Advanced Microeconomics (ECON 401)
Lecture 3
Market Equilibrium

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What we discussed thus far is broadly what is commonly termed a *partial equilibrium* analysis, since we have discussed individual agent, and firm choices to the exclusion of the other party or parties in the market exchange. In doing so, we have assumed the clearing prices have been fixed. However, to truly understand what goes on in the market, we need to allow prices to adjust, so as to clear all the markets, what is commonly referred to as *general equilibrium* analysis. Our discussion will first examine how individual agents interact with each other (*pure exchange*), before the inclusion of the firms. In the *pure exchange* economy, the agents will exchange goods with each other, guided by the their own preferences, and rules that guide their exchange, to enhance their own wellbeing.

1 Walrasian Equilibrium

We will take a very broad definition of goods here, and these goods may be distingushed from each in other in terms of time, location, and/or the state of the world the agents find themselves in. So for instance, labour services can be construed as a type of good traded here. What is of importance to note here is that each good will have a market where their price will be determined.

In the pure exchange model of an economy, the market is completely constituted by agents, completely described by their preference relations $\succsim$, and their individual initial endowments of k commodities, ω_i. The agents all behave competitively, in that they take the prices as given, and understand that it is independent of their actions. The agent then makes their choice on their most preferred bundle of goods they can afford.

What is of essence in *general equilibrium* is for us to gain an understanding into how goods are ultimately allocated between the agents. Denote the amount of good j that an agent i hold be x_i^j. Then her vector of consumption bundle is $\mathbf{x}_i = [x_i^1, x_i^2, \ldots, x_i^k]'$. Then an allocation $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]'$ is a collection of consumption bundles held by the n constituents of the exchange economy. A *feasible allocation* is then one where all goods within the econom t are completely used, $\sum_{i=1}^n \mathbf{x}_i = \sum_{i=1}^n \omega_i$, or more generally, $\sum_{i=1}^n \mathbf{x}_i \leq \sum_{i=1}^n \omega_i$. You would have seen this earlier in discussions within a exchange economy with only 2 agents, depicted with *Edgeworth Box*.

Let the price for each good within the exchange economy be p_i, so that the market price vector is $\mathbf{p} = [p_1, p_2, \ldots, p_k]'$. Then when each agent takes these prices as given, beyond her actions can alter, she solves,

$$\begin{aligned} & \max_{\mathbf{x}_i} u_i(\mathbf{x}_i) \\ \text{subject to} \quad & \mathbf{p}\mathbf{x}_i = \mathbf{p}\omega_i (= m_i) \end{aligned}$$

so that the solution, as you are familiar by now, is $\mathbf{x}(\mathbf{p}, \mathbf{p}\omega_\mathbf{i})$, and is the individual's demand function. Note that $\mathbf{p}\omega_i$ is just the market value of the individual's endowment.

Note that the problem is not a trivial one, since given a price vector $\mathbf{p}$, the aggregate demand of $\sum_i \mathbf{x}(\mathbf{p}, \mathbf{p}\omega_\mathbf{i})$ need not be equal to $\sum_i \omega_i$. Particularly if some goods are undesirable, so that those goods might exhibit excess supply.

Definition 1 *A Walrasian Equilibrium is a couplet* $(\mathbf{p}^*, \mathbf{x}^*)$ *such that,*

$$\sum_i \mathbf{x}_i(\mathbf{p}^*, \mathbf{p}^*\omega_i) \quad \leq \quad \sum_i \omega_i$$

So that we may interpret a *Walrasian Equilibrium* as a $\mathbf{p}^*$ such that there is no positive excess demand. However, when all goods are desirable, the inequality will hold with strict equality at the market clearing price.

1.1 Existence of Walrasian Equilibrium

The next question we wish to ask would be whether such an equilibrium actually exists. In order to adequately answer the question, we will need some preliminaries. Firstly, recall from our discussion of consumer theory the the Walrasian demand is homogeneous

of degree 0, so that $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\omega_i) = \mathbf{x}_i(\alpha\mathbf{p}, \alpha\mathbf{p}\omega_i)$, for all $\alpha > 0$. Then we can define the excess demand function as,

$$\mathbf{z}(\mathbf{p}) = \sum_{i=1}^{n}[\mathbf{x}_i(\mathbf{p}, \mathbf{p}\omega_i) - \omega_i]$$

and it will likewise be homogeneous of degree 0. It is useful to keep in mind that the endowments of each individual i is fixed throughout our analysis.

