Advanced Microeconomics (ECON 401)
Lecture 2
Theory of the Firm

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We have now completed our discussion of how individual agents make their consumption choices, and are now ready to examine the other side of the market, that of the firm. Before beginning examining how these firms make their choice, we need, as before, to delineate the premise of our discussion, beginning with the technology that bounds production, its commensurating cost.

1 Production Set

In describing production in an economy, first let there be K commodities. All of which are either used in the production or not, or are outputs in the production process. In other words, let the vector,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}$$

be the *production/input-output/netput/production plan* vector, where $\mathbf{y} \in \mathbb{R}^K$, and it describes the *net* outputs of the commodities in in the production process. This then means that an input would be negative numbers, while outputs would be positive numbers, and zeros simply denote that the commodity is not produced in the production process in discussion.

Necessarily, not everything is feasible in $\mathbb{R}^K$ since it is dependent on the technology available. But it is also possible, as in our discussion of agent choices, that this constraints

on the production set could be legal or contractual in nature. In consequence, elements that are feasible are known as being in the *Production Set*. Let us denote the production set as $Y \subset \mathbb{R}^K$, so that $\mathbf{y} \in Y$ is feasible, and if $\mathbf{y} \notin Y$ is not feasible.

A convenient way to describe the production set is through the *production function/transformation function* $F(.)$, and it has the property that, $Y = \{\mathbf{y} \in \mathbb{R}^K : F(\mathbf{y}) \leq 0\}$, noting that $F(\mathbf{y}) = 0 \Leftrightarrow \mathbf{y}$ is on the *transformation frontier*. For a point on the frontier $\mathbf{y}^*$, such that $F(\mathbf{y}^*) = 0$, and for a twice differentiable transformation function, the gradient or the marginal rate of technical substitution, is thus,

$$\begin{aligned} dF(\mathbf{y}^*) &= \frac{\partial F(\mathbf{y}^*)}{\partial y_j} dy_j + \frac{\partial F(\mathbf{y}^*)}{\partial y_k} dy_k = 0 \\ \Rightarrow MRTS_{jk} = \frac{dy_k}{dy_j} &= \frac{\frac{\partial F(\mathbf{y}^*)}{\partial y_j}}{\frac{\partial F(\mathbf{y}^*)}{\partial y_k}} \end{aligned}$$

The interpretation is exactly as before and in your intermediate microeconomics.

However, you would be more familiar with the case where the inputs and outputs are distinct commodities. In which case, distinguish the elements all as positive numbers, and separate two sets of vectors, where $\mathbf{q}$ denotes the M vector of outputs, and the remaining $K - M$ vector of inputs $\mathbf{z}$, and where their typical elements would be $q_m \geq 0$ and $z_j \geq 0$, for $m = 1, \ldots, M$ and $j = 1, \ldots, K - M$ respectively. As an illustration, consider the single output case, then $\mathbf{q} = q \geq 0$, and $\mathbf{z}$ would be a $K - 1$ vector. Then denote the transformation function as $f(\mathbf{z})$. Here the production set can be characterized as,

$$Y = \{(-z_1, \ldots, -z_{K-1}, q) : q - f(\mathbf{z}) \leq 0 \ \& \ \mathbf{z} \geq 0\}$$

So what are some of the assumptions required?

1. $Y \neq \emptyset$: Which says that the firm will always have a production plan.

2. *Y is a closed set*: This just ensures that the limit of sequence in Y will also be in Y. Precisely, $\mathbf{y}^n \to \mathbf{y}$, and that if $\mathbf{y}^n \in Y \Rightarrow \mathbf{y} \in Y$.

3. *No free lunch*: For $\mathbf{y} \in Y$, and $\mathbf{y} \geq 0$, so that the process does not use any inputs. Then, output must also be zero, i.e. $\mathbf{y} = 0$, since it is not possible to produce something from nothing.

4. *Possibility of inaction*: $0 \in Y$, and basically means that complete shutdown is possible.

