq u(\mathbf{x})$.

Definition 8 ***Generalized Axiom of Revealed Preference (GARP):*** $\mathbf{x}'R\mathbf{x} \Rightarrow \mathbf{x} \not P^D \mathbf{x}'$. *Put another way,* $\mathbf{x}'R\mathbf{x} \Rightarrow \mathbf{px} \leq \mathbf{px}'$.

A key point to the definition is that *GARP* allows segments of the indifference curve to be flat.

The punchline is coming. It turns out that if the observed information of the couplet $(\mathbf{p}, \mathbf{x})$ were generated by a utility maximizing agent wih non-satiated preferences, it must satisfy *GARP*. Put another way, this is saying that if the information/data we see is consistent with *GARP*, then a utility function will exist, and it will rationalize that behavior. This is indeed a very powerful idea, which may be shown by the following theorem.

Theorem 8 ***Afriat's Theorem:*** *Let* $(\mathbf{p}^k, \mathbf{x}^k)$ *for* $k = 1, \ldots, K$ *be a finite set of oberved price and associated vector of goods consumed. Then the following conditions are equivalent.*

1. *There exists a locally nonsatiated utility function that rationalizes the data;*
2. *The observed data satisfies* ***GARP****.*
3. *There exist positive numbers* (u^k, β^k) *for* $k = 1, \ldots, K$ *such that Afriat's inequalities are satisfied:*

$$u^m \leq u^k + \beta^k \mathbf{p}^k (\mathbf{x}^m - \mathbf{x}^k)$$

   $\forall k, m$.

4. *There exists a locally nonsatiated, continuous, concave, monotonic utility function that rationalizes the data.*

Proof. In describing the revealed preference relation, we had already shown the relationship between 1 and 2. You may read the recommended text for the reference there in for the proof between 2 and 3. We will proof only that 3 implies 4 following the text.

First, define

$$u(\mathbf{x}) = \min_k \{u^k + \beta^k \mathbf{p}^k (\mathbf{x} - \mathbf{x}^k)\}$$

By assumption this function is continuous, and for $\mathbf{p} \geq \mathbf{0}$ and none of the price vectors $\mathbf{p}^k = \mathbf{0}$, this function will be locally nonsatiated, monotonic, and concave (you can show yourself this last one). This is essentially the lower envelope or hull created by the minimum of all the functions.

First, observe that $u(\mathbf{x}^k) = u^k$, failing which,

$$u(\mathbf{x}^k) \quad = \quad u^m + \beta^m \mathbf{p}^m (\mathbf{x}^k - \mathbf{x}^m) < u^k$$

which violates Afriat's inequality.

Next, assume that $\mathbf{p}^n \mathbf{x}^n \geq \mathbf{p}^n \mathbf{x}$. This then implies that,

$$u(\mathbf{x}) = \min_k \{u^k + \beta^k \mathbf{p}^k (\mathbf{x} - \mathbf{x}^k)\} \quad \leq \quad u^n + \beta^n \mathbf{p}^n (\mathbf{x} - \mathbf{x}^n) \leq u^n = u(\mathbf{x}^n)$$

where the result shows that since $u(\mathbf{x}^n) \geq u(\mathbf{x}) \; \forall \mathbf{x}$, which in turn means that $\mathbf{p}^n \mathbf{x}^n \geq \mathbf{p}^n \mathbf{x}$. In other words, if $\mathbf{x}^n$ is the observed choice, then all other choices must be within the budget set, yielding a lower level of welfare/utility, and in turn, the utility function thus rationalizes the observed choices.

■

To see the relationships within Afriat's inequality, first note that if we assume $u(\mathbf{x})$ is concave and differentiable in the elements of $\mathbf{x}$. This in turn gives us K first order conditions,

$$\mathbf{D}u(\mathbf{x}^k) \quad = \quad \beta^k \mathbf{p}^k$$

Further, since it is concave,

$$u(\mathbf{x}^k) \quad \leq \quad u(\mathbf{x}^m) + \mathbf{D}u(\mathbf{x}^m)(\mathbf{x}^k - \mathbf{x}^m)$$

where the inequality is from the definition of concavity. This basically then tells us that u^k is the level of utility, and β^k is just the marginal utility.