Observe next that if the Walrasian demand is continuous, then so too the excess demand function is continuous in $\mathbf{p}$. Further, it will have to satisfy *Walras' Law*

Theorem 1 *For any price vector* $\mathbf{p}$, $\mathbf{pz}(\mathbf{p}) = 0$.

Proof.

$$\begin{aligned}\mathbf{pz}(\mathbf{p}) &= \mathbf{p}\left[\sum_{i=1}^{n}\mathbf{x}_i(\mathbf{p}, \mathbf{p}\omega_i) - \sum_{i=1}^{n}\omega_i\right] \\ &= \sum_{i=1}^{n}[\mathbf{p}\mathbf{x}_i(\mathbf{p}, \mathbf{p}\omega_i) - \mathbf{p}\omega_i] = 0\end{aligned}$$

The equality follows since the Walrasian demand $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\omega_i)$ must satisfy their budget constraint, $\forall i$. ■ Walras' Law basically says that the value of excess demand will always zero, since each agent's excess demand will always be zero, and that this is for all price vectors $\mathbf{p}$.

This then gives us the following propositions.

Proposition 2 ***Market Clearing.*** *If demand equals supply for* $k-1$ *markets, and* $p_k > 0$*, so that the* k^{th} *market's good is desirable, then the demand must equal supply in the* k^{th} *market.*

The *Market Clearing* proposition must clearly be met, failing which *Walras' Law* would be violated.

Proposition 3 ***Free Goods.*** *If* $\mathbf{p}^*$ *is a Walrasian Equilibrium, and* $z_j(\mathbf{p}^*) < 0$*, then* $p_j^* = 0$*, so that the* j^{th} *good must be a free good.*

Proof. Since $\mathbf{p}^*$ is a Walrasian Equilibrium, we have $\mathbf{z}(\mathbf{p}^*) \leq 0$. Since prices are nonnegative, $\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) = \sum_{i=1}^{k} p_i z_i(\mathbf{p}^*) \leq 0$. But if $z_j(\mathbf{p}) < 0$, while $p_j^* > 0$, this would imply that $\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) < 0$, which would contradict Walras' Law, so that the proposition follows. ■

Definition 2 ***Desirability.*** *If $p_i = 0$, then $z_i(\mathbf{p}) > 0$ for $i = 1, \ldots k$.*

The definition thus defines what is a desirable good, which is simply one where should its price be zero, we would observe a strictly positive excess demand. This then leads to the following proposition.

Proposition 4 ***Equality of Demand and Supply.*** *If all goods are desirable, and $\mathbf{p}^*$ is a Walrasian Equilibrium, then $\mathbf{z}(\mathbf{p}^*) = 0$*

Proof. Suppose that $z_i(\mathbf{p}^*) < 0$. Then by proposition 3, it must be that $p_i = 0$. However, since good i is desirable, this would mean instead that $z_i(\mathbf{p}^*) > 0$, leading to a contradiction, so that the proposition follows. ■

To begin our discussion of the existence of an equilibria, let us first normalize our prices so that the price vector is in $\mathbb{R}_+^{k-1}$. Denote the absolute prices by $\widehat{p}_i$, then the normalized price for good i is just,

$$p_i = \frac{\widehat{p}_i}{\sum_{j=1}^{k} \widehat{p}_j}$$

In consequence, $\sum_{i=1}^{k} p_i = 1$. This allows us to restrict attention to the $k-1$ dimensioned simplex,

$$S^{k-1} = \left\{ \mathbf{p} \in \mathbb{R}_+^k : \sum_{i=1}^{k} p_i = 1 \right\}$$

In order to prove existence, we will need the following axiom, and theorems.