5. *Free Disposal*: If $\mathbf{y} \in Y$, and $\mathbf{y}' \leq \mathbf{y}$, then $\mathbf{y}' \in Y$. This can also be stated as $Y - \mathbb{R}^K_+ \subset Y$.

6. *Irreversibility*: If $\mathbf{y} \in Y$ and $\mathbf{y} \neq 0$, $\Rightarrow -\mathbf{y} \in Y$. This basically says that it is technologically not possible to reverse engineer the same amount of inputs from the outputs.

7. *Nonincreasing returns to scale*: A production set has this when for $\mathbf{y} \in Y$, $\alpha\mathbf{y} \in Y$ for $\alpha \in [0, 1]$, and says that any feasible process can be scaled down. It also implies that inaction is always possible.

8. *Nondecreasing returns to scale*: A production set has this when for $\mathbf{y} \in Y$, $\alpha\mathbf{y} \in Y$ for $\alpha \geq 1$, and says that a feasible process can be scaled up.

9. *Constant returns to scale*: A production set has this when for $\mathbf{y} \in Y \Rightarrow \alpha\mathbf{y} \in Y$ for $\alpha \geq 0$, and geometrically, Y is a *cone*.

10. *Additivity (or free entry)*: For $\mathbf{y} \in Y$ and $\mathbf{y}' \in Y$, then

$$\mathbf{y} + \mathbf{y}' \in Y$$

, or $Y + Y \subset Y$, or $\beta\mathbf{y} \in Y$ for positive integer β. Basically, this says that if both processes are feasible, we can always create to plants to produce it. Put another way, the *aggregate production set* must satisfy *additivity* when there is *free entry*.

11. *Convexity*: If $\mathbf{y}$ $\mathbf{y}' \in Y$, then $\alpha\mathbf{y} + (1 - \alpha)\mathbf{y}' \in Y$, where $\alpha \in [0, 1]$. This assumption captures several ideas. Firstly, it includes nonincreasing returns to scale. Secondly, "unbalanced" input combinations cannot be mpore productive than balanced ones.

12. *Y is a convex cone*: This is a combination of the convexity and constant returns to scale assumption. Formally, Y is a convex cone if for $\mathbf{y}$, $\mathbf{y}' \in Y$, then $\alpha\mathbf{y} + \beta\mathbf{y}' \in Y$ for α, $\beta \geq 0$.

This then allows us to prove the following proposition.

Proposition 1 *The production set Y is additive and satisfies the nonincreasing returns to scale condition if and only if it is a convex cone.*

Proof. First note that by definition of a convex cone implies nonincreasing returns to scale, and additivity. To show the reverse, let $\gamma > \max\{\alpha, \beta\}$ for $\alpha > 0$ and $\beta > 0$. By additivity, $\gamma\mathbf{y} \in Y$, and $\gamma\mathbf{y}' \in Y$. Since $\frac{\alpha}{\gamma} < 1$ and $\alpha\mathbf{y} = \left(\frac{\alpha}{\gamma}\right)\gamma\mathbf{y}$, the nonincreasing assumption implies that $\alpha\mathbf{y} \in Y$, and the same is true for $\beta\mathbf{y}$. Then by additivity, $\alpha\mathbf{y} + \beta\mathbf{y}' \in Y$. ■

The latter part of the proof is important since it is saying as long as we have nonincreasing technology (natural), and additivity (also natural), we have convexity! So that the convexity assumption is natural, and not "manufactured" or an imposition! However, it should be noted that additivity is quite strong since it assumes production of one does not interfere with one another. But we can assuredly concieve of situations when this does not hold.

