Intermediate Microeconomics II, ECON 301
General Equilibrium I: Exchange

The equilibrium concepts you have used till now in your Introduction to Economics, and Intermediate Microeconomics I classes is what is known as partial equilibrium. As its name suggests, the equilibrium is achieved holding what occurs in other markets constant. This assumption would be correct if markets are not correlated with each other (which is true for some basket of markets, but not all. It all depends on the context in which we are thinking about). However, we know reality is far more complex than that. General Equilibrium attempts to circumvent that by examining how equilibrium is achieved in all markets simultaneously. For example, consider the effect of the extraction of the natural gas reserves off the east coast of Nova Scotia on the near by communities and business such as Antigonish, New Glasgow, and Guysborough.

We will be adopting 3 essential assumptions that will simplify our analysis. Nonetheless, the results are still applicable in a general case.

1. Assume that consumers and producers are in competitive markets, implying that all agents are price takers, and optimize given those prices.
2. Assume there are only two goods market with either two consumers or producers.
3. In our first discussion, we will first ignore production, and producers. Instead we will assume that the two consumers are each endowed (born with) a certain quantity of the two goods. We will then examine how they achieve equilibrium through trade with one another. This is what is typically termed a **Pure Exchange** economy.

1 Edgeworth Box

A useful tool in general equilibrium analysis is the **Edgeworth Box** used to analyze the trading of goods. Essentially, it merges the indifference map between the parties in the trade by inverting one of the agents diagram.

Setting up the study:

1. Let their be two agents, A & B, and two goods 1 & 2.
2. Denote A's and B's consumption bundle be $X_A = (x_A^1, x_A^2)$ & $X_B = (x_B^1, x_B^2)$ respectively.
3. Let A & B be endowed or born with an initial endowment of the two goods which we call the initial endowment, $\Omega_A = (\omega_A^1, \omega_A^2)$ & $\Omega_B = (\omega_B^1, \omega_B^2)$

Some key definitions:

- A pair of consumption bundles $X_A \& X_B$ is an **allocation**

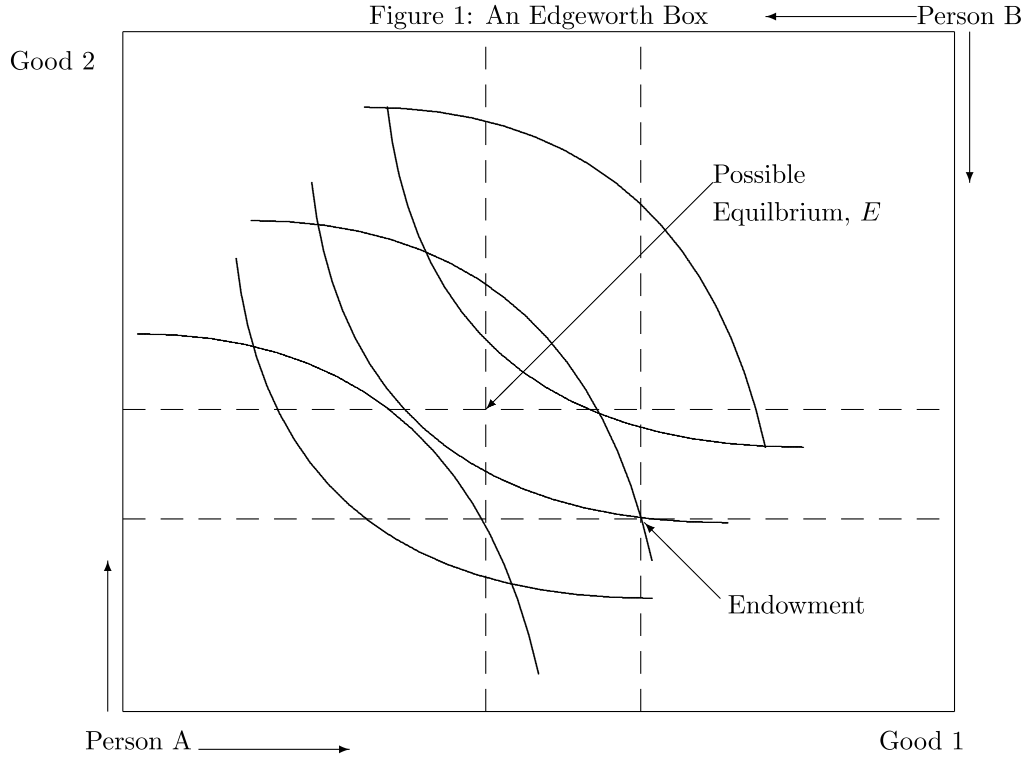
- An **allocation** is a **feasible allocation** $\Leftrightarrow$

$$x_A^1 + x_B^1 = \omega_A^1 + \omega_B^1$$

$$x_A^2 + x_B^2 = \omega_A^2 + \omega_B^2$$

An edgeworth box can now be drawn to illustrate how agents could arrive at a mutually beneficial consumption bundle:

Figure 1: An Edgeworth Box



The diagram neatly includes the endowment for both agent A & B. The set of rectangular hyperbolas that expand away from the origin are those of agent A's. Whereas the set of concave curves extending from the top right corner are those of B's. What do you think would be an equilibrium point at which both agents would be better off than they would have been if they had retained just their endowment levels of consumption? It should be noted that the beauty of the Edgeworth Box is that it neatly depicts all possible consumption bundles for both consumers under examination (i.e. all feasible allocations), as well as both preferences of both individuals.

2 Trade

The above diagram suggests a possible equilibrium point, but is that a unique point? Let's first examine the endowment point where each agent is initially at. We know that from Intermediate Microeconomics I, agent A & B are both indifferent to their endowment compared to another point along the indifference curve that passes the endowment point. Further we also know that all consumption bundles to the north-east of the indifference curve that passes through the endowment point yield a higher level of utility for the agent A. Similarly, all point to the south-west of the inverted indifference curve passing through the endowment point in the diagram is preferred to by agent B. This means there might be a set of points enveloped by the two indifference curves that passes through the endowment point that could make both agent A and B better off compared to their initial endowment.

Suppose our conjectured equilibrium E is indeed a possible equilibrium. It is easy to see that A could achieve it by trading her endowment of good 1 to agent B in return for her endowment of good 2. That is based on the diagram, agent A could reach a higher level of utility by consuming more good 2 than she is endowed with. Without trade, how consumption would lead to a level of utility measured by the indifference curve that passes her endowment only. With trade, she gets more of good 2 to consume, thereby making herself better off.

Let E be the point (x_A^1, x_A^2) for agent A, and (x_B^1, x_B^2) for agent B. Algebraically, agent A trades $|x_A^1 - \omega_A^1|$ for $|x_A^2 - \omega_A^2|$. While similarly, agent B trades $|x_B^2 - \omega_B^2|$ for $|x_B^1 - \omega_B^1|$. At this point, it is necessary that both indifference curves for A & B are tangent to each other, otherwise it is still possible for them to trade to another level within an area of a similar lens-like shape in figure 1, and where both of them are better off. Such an equilibrium is depicted in the diagram below, figure 2.

3 Pareto Efficient Allocations

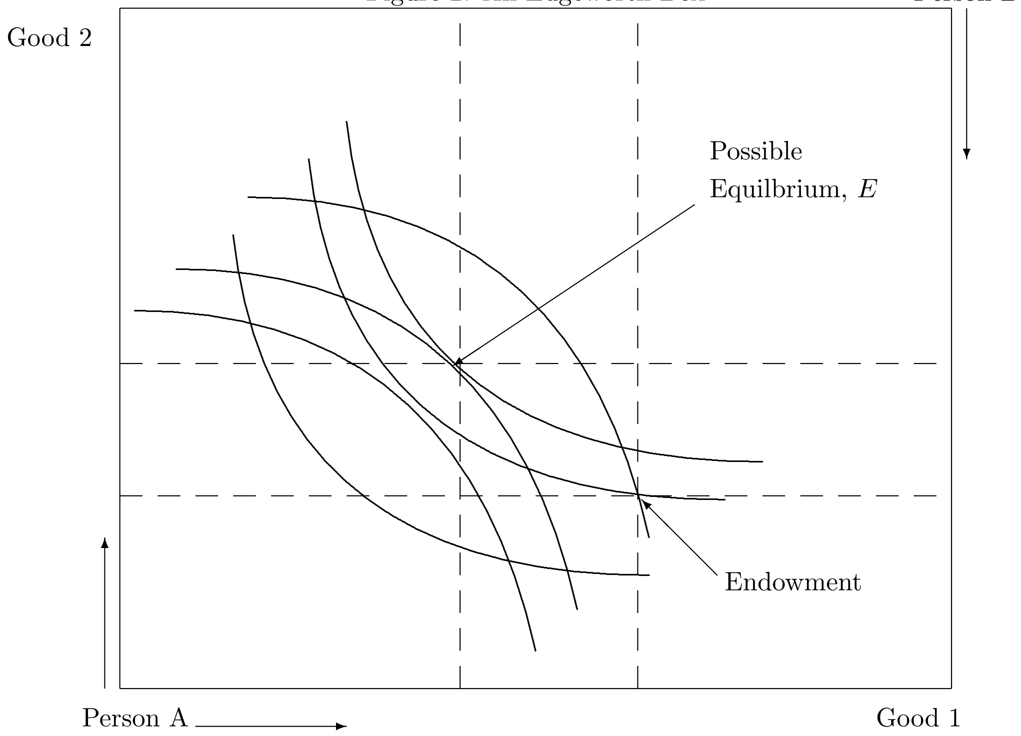
A equilibrium point such as E is a point where no one can be made better off without making the other party worse off. We call such an allocation a **Pareto Efficient Allocation**.