Axiom 5 ***Axiom of Continuity:*** *Suppose that an infinite sequence $\{x_n\}_{n=1}^{\infty}$ is such that $x_{n+1} \geq x_n \ \forall n$, and there is a number M such that $x_n \leq M \ \forall n$. Then there is a number $L \leq M$ such that $x_n \to L$ as $n \to \infty$ and $x_n \leq L \ \forall n$.*

Theorem 6 ***Nested Intervals Theorem:*** *Suppose that,*

$$I_n = \{x : a_n \leq x \leq b_n\}$$

for $n = 1, 2, \ldots,$ is a sequence of closed intervals such that $I_{n+1} \subset I_n$ for each n. If $\lim_{n\to\infty}(b_n - a_n) = 0$, then there is one and only one number x_0 which is in every I_n.

Proof. By the statement, we have $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$. Since $a_n < b_n \ \forall n$, the sequence $\{a_n\}$ is nondecreasing and bounded from above by b_1. Similarly, the sequence $\{b_n\}$ is nonincreasing and bounded from below by a_1. Then by the Axiom of Continuity, there must be numbers x_0 and x_0' such that $a_n \to x_0$, $a_n \leq x_0$, and $b_n \to x_0'$, $b_n \geq x_0'$, as $n \to \infty$. This then means that $b_n - a_n \to 0$ as $n \to \infty$, so that $x_0 = x_0'$, and $x_0 \in [a_n, b_n]$ $\forall n$, and $x_0 \in I_n \ \forall n$.

To demonstrate that x_0 is the only number in every I_n, suppose there is another number x_1 likewise in I_n. Define $\epsilon = |x_1 - x_0|$. Since $a_n \to x_0$ and $b_n \to x_0$, there is an integer N_1 such that $a_{N_1} > x_0 - \epsilon$. Similarly, there is also another integer N_2 such that $b_{N_2} < x_0 + \epsilon$. These inequalities then imply that I_n cannot have x_1 for any n beyond N_1 and N_2. Since these are completely arbitrary, this means that x_1 is not in every I_n. ■

Theorem 7 *Suppose f is continuous on $[a, b]$ $x_n \in [a, b]$ for each n, and $x_n \to x_0$. Then $x_0 \in [a, b]$, and $f(x_n) \to f(x_0)$.*

Theorem 8 ***Intermediate-value Theorem:*** *Suppose f is continuous on $[a, b]$, $c \in \mathbb{R}^1$, $f(a) < c$ and $f(b) > c$. Then there is at least one number x_0 on $[a, b]$ such that $f(x_0) = c$.*

Theorem 9 ***Brouwer Fixed-Point Theorem:*** *If $f : S^{k-1} \to S^{k-1}$ is a continuous function from the unit simplex to itself, there is some $\mathbf{x} \in S^{k-1}$ such that $\mathbf{x} = f(\mathbf{x})$.*

Proof. We will illustrate the more general proof through the case of $k = 2$. When $k = 2$, we can then identify the unit 1-dimensional simplex S^1 as a unit interval, $[0, 1]$, so that we have a function $f : [0, 1] \to [0, 1]$. Next, define a function $g(x) = f(x) - x$. A fixed point, x^*, here is just where $g(x^*) = 0$, when $f(x^*) = x^*$. Observe next that $g(0) = f(0) - 0 \geq 0$ since $f(0) \in [0, 1]$. Similarly, $g(1) = f(1) - 1 \leq 1$ since $f(1) \in [0, 1]$. Thensince f is continuous, by the Intermediate Value Theorem, there must exist $x \in [0, 1]$ such that $g(x) = f(x) - x = 0$. ■

Theorem 10 ***Existence of Walrasian Equilibria:*** *If $\mathbf{z} : S^{k-1} \to \mathbb{R}^k$ is a continuous function that satisfies Walras' law, $\mathbf{pz}(\mathbf{p}) \equiv 0$, then there exists some $\mathbf{p}^*$ in S^{k-1} such that $\mathbf{z}(\mathbf{p}^*) \leq 0$.*

Proof. Define a mapping $g : S^{k-1} \to S^{k-1}$ by,

$$g_i(\mathbf{p}) \quad = \quad \frac{p_i + \max\{0, z_i(\mathbf{p})\}}{1 + \sum_{j=1}^{k} \max\{0, z_j(\mathbf{p})\}}$$

for $i = 1, \ldots, k$. Next observe that since $z(.)$ and $\max\{.\}$ are continuous, so too would be $g(.)$. Further, $\mathbf{g}(\mathbf{p})$ is a point in the simplex S^{k-1} since $\sum_{i=1}^{k} g_i(\mathbf{p}) = 1$. Economically, $g(.)$ is saying that if there is excess demand in a market so that $z_i(\mathbf{p}) \geq 0$ then the relative price of i would be increased.