2 Profit Maximization

To delineate the boundaries of our discussion now regarding the firm decisions, we will focus on the key assumptions for this examination here:

1. Let $\mathbf{p} \gg 0$ be a K price vector for the K commodities.
2. These prices are give $\Rightarrow$ *price taking assumption.*
3. Firms are profit maximizing.
4. Production set Y satisfies *nonemptiness*, *closedness*, and *free disposal.*

We can state the firm's *profit maximizing problem* as,

$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{p}'\mathbf{y} \\ \text{subject to:} \quad & \mathbf{y} \in Y \end{aligned}$$

Observe that given the sign convention we have adopted, $\mathbf{p}'\mathbf{y}$ is precisely the profit. And the technology bounding the production process is captured completely by Y. Of course we could state the above using the *transformation function.*

$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{p}'\mathbf{y} \\ \text{subject to:} \quad & F(\mathbf{y}) \leq 0 \end{aligned}$$

Then the act of profit maximization simply implies that,

$$\pi(\mathbf{p}) \quad = \quad \max\{\mathbf{p}'\mathbf{y} : \mathbf{y} \in Y\}$$

From which we obtain the vector of *supply correspondence* evaluated at $\mathbf{p}$,

$$\mathbf{y}(\mathbf{p}) \quad = \quad \{\mathbf{y} \in Y : \mathbf{p}'\mathbf{y} = \pi(\mathbf{p})\}$$

You can think of the equation $\mathbf{p}'\mathbf{y} = \pi(\mathbf{p})$ as a *isoprofit line*, so that in equilibrium, the *isoprofit line* is just tangent to the boundary of the *production set*, or the frontier of the *transformation function*. Note that the price vector $\mathbf{p}$ is just the *normal vector* and will be orthogonal to the *isoprofit line*.

If we assume that the transformation function is continuous and twice differentiable, we can write the Lagrangian of the contrained optimization problem as,

$$\mathcal{L} \quad = \quad \mathbf{p}'\mathbf{y} - \lambda F(\mathbf{y})$$

so that we have the following first order conditions,

$$\lambda \frac{\partial F(\mathbf{y}^*)}{\partial y_k} \quad = \quad p_k$$

$\forall k = 1, \ldots, K$. The above can also be written in matrix form as,

$$\mathbf{p} \quad = \quad \lambda \nabla F(\mathbf{y}^*)$$

which just says that the price vector will be proportionally equal to the Jacobian vector. You will also realize that you can substitute away the multiplier to obtain the equilibrium condition you are familiar with.

$$\frac{p_k}{p_{k'}} \quad = \quad \frac{\frac{\partial F(\mathbf{y}^*)}{\partial y_k}}{\frac{\partial F(\mathbf{y}^*)}{\partial y_{k'}}} = MRTS_{k'k}(\mathbf{y}^*)$$

$\forall k, k' = 1, \ldots, K$, and gives you the tangency condition that you have known.

In the simples case, as you have learned, is for there to be a single output, with a production function of $f(.)$, using input vector $K-1$ vector of inputs $\mathbf{z}$, with a scalar output price of p, with a $K-1$ vector of input prices $\mathbf{w} \gg 0$. Then the firm chooses the optimal level of inputs $\mathbf{z}$,

$$\max_{\mathbf{z} \geq 0} pf(\mathbf{z}) - \mathbf{w}'\mathbf{z}$$

This will give the following first order conditions:

$$p\frac{\partial f(\mathbf{z}^*)}{\partial z_k} \leq w_k$$

and the condition holds with equality if $z_k > 0$, for $k = 1, \ldots, K-1$. This the case since this is a constrained optimization problem, where the constraints are the nonnegativity conditions on the inputs. You should use the complimentary slackness conditions to show yourself that this is the case. This condition can be written in matrix form as,

$$p\nabla f(\mathbf{z}^*) \leq \mathbf{w} \quad \& \quad [p\nabla f(\mathbf{z}^*) - w]\mathbf{z}^* = 0$$

which are just the Kuhn-Tucker conditions for constrained optimization with inequality constraints. Of course, these conditions could similarly be written as Complimentary Slackness conditions. The Lagrangian can be written as,

$$\mathcal{L} \quad = \quad pf(\mathbf{z}) - \mathbf{wz} + \Lambda(\mathbf{z})$$

Then the first order and complimentary slackness conditions can be written as,

$$\begin{aligned} p\nabla f(\mathbf{z}^*) - \mathbf{z}^* + \Lambda \quad &= \quad 0 \\ \mathbf{z}^* \geq 0 \quad \Lambda \geq 0 \quad \Lambda\mathbf{z}^* &= 0 \end{aligned}$$

Here, we have $p\nabla f(\mathbf{z}^*) = \mathbf{z}^*$ when $\mathbf{z}* > 0$, the latter implies that $\Lambda = 0$.