Specifically, a **Pareto Efficient Allocation** is an allocation where:

1. *No one can be made better off.*
2. *All gains from trade are exhausted*:This is the tangency of indifference curve condition.
3. *No way to make some one better off without making someone else worse off*: Basically, if we are at a tangency point, for us to raise the utility of one agent can only be achieved by reducing the utility of the other agent.
4. *There are no mutually advantageous trades to be made*: This essentially means that if you were to allocate a equilibrium to the agents, no one could unilaterally negotiate a better arrangement for everyone.

It should be obvious as well that a **Pareto Efficient** point suggested above is not unique. In fact, there is an infinite number of these points. The set of such points is called a **Pareto Set** or the **Contract Curve**. This curve will stretch from A's origin to that of B's. The **Pareto Set** describes all possible outcomes of mutually advantageous trade from starting anywhere in the Edgeworth Box. Note that although the relevant segment of the Pareto Set where we will find the Pareto Efficient solution is dependent on the initial endowment, thereby giving us a subset of the Contract curve, the Pareto Set is not dependent on initial endowment.

Figure 2: An Edgeworth Box

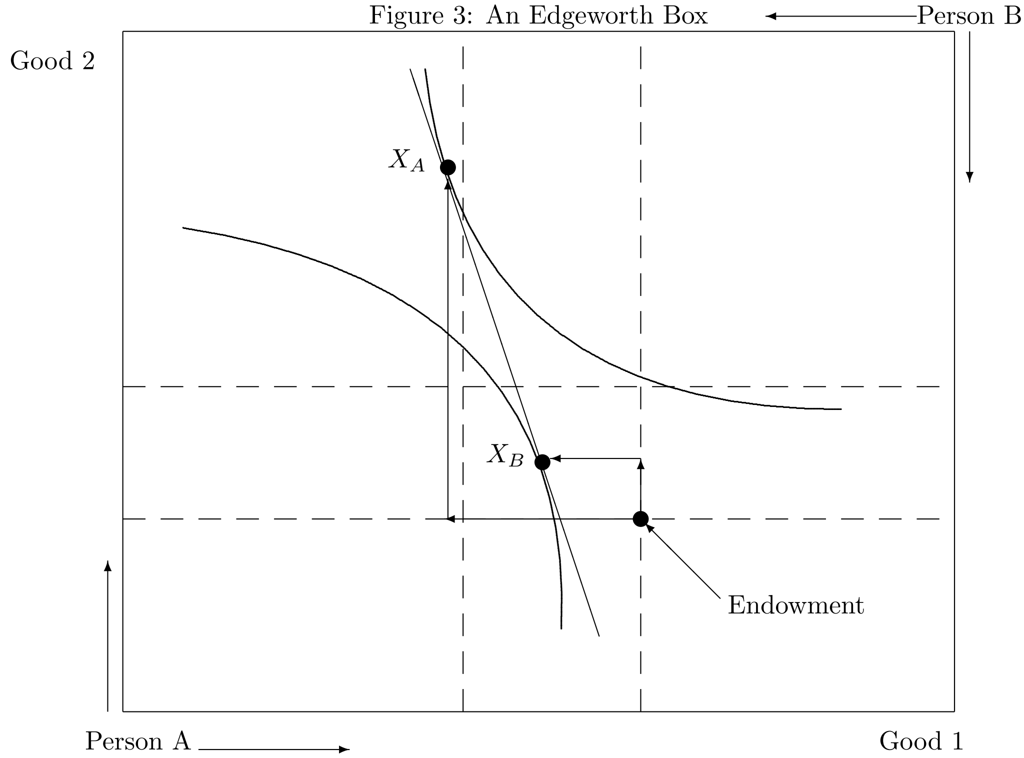


4 Market Trade

We will now discuss the process/mechanism by which an equilibrium such as E is derived. We can think of the process as either bargaining process where each individual agent given the demand for her endowment decides on a particular price competitively (If you find the idea that just two players could lead to a competitive structure, you could think consumer as being a group of consumers of a particular type), or you could imagine a social planner (higher authority or auctioneer) who sets the prices and tries to get both agents A & B to come to an equilibrium point on the **Contract Curve**.

You would recall that the slope of the individual's budget constraint, or the marginal rate of transformation is just the price ratio of the two goods for a consumer, $\frac{p_1}{p_2}$. In consumer theory, an individual finds her optimal consumption bundles when her indifference curve is tangent to her budget constraint. As you can imagine, not all price ratio would bring the two agents onto the contract curve as depicted in the diagram below.

Figure 3: An Edgeworth Box