By Brouwer's fixed-point theorem, there is a $\mathbf{p}^*$ such that $\mathbf{p}^* = g(\mathbf{p}^*)$, so that,

$$g_i(\mathbf{p}^*) = \mathbf{p}^* \quad = \quad \frac{p_i^* + \max\{0, z_i(\mathbf{p}^*)\}}{1 + \sum_{j=1}^{k} \max\{0, z_j(\mathbf{p}^*)\}}$$

To show that $\mathbf{p}^*$ is a Walrasian equilibrium, observe that,

$$\begin{aligned} p_i^* \sum_{j=1}^{k} \max\{0, z_j(\mathbf{p}^*)\} &= \max\{0, z_i(\mathbf{p}^*)\} \\ \Rightarrow z_i(\mathbf{p}^*) p_i^* \sum_{j=1}^{k} \max\{0, z_j(\mathbf{p}^*)\} &= z_i(\mathbf{p}^*) \max\{0, z_i(\mathbf{p}^*)\} \\ \Rightarrow \left[\sum_{i=1}^{k} p_i^* z_i^*(\mathbf{p}^*)\right] \sum_{j=1}^{k} \max\{0, z_j(\mathbf{p}^*)\} &= \sum_{i=1}^{k} z_i(\mathbf{p}^*) \max\{0, z_i(\mathbf{p}^*)\} \end{aligned}$$

for $i = 1, \ldots, k$. But we know $\sum_{i=1}^{k} p_i^* z_i(\mathbf{p}^*) = 0$ by Walras' law, so that,

$$\sum_{i=1}^{k} z_i(\mathbf{p}^*) \max\{0, z_i(\mathbf{p}^*)\} \quad = \quad 0$$

And for the equality to hold, we need $z_i(\mathbf{p}^*) \leq 0$, for $i = 1, \ldots, k$. ■

The key insight here is that we have obtained existence merely through the assumption that the excess demand function be continuous, and that it satisfies Walras' law. These are not unreasonable assumptions given what we have learned thus far. Walras' law simply requires that agents fulfil a budget constraint in their decision process, which is natural in economic decisions. Similarly, we have learned that for consumers with strict convex preferences, demand functions would naturally be well defined, and continous. Think back to the Cobb-Douglas, Leontieff and linear utility functions, all of which have exhibits convex preferences, and recall how the Walrasian demand functions were all unanimously continuous. The only issue arises at the boundary of the demand when there are elements of the price vector which are zero. There, demand may not be defined. We will not discuss those possibilities, rest assured, those circumstances can likewise be handled.

Example 1 *Let the utility function of two agents, 1 and 2, be* $x_{1,1}^a x_{1,2}^{1-a}$ *and* $x_{2,1}^b x_{2,2}^{1-b}$ *respectively. Further, let their endowments be* $w_1 = (1, 0)$ *and* $w_2 = (0, 1)$ *respectively. We know the demand function for good 1 for agent 1 is* $x_{1,1} = \frac{am_1}{p_1}$*. The agent's income in turn is* $m_1 = p_1$*, so that* $x_{1,1} = a$*. Similarly, agent 2's demand is* $x_{2,1} = \frac{bm_2}{p_1}$*, and the agent's income is* $m_2 = p_2$*, so that* $x_{2,1} = \frac{bp_2}{p_1}$*. The reason we are focusing on a single good is because by Walras' law, market clearing in one market, implies market clearing in the other, in this 2 good case. Thus, we have,*

$$\begin{aligned} x_{1,1}(p_1, p_2) + x_{2,1}(p_1, p_2) &= 1 \\ \Rightarrow a + \frac{bp_2}{p_1} &= 1 \\ \Rightarrow \frac{p_2}{p_1} &= \frac{1-a}{b} \end{aligned}$$

which is the market clearing market price ratio.