If you are curious about where there is a boundary condition here in this formulation, versus the previous one with the production process vector, this is because the values of $\mathbf{y}$ span $\mathbb{R}^K$, in other words, each element can take on positive and negative values, but not in the above formulation.

What you should notice here is that in equilibrium, for $\mathbf{z} \gg 0$, we have the condition you are familiar with, that the marginal product of labour must be equal to the price of the input, in terms of the output, or "real input price". Putting two of these conditions together, would give you the condition of,

$$MRTS_{kk'} \quad = \quad \frac{\frac{\partial f(\mathbf{z})}{\partial z_k}}{\frac{\partial f(\mathbf{z})}{\partial z_{k'}}} = \frac{w_k}{w_{k'}}$$

Finally, note that if the production set Y is convex, then all the first order conditions we have discussed, for a single output and multiple general case, are then *necessary and sufficient conditions* for a solution.

With the above discussion as background, we can now reveal the following features, of the profit function, and the supply correspondence.

Proposition 2 *For a profit function $\pi(.)$ of a production set Y, and its associated supply correspondence $\mathbf{y}(.)$, with Y being a closed set, and satisfies the free disposal property. Then,*

1. *$\pi(.)$ is homogeneous of degree one.*
2. *$\pi(.)$ is convex.*
3. *If Y is convex, then $Y = \{y \in \mathbb{R}^K : \mathbf{p}'\mathbf{y} \leq \pi(\mathbf{p}) \forall \mathbf{p} \gg 0\}$.*
4. *$\mathbf{y}(.)$ is homogeneous of degree zero.*
5. *If Y is convex, then $\mathbf{y}(\mathbf{p})$ is a convex set $\forall \mathbf{p}$. Further, if Y is strictly convex, then $\mathbf{y}(\mathbf{p})$ is single-valued (if nonempty).*
6. *(Hotelling's Lemma) If $\mathbf{y}(\mathbf{p})$, then $\pi(.)$ is differentiable at $\mathbf{p}$ and $\nabla\pi(\mathbf{p}) = \mathbf{y}(\mathbf{p})$.*
7. *If $\mathbf{y}(.)$ is differentiable function at $\mathbf{p}$, then $\nabla\mathbf{y}(\mathbf{p}) = \nabla^2\pi(\mathbf{p})$ is a symmetric and positive semidefinite matrix with $\nabla\mathbf{y}(\mathbf{p})\mathbf{p} = 0$.*

Proof.