As may be discerned from the diagram, an arbitrary set of prices (p_1, p_2) cannot guarantee that demand (aggregate demand of both agent A & B) equate with the supply (sum of the agent's individual endowment of the two goods). That is the excess demand (or net demand) of A(B) $e_A^i = x_A^i - \omega_A^i$ ($e_B^i = x_B^i - \omega_B^i$), where $i \in 1, 2$, need not be what the other agent B(A) wishes to sell. Or in terms of aggregate demand of the two agents, the total demand need not be equal to the total endowment of the two goods. In that sense, the above diagram shows an exchange market in disequilibrium. **Note that in equilibrium, the budget line must pass through the endowment point principally for the reason that the income of each agent is determined by her own endowment.**

The social planner or auctioneer would then have to recalibrate prices to the point where aggregate demand and supply of the market equates, i.e. when the indifference curves become tangent to each other at the going prices. That is the equilibrium would then be on the **Contract Curve**. Only then is the market in a competitive equilibrium. This equilibrium is also called **Walrasian Equilibrium**. It should further be noted that at equilibrium, due to the tangency of the indifference curves, the following must be true:

$$MRS_A = \frac{MU_A^1}{MU_A^2} = \frac{MU_B^1}{MU_B^2} = MRS_B = \frac{p_1}{p_2}$$

5 The Algebra of a General Equilibrium Solution

Let the demand function of each agent be $x^i_J(p_1, p_2)$, where $J \in \{A, B\}$ & $i \in \{1, 2\}$. Then what we mean when we say that aggregate demand must equate with aggregate supply in equilibrium is that:

$$x^1_A(p^*_1, p^*_2) + x^1_B(p^*_1, p^*_2) = \omega^1_A + \omega^1_B$$

$$x^2_A(p^*_1, p^*_2) + x^2_B(p^*_1, p^*_2) = \omega^2_A + \omega^2_B$$

We can rearrange the above equation so that we express it in terms of the sum of excess demand of the two agents.

$$[x^1_A(p^*_1, p^*_2) - \omega^1_A] + [x^1_B(p^*_1, p^*_2) - \omega^1_B] = 0$$

$$[x^2_A(p^*_1, p^*_2) - \omega^2_A] + [x^2_B(p^*_1, p^*_2) - \omega^2_B] = 0$$

The equation essentially says that in equilibrium, the sum of excess demand for both agents must be zero. Put another way, it says that what one agent demands will equate with what the other agent chooses to supply.

Let us denote excess demand as

$$e^i_J(p_1, p_2) = [x^i_J(p^*_1, p^*_2) - \omega^i_J]$$

and it essentially measures agent J's excess demand. Further let

$$z_i(p_1, p_2) = e^i_A(p_1, p_2) + e^i_B(p_1, p_2)$$

Then the equilibrium condition can be stated algebraically more succinctly as

$$z_i(p_1, p_2) = 0, \forall i \in \{1, 2\}$$

6 Walras' Law

Walras' Law says that

$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) \equiv 0$$

in words it says that the value of aggregate excess demand is identically zero. This means that for all prices (not just equilibrium prices), the value of aggregate excess demand is always zero.