1.2 First Welfare Theorem

With the above, we can now examine the implications on efficiency. To start, we need first some definitions.

Definition 3 ***Pareto Efficiency:*** *A feasible allocation* $\mathbf{x}$ *is a* ***weakly Pareto efficient*** *allocation if there is no feasible allocation* $\mathbf{x}'$ *such for all agents,* $\mathbf{x}' \succ \mathbf{x}$*. A feasible allocation* $\mathbf{x}$ *is a* ***strongly Pareto efficient*** *allocation if there is no feasible allocation* $\mathbf{x}'$ *such that for all agents,* $\mathbf{x}' \succsim \mathbf{x}$*, and for some,* $\mathbf{x}' \succ \mathbf{x}$*.*

This implies also that is an allocation is **strongly Pareto efficient**, it is **weakly Pareto efficient**. The reverse however needs additional assumptions or qualifications.

Proposition 11 ***Equivalence of weak & strong Pareto efficiency:*** *Suppose that preferences are continuous and monotonic. Then an allocation is weakly Pareto efficient if and only if it is strongly Pareto efficient.*

Proof. If an allocation is strongly Pareto efficient, then it is weakly Pareto efficient. Put another way, if you cannot make some people better off, without making others worse off, you definitively cannot make everyone better off. Now, we need to show that if an allocation is weakly Pareto efficient, then it is also strongly Pareto efficient. This can be done by showing that if an allocation is not strongly efficient, then it is also not weakly

efficient. Suppose that is the case, so that we can make an agent i better off without making everyone else worse off (since it is not strongly efficient). Then to prove the above statement, we need now to show that it is possible to make everyone better off. This can be done by reducing the new allocation x_i by a factor $(1-\theta)$, and redistributing, so that for the jth agent, her allocation is $x_j + \frac{(1-\theta)x_i}{n-1}$. By continuity, θ can be made infinitesimally close to 1, so that i remains better off than her original allocation, and yet, by monotonicity, all other agents made strictly better off. ■

Due to the above advantage with the continuity, and monotonicity assumption, we will maintain it through out the rest of our discussion.

It should be noted here that Pareto efficiency is a weak normative concept, since an allocation where one agent gets everything in the economy, and the rest gets nothing will be Pareto efficient as long as we assume that the lucky agent who has everything is never satiated!

We can now examine how a Pareto efficient allocation can be obtained. Focusing on a two person exchange economy that can be represented in an Edgeworth box, it is clear that it is any point where we have tangency between two individuals indifference curve. This then suggests that the program to obtain this sequence of outcomes can be written as,

$$\begin{aligned} & \max_{\mathbf{x}_1, \mathbf{x}_2} u_1(\mathbf{x}_1) \\ \text{such that} \quad & u_2(\mathbf{x}_2) \geq \overline{u}_2 \\ & \mathbf{x}_1 + \mathbf{x}_2 = \omega_1 + \omega_2 \end{aligned}$$

So that in equilibrium, we would have $MRS_1 = MRS_2$. This is easily derived from our earlier discussion likewise of the Walrasian equilibrium where the marginal rate of substitution must equate with the price ratio. The set of these Pareto efficient points are the **Pareto set** and are the locus of tangencies. The **Pareto set** is also known as the **Contract Curve**, denoting the set of efficient contracts tha clears the market.

We can now restate the Walrasian equilibrium in the following manner:

Definition 4 *An allocation-price couplet* $(\mathbf{x}, \mathbf{p})$ *is a* ***Walrasian Equilibrium*** *if,*

1. *the allocation is feasible, or more precisely,*

$$\sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} \omega_i$$

and,

2. *each agent is making an optimal choise from her budget set. That is if* $\mathbf{x}_i' \succsim_i \mathbf{x} \Rightarrow \mathbf{p}\mathbf{x}_i' > \mathbf{p}\omega_i$.