1. Since $\pi(\mathbf{p}) = \max\{\mathbf{p}'\mathbf{y} : \mathbf{y} \in Y\}$, then $\pi(\alpha\mathbf{p}) = \max\{\alpha\mathbf{p}'\mathbf{y} : \mathbf{y} \in Y\}$, which can be written as $\pi(\mathbf{p}) = \alpha \max\{\mathbf{p}'\mathbf{y} : \mathbf{y} \in Y\} = \alpha\pi(\mathbf{p})$, so that $\pi(\alpha\mathbf{p}) = \alpha\pi(\mathbf{p})$.
2. Let $\mathbf{y}^* = \alpha\mathbf{y} + (1-\alpha)\mathbf{y}'$, and $\mathbf{p}^* = \alpha\mathbf{p} + (1-\alpha)\mathbf{p}'$. Define then $\pi(\mathbf{p}^*) = \mathbf{p}^*\mathbf{y}^* = \alpha\mathbf{p}\mathbf{y}^* + (1-\alpha)\mathbf{p}'\mathbf{y}^*$. We know however that $\mathbf{p}\mathbf{y}^* \leq \mathbf{p}\mathbf{y} \leq \pi(\mathbf{p})$. Similarly, $\mathbf{p}'\mathbf{y}^* \leq \mathbf{p}'\mathbf{y}' \leq \pi(\mathbf{p}')$. Therefore, $\pi(\mathbf{p}^*) = \alpha\mathbf{p}\mathbf{y}^* + (1-\alpha)\mathbf{p}'\mathbf{y}^* \leq \alpha\pi(\mathbf{p}) + (1-\alpha)\pi(\mathbf{p}')$
3. This property basically says that the if Y is closed, convex, and satisfies free disposal, $\pi(\mathbf{p})$ is an alternative description of the technology.
4. This can be proven using Hotelling's lemma, and the fact that the profit function is homogeneous of degree 1. Since $\pi(\mathbf{p})$ is homogeneous of degree 1, this implies that $\pi(\alpha\mathbf{p}) = \alpha\pi(\mathbf{p})$, $\Rightarrow \alpha\nabla\pi(\alpha\mathbf{p}) = \alpha\nabla\pi(\mathbf{p})$, $\Rightarrow \nabla\pi(\alpha\mathbf{p}) = \nabla\pi(\mathbf{p}) = \alpha^0\nabla\pi(\mathbf{p})$, which means that the first order derivative of $pi(.)$ is homogeneous of degree zero. But by Hotelling's Lemma, $\nabla\pi(\mathbf{p}) = \mathbf{y}(\mathbf{p})$. So that $\mathbf{y}(\alpha\mathbf{p}) = \nabla\pi(\alpha\mathbf{p}) = \nabla\pi(\mathbf{p}) = \mathbf{y}(\mathbf{p})$, so that the supply correspondence $\mathbf{y}$ is homogeneous of degree 0.
5.

6. Let $\mathbf{y}(\mathbf{p})^*$ be the profit maximizing netput vector. Define the following function,

$$g(\mathbf{p}) = \pi(\mathbf{p}) - \mathbf{p}\mathbf{y}^*$$

Observe that $\pi(\mathbf{p})$ will always be at least as high as the plan at $\mathbf{y}$, but will be exactly equal to 0 at $\mathbf{p}^*$, implying this function is convex, and has a interior minimum on $\mathbf{R}$. Thus,

$$\begin{aligned} \nabla g(\mathbf{p}^*) &= \nabla \pi(\mathbf{p}^*) - \mathbf{y}^* = 0 \\ \Rightarrow \nabla \pi(\mathbf{p}^*) &= \mathbf{y}^* \end{aligned}$$

$\forall \mathbf{p}^*$.

■

3 Cost Minimization

There is a direct parallel between profit maximization and cost minimization, in that profit cannot be maximized without the cost being minimized, so that a necessary condition for profit maximization is cost minimization. The analysis will also reveal several interesting facts.

As before, let $\mathbf{z}$ be the nonnegative vector of inputs, and the production function utilizing the inputs is $f(\mathbf{z})$, producing output q, and given the input price vector $\mathbf{w} \gg 0$. The cost minimization problem can be written as,

$$\begin{aligned} \min_{\mathbf{z} \geq 0} \quad & \mathbf{w}\mathbf{z} \\ \text{s.t} \quad & f(\mathbf{z}) \geq q \end{aligned}$$

The corresponding input demand, or more precisely the *conditional factor demand correspondence/function* obtained can be denoted as $\mathbf{z}(\mathbf{w}, q)$, so that the optimized cost is just $c(\mathbf{w}, q)$. The qualifier of *conditional* is because the solution is conditional on output q.