To show this, first note that since demand for any of each agent must satisfy their budget constraint, the following must be true

$$p_1 x^1_A(p_1, p_2) + p_2 x^2_A(p_1, p_2) \equiv p_1 \omega^1_A + p_2 \omega^2_A$$

$$\Rightarrow p_1\{x^1_A(p_1, p_2) - \omega^1_A\} + p_2\{x^2_A(p_1, p_2) - \omega^2_A\} \equiv 0$$

$$\Rightarrow p_1 e^1_A(p_1, p_2) + p_2 e^2_A(p_1, p_2) \equiv 0$$

The same is true for agent B

$$\Rightarrow p_1 e^1_B(p_1, p_2) + p_2 e^2_B(p_1, p_2) \equiv 0$$

Summing both of the final identities yields the following,

$$p_1 e_A^1(p_1, p_2) + p_2 e_A^2(p_1, p_2) + p_1 e_B^1(p_1, p_2) + p_2 e_B^2(p_1, p_2) \equiv 0$$

$$\Rightarrow p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) \equiv 0$$

And Walras' Law is derived. Essentially, since the value of each agent's excess demand equals to zero, the sum of everyone's excess demand must still be zero.

Note that in the previous section, we noted that in equilibrium, the aggregate excess demand must be zero for each and every market (at equilibrium prices). This conclusion is in truth stronger than necessary. We can use Walras' Law to show this. The idea is the following, as long as $p_i > 0$ aggregate demand in any one of the markets at the equilibrium prices, 1 or 2, necessarily it would imply that the remaining market's aggregate excess demand must also be zero. Put another way, the significance of this is because it says that for given equilibrium prices that bring about equating of demand and supply in any one of the markets necessarily would imply that the remaining market must also have their aggregate demand equate with their aggregate supply.

More generally, what this says is that in an economy with n markets, we only need to find a set of prices where $n-1$ of the markets are in equilibrium, and Walras' Law says that the n^{th} market must also have demand and supply equate there too.

7 Relative Prices

The last statement essentially says that given an economy with n goods all we need to find are $n-1$ equilibrium prices. However, on further thought, how can it be possible that we could solve for n unknowns with $n-1$ equations. Essentially, the point is that we have only $n-1$ independent prices. This means that we can choose one of the prices as the normalizing price so that all other prices are measured relative to it. Suppose we decide to have the price of the n^{th} good to be the numeraire price (the normalizing price), all we need to do is to divide p_i, $i \in \{1, 2, ..., n-1\}$ by p_n. This is possible because by Walras' Law all markets would be in equilibrium for any set of prices, so that if $(p_1, p_2, ..., p_n)$ are the equilibrium prices, then for any constant $t \in R^+$, $(tp_1, tp_2, ..., tp_n)$ are also equilibrium prices that would bring about equilibrium in all markets.

8 Existence of Equilibrium

Although we could solve for these equilibrium prices given consumers demand, in truth we typically do not know the functional form of consumers' demand. In that case how do we know that there really is a set of prices that would really bring about equilibrium in all markets. This is the question the "**Existence of a Competitive Equilibrium**". Your basic algebra might lull you into believing that just because we have the same number of equations as there are unknowns once we have normalized the prices, a solution would exist. The fact is that that idea is not sufficient. The crucial assumption is however that we need the consumers' demand function to be continuous for existence.

What do we need then to have a continuous demand function?

1. As long as consumers have convex preferences, we will get continuous demand.
2. The more general condition is that even if individual demand for each consumer is discontinuous. As long as each consumer is only a small component of the market, the aggregate demand would be continuous, and the solution would exist.

9 The Algebra of Efficiency in General Equilibrium & Equilibrium and Efficiency

We have argued that the equilibrium is Pareto Efficient. We will show this to be true algebraically. We will do so by contradiction.