The equivalency between this definition, and the previous is maintained as long as the goods are desirable. In this case here, we do not have to be concerned with free goods.

Theorem 12 ***First Theorem of Welfare Economics:*** *If* $(\mathbf{x}, \mathbf{p})$ *is a Walrasian equilibrium, then* $\mathbf{x}$ *is Pareto efficient.*

Proof. We will prove by contradiction. Suppose that is not the case, so that $\mathbf{x}'$ is feasible, and all agents $\mathbf{x}' \succsim \mathbf{x}$. Then by the second property of a Walrasian equilibrium, we have

$$\mathbf{p}\mathbf{x}_i' > \mathbf{p}\omega_i$$

$\forall\ i = 1, \ldots, n$. Therefore,

$$\begin{aligned} \text{By feasibility } \sum_{i=1}^{n} \omega_i &= \sum_{i=1}^{n} \mathbf{x}_i' \\ \Rightarrow \mathbf{p}\sum_{i=1}^{n} \omega_i &= \mathbf{p}\sum_{i=1}^{n} \mathbf{x}_i' \\ &= \sum_{i=1}^{n} \mathbf{p}\mathbf{x}_i' > \sum_{i=1}^{n} \mathbf{p}\omega_i \end{aligned}$$

and we have a contradiction. ■ The theorem basically highlights that as long as the behavioral assumptions of the model are satisfied, our market equilibrium will be efficient. However, there is nothing here that suggests that the equilibrium will be ethical, or fair, since the entire outcome is dependent on the initial distribution of endowments.

The next question you may have is whether the reverse is true, which leads us to the following theorem. But it would need the following theorem.

Theorem 13 ***Separating Hyperplane:*** *For two non-empty, disjoint, convex sets* A *and* B *in* $\mathbf{R}^n$*, there will exist a linear functional* $\mathbf{p}$ *such that* $\mathbf{p}\mathbf{x} \geq \mathbf{p}\mathbf{y}$*,* $\forall \mathbf{x} \in A$*, and* $\mathbf{y} \in B$.

Then,

Theorem 14 *If $\mathbf{x}^*$ is a Pareto efficient allocation where each agent has in possession a positive amount of each good. Assuming that preferences are convex, continuous, and monotonic. Then $\mathbf{x}^*$ is a Walrasian equilibrium for the initial endowments $\omega_i = \mathbf{x}_i^*$ $\forall$ $i = 1, \ldots, n$.*

Proof. Define the set of all comsumption bundles prefered by an agent i to $\mathbf{x}_i^*$ be,

$$P_i \;\; = \;\; \{\mathbf{x}_i \in \mathbb{R}^k : \mathbf{x}_i \succ_i \mathbf{x}_i^*\}$$

Then the set of all bundles of goods that could be redistributed between all the n agents can be denoted as,

$$P = \sum_{i=1}^{n} P_i \;\; = \;\; \left\{ \mathbf{z} : \mathbf{z} = \sum_{i=1}^{n} \mathbf{x}_i \text{ for } \mathbf{x}_i \in P_i \right\}$$

Since each preference set P_i is convex, then their sum, P would likewise be convex.

Without loss of generality, let $\omega = \sum_{i=1}^{n} \mathbf{x}_i^*$ be the current allocation. Since this allocation $\mathbf{x}^*$ is Pareto efficient, welfare enhancing redistribution will not be possible, and in turn implies that ω not in P. In turn, this thus imply that by the separating hyperplance theorem, there exists a normal vector $\mathbf{p} \neq \mathbf{0}$ such that,

$$\begin{aligned} \mathbf{pz} &\geq \mathbf{p} \sum_{i=1}^{n} \mathbf{x}_i^* \; \forall \mathbf{z} \in P \\ \Rightarrow \mathbf{p} \left(\mathbf{z} - \sum_{i=1}^{n} \mathbf{x}_i^* \right) &\geq \mathbf{0} \; \forall \mathbf{z} \in P \end{aligned}$$

To complete the proof, we need to demonstrate that $\mathbf{p}$ is an equilibrium price vector.