In the case of two inputs, the solution can be characterized as the tangency between the isoquant, and the isocost line. If the production function is continuous and twice differentiable, we can then characterize the solution with the equilibrium condition of,

$$w_k \geq \lambda \frac{\partial f(\mathbf{z}^*)}{\partial z_k}$$

with equality if $z_k^* > 0$. In matrix notation, this can be written as,

$$\begin{aligned} \mathbf{w} &\geq \lambda \nabla f(\mathbf{z}^*) \\ [\mathbf{w} - \lambda \nabla f(\mathbf{z}^*)]\, \mathbf{z}^* &= 0 \end{aligned}$$

which is just the Kuhn-Tucker conditions as before in the Profit Maximization Problem. Combining these conditions for any pair of inputs then gives,

$$MRTS_{kk'} = \frac{MP_{k'}}{MP_k} = \frac{w_k}{w_{k'}}$$

which is the same condition as that from profit maximization as mentioned earlier.

We can now discuss the properties of the cost function.

Proposition 3 *Let $c(\mathbf{w}, q)$ be the cost function of a single output technology represented by Y, with production function $f(.)$, and conditional factor demand correspondence associated with the cost function is $\mathbf{z}(\mathbf{w}, q)$. Further, assume that Y is closed, and satisfies free disposal. Then,*

1. *$c(.)$ is homogeneous of degree one in $\mathbf{w}$, and nondecreasing in q.*
2. *$c(.)$ is a concave function of $\mathbf{w}$.*
3. *If the sets $\{\mathbf{z} \geq 0 : f(\mathbf{z} \geq q)\}$ are convex for every q, then $Y = \{(-\mathbf{z}, q) : \mathbf{wz} \geq c(\mathbf{w}, q)\}\ \forall \mathbf{w} \gg 0$.*
4. *$\mathbf{z}(.)$ is homogeneous of degree 0 in $\mathbf{w}$.*
5. *If the set $\{\mathbf{z} \geq 0 : f(\mathbf{z} \geq q)\}$ is convex, then $\mathbf{z}(\mathbf{w}, q)$ is a convex set, and if it is strictly convex, then $\mathbf{z}(\mathbf{w}, q)$ is single valued.*
6. *(Sheppard's Lemma) If $\mathbf{z}(\mathbf{w}^*, q)$ consists of a single point, then $c(.)$ is differentiable with respect to $\mathbf{w}$ at $\mathbf{w}^*$, and $\nabla_w c(\mathbf{w}^*, q) = \mathbf{z}(\mathbf{w}^*, q)$.*
7. *If $\mathbf{z}(.)$ is differentiable at $\mathbf{w}^*$, then $\nabla_w \mathbf{z}(\mathbf{w}, q) = \nabla_w^2 c(\mathbf{w}^*, q)$ is a symmetric and negative semidefinite matrix with $\nabla_w \mathbf{z}(\mathbf{w}^*, q)\mathbf{w}^* = 0$.*
8. *If $f(.)$ is homogeneous of degree one (has constant returns to scale), then $c(.)$ and $\mathbf{z}(.)$ are homogeneous of degree 0 in q.*
9. *If $f(.)$ is concave, then $c(.)$ is a convex function of q (and marginal cost is nondecreasing in q).*

Although there the idea of utilizing cost functions does not carry more information than from profit maximization, it has some benefits. For instance, if $\mathbf{y}(.)$ is not single valued for any $\mathbf{p}$ for a nonzero production, so that Hotelling's lemma cannot be used, we can still resort to Shepard's lemma.

Finally, given the above discussion, we can rewrite the firm's problem now as,

$$\max_{q \geq 0} pq - c(\mathbf{w}, q)$$

So that the Kuhn-Tucker first order condition for q^* is,

$$p - \frac{\partial c(\mathbf{w}, q)}{\partial q} \quad \leq \quad 0$$

where the inequality holds as equality if $q^* > 0$, and when that happens, we have an interior solution, and the equilibrium is described by price equal to marginal cost, as we have always known. Further, if $c(\mathbf{w}, q)$ is convex. the first order condition above is also a sufficient condition for q^* to be the optimal choice.