1. First, we will show that $\mathbf{p} \geq \mathbf{0}$.

   Let $\mathbf{e}_i = [0, \ldots, 1, \ldots, 0]'$ with 1 at the ith element. Then the monotonicty of preferences would imply that $\omega + \mathbf{e}_i \in P$. This in turn implies that,

   $$\begin{aligned} \mathbf{p} \left(\omega + \mathbf{e}_i - \sum_{i=1}^{n} \mathbf{x}_i^* \right) &\geq \mathbf{0} \\ \Rightarrow \mathbf{p}(\omega + \mathbf{e}_i - \omega) &\geq \mathbf{0} \\ \Rightarrow \mathbf{p}\mathbf{e}_i &\geq \mathbf{0} \end{aligned}$$

   so that $p_i \geq 0 \; \forall i = 1, \ldots, k$.

2. Next, we need to show that any other allocation that is preferred to $\mathbf{x}_j^*$ will be more costly. Without lost of generality, let $\mathbf{y}_j \succ \mathbf{x}_j^*$. Then based on the preference set described about, we have,

$$\mathbf{p}\sum_{i=1}^{n}\mathbf{y}_i \;\geq\; \mathbf{p}\sum_{i=1}^{n}\mathbf{x}_i^*$$

Suppose only for agent j that $\mathbf{y}_j \succ \mathbf{x}_j^*$. Then we could try to redistribute some of these difference infinitesimally so that everyone is made better off relative to the $\mathbf{x}_i$ allocation. Let us take θ from $\mathbf{y}_j$, leaving her with $(1-\theta)\mathbf{y}_j$. So the new definition of $\mathbf{z}$ is,

$$\begin{aligned}\mathbf{z}_j &= (1-\theta)\mathbf{y}_j \\ \mathbf{z}_i &= \mathbf{x}_i^* + \frac{\theta}{n-1}\mathbf{y}_i \;\forall\; i \neq j\end{aligned}$$

However, by strong monotonicty, $\mathbf{z} \succ \mathbf{x}^*$, so that $\mathbf{z} \in P$.

$$\begin{aligned}\mathbf{p}\sum_{i=1}^{n}\mathbf{z}_i &\geq \mathbf{p}\sum_{i=1}^{n}\mathbf{x}_i^* \\ \mathbf{p}\left[(1-\theta)\mathbf{y}_j + \sum_{i\neq j}\mathbf{x}_i^* + \theta\mathbf{y}_j\right] &\geq \mathbf{p}\left[\mathbf{x}_j^* + \sum_{i\neq j}\mathbf{x}_i^*\right] \\ \mathbf{p}\mathbf{y}_j &\geq \mathbf{p}\mathbf{x}_j^*\end{aligned}$$

3. Finally, we show that this inequality hold with strict inequality. That means that we need to show that $\mathbf{p}\mathbf{y}_j = \mathbf{p}\mathbf{x}_j^*$ cannot hold. Proving by contradiction, suppose it does. By continuity of preferences, we can find another $\beta \in (0,1)$ such that $\beta'\mathbf{y}_j \succ \mathbf{x}_j^*$, which as before implies that $\beta\mathbf{p}\mathbf{y}_j \geq \mathbf{p}\mathbf{x}_j^*$. Further, since $\mathbf{x}_j^* > 0$, this implies that $\mathbf{p}\mathbf{x}_j^* > 0$. However, since $\mathbf{p}\mathbf{y}_j - \mathbf{p}\mathbf{x}_j^* = 0$, we must have $\beta\mathbf{p}\mathbf{y}_j < \mathbf{p}\mathbf{x}_j^*$ by continuity, which then leads to the contradiction.

■

The theorem can be alternatively stated and proved using revealed preference arguments.

Theorem 15 ***Second Theorem of Welfare Economics:*** *Let $\mathbf{x}^*$ be a Pareto efficient allocation, and the associated preferences be nonsatiated. Further, suppose that a competitive equilibrium exists from the initial endowments $\omega_i = \mathbf{x}_i^*$, and let it be $(\mathbf{p}', \mathbf{x}')$. Then, $(\mathbf{p}', \mathbf{x}^*)$ is a competitive equilibrium.*

Proof. By construction, $\mathbf{x}_i^*$ is in i's budget set, so that $\mathbf{x}_i' \succsim_i \mathbf{x}_i^*$. Since $\mathbf{x}^*$ is Pareto efficient, this implies that $\mathbf{x}_i^* \sim \mathbf{x}_i'$. In turn, this means that if $\mathbf{x}_i'$ is optimal, so too must $\mathbf{x}_i^*$. Thus $(\mathbf{p}', \mathbf{x}^*)$ is a Walrasian Equilibrium. ■

To summarize the proof argues that if a competitive equilibrium exists from a Pareto efficient allocation, then the Pareto efficient allocation itself is a competitive equilibrium for some distribution of endowments. In turn this requires mainly continuity of aggregate demand of the market, which in turn flows from either or both convexity of individual preferences, or a "large" economy.

With these technicalities out of the way, we can focus instead on the practicalities. For instance, what first order, or equilibrium conditions need to hold for market equilibria and Pareto efficiency? We first examine the market equilibrium and Pareto efficiency independently.

Theorem 16 ***Calculus characterization of equilibrium:*** *For a market equilibrium* $(\mathbf{x}^*, \mathbf{p}^*)$ *where each consumer holds a positive amount of every good, there exists a set of numbers* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *for each of the* n *individuals, such that,*

$$\nabla u_i(\mathbf{x}^*) \quad = \quad \alpha_i \mathbf{p}^*$$

Proof. Within a market equilibrium, every agent would have maximized on her budget set, and the above is just the first order conditions from the usual constrained optimization, where in turn α_i's are just the marginal utility of income. ■

Theorem 17 ***Calculus characterization of Pareto efficiency:*** *For a feasible allocation* $\mathbf{x}^*$*, it is Pareto efficient if and only if it solves the following constrained problem for each of the* n *individuals.*

$$\begin{aligned} \max_{x_i^g, x_j^g} \quad & u_i(\mathbf{x}_i) \\ \text{subject to} \quad & \sum_{h=1}^{n} x_h^g \leq \omega^g \qquad g = 1, \ldots, k \\ & u_j(\mathbf{x}_j^*) \leq u_j(\mathbf{x}_j) \quad i \neq j \end{aligned}$$

Proof. If $\mathbf{x}^*$ solves all the n maximization problems, but $\mathbf{x}^*$ is not Pareto efficient, then there is another allocation $\mathbf{x}'$ where everyone is better off. But this would contradict $\mathbf{x}^*$ being utility maximizing!

If instead $\mathbf{x}^*$ is Pareto efficient, but does not solve the utlity maximization problem for one of the individuals, then it would mean that there is another choice $\mathbf{x}'$ such that that

individual would be better off, and yet maintaining the wellbeing of the others, which in turn would contradict $\mathbf{x}^*$ being Pareto efficient. ■ Keep in mind that the above problem is for n individuals, with k goods, so that we have $k \times n$ choice variables to choose. Further, each program has $k + n - 1$ constraints. With that note, we can write the Lagrangian as,

$$\mathcal{L} = u_i(\mathbf{x}_i) - \sum_{g=1}^{k} \gamma^g \left(\sum_{h=1}^{n} x_h^g - \omega^g \right) - \sum_{i \neq j} \lambda_j (u_j(\mathbf{x}_j^*) - u_j(\mathbf{x}_j))$$

This then gives us these first order conditions,

$$\begin{aligned} \frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g} - \gamma^g &= 0 \text{ for } g = 1, \dots k \\ \lambda_j \frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g} - \gamma^g &= 0 \text{ for } i \neq j; j = 1, \dots, n; g = 1, \dots, k \end{aligned}$$

Noting that this are the same conditions for each $i = 1, \dots, n$,

$$\frac{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^g}}{\frac{\partial u_i(\mathbf{x}_i^*)}{\partial x_i^h}} = \frac{\gamma^g}{\gamma^h}$$

for $i = 1, \dots, n$ and $g, h = 1, \dots, k$. We can think of λ_j as essentially the weights allocated to the other individuals in individual i's program, and that we have implicitly set $\lambda_i = 1$, so that both of the prior first order conditions would accord.