Let our soultion we found be (x_A^1, x_A^2) & (x_B^1, x_B^2). Suppose this equilibrium is not Pareto Efficient. This means that there is another solution for agents A and B which they prefer strictly, (y_A^1, y_A^2) & (y_B^1, y_B^2). That is,

$$(y_A^1, y_A^2) \succ_A (x_A^1, x_A^2)$$
$$(y_B^1, y_B^2) \succ_B (x_B^1, x_B^2)$$

Further, this new solution must also be feasible in the sense that what is demanded will be equal to what is supplied on the aggregate:

$$y_A^1 + y_B^1 = \omega_A^1 + \omega_B^1$$

$$y_A^1 + y_B^1 = \omega_A^1 + \omega_B^1$$

However, since the original solution is on the individual agents budget set, that is it is tangent to the budget constraint, if (y_A^1, y_A^2) and (y_B^1, y_B^2) are strictly preferred to the original solution, necessarily the following must be true (the new solution must be greater than the individual agent's income;

$$p_1 y_A^1 + p_2 y_A^2 > p_1 \omega_A^1 + p_2 \omega_A^1$$

$$p_1 y_B^1 + p_2 y_B^2 > p_1 \omega_B^1 + p_2 \omega_B^1$$

Which based on the feasibility condition implies that,

$$p_1\{y_A^1 + y_B^1\} + p_2\{y_A^2 + y_B^2\} > p_1\{\omega_B^1 + \omega_A^1\} + p_2\{\omega_A^1 + \omega_B^1\}$$

$$\Rightarrow p_1\{y_A^1 + y_B^1\} + p_2\{y_A^2 + y_B^2\} > p_1\{y_A^1 + y_B^1\} + p_2\{y_A^2 + y_B^2\}$$
$$\Rightarrow p_1\{\omega_B^1 + \omega_A^1\} + p_2\{\omega_A^1 + \omega_B^1\} > p_1\{\omega_B^1 + \omega_A^1\} + p_2\{\omega_A^1 + \omega_B^1\}$$

Which is a contradiction since no matter how you see it, both the right hand side and left hand side of the inequality are actually all the same. This gives us the following important theorem.

First Theorem of Welfare Economics: *All market equilibria or competitive equilibria are Pareto Efficient.*

The significance of the First Welfare Theorem is that it guarantees that a competitive equilibrium will exhaust all of the gains from trade, and be consequently Pareto Efficient. However, the First Welfare Theorem says nothing about the distribution of economic benefits among the agents of a society. That is it says nothing about the distributive effects of the allocation. Some would believe that all individuals are born equal, and consequently should have equal share of all resources and an economy's output. This is the common socialistic notion of "fairness". But if someone is born rich, is it fair for a social planner to level the playing field by reallocating their endowment?

10 Efficiency and Equilibrium

We now consider the arguement from efficiency to equilibrium, that is we ask the question "If we have a Pareto Efficient Allocation, can we find the set of prices that this equilibrium is also the competitive or market equilibrium?" The answer is as long as individual preferences are convex, it is always true, and this is known as the **Second Welfare Theorem**.

Second Theorem of Welfare Economics: *If all agents have convex preferences, then there will always be a set of prices such that each Pareto Efficient Allocation is a market equilibrium for an appropriate assignment of endowments.*

11 Implications of the First & Second Welfare Theorem

Both the welfare theorems have important messages about the ways we can allocate resources

1. **First Welfare Theorem:**

    - Our basic consumer theory assumes that all agents are in a sense selfish, and this has been extended here implicitly. If agents do care about each other, we have **consumption externality**, and its presence may imply that the competitive equilibrium needn't be Pareto Efficient.
    - We have implicitly assumed that each agent will behave competitively. But we have come to understand that competition is possible if everyone constitute small proportion of the entire market. So that if we take our result and model naively, competitive equilibrium may not be all that possible.

    Given the caveats, you should realize the conclusions of the theorem are rather strong. Its importance is that it provides for a mechanism (competition) to achieve Pareto Efficiency. The use of competitive markets essentially economizes on the information that any one agent needs to have, since the only thing she really needs to make her decision are the prices of the goods.

This then bodes well for competition as an important mechanism by which we can efficiently allocate resources.

2. **Second Welfare Theorem:**

   The importance of the Second Welfare Theorem says that the problems of efficiency and distribution can be separated. So what this means is that if you want a particular equilibrium because it is desirable, the competitive market will make it a reality. What do we really mean? Well, suppose you deem a particular initial endowment of resources as particularly appealing for an economy on account of your altruism. You could reallocate this initial endowment, and use prices to indicate relative scarcity and consequently justify the reallocation.

   What this theorem essentially says is that to achieve some distributive rationale, we do not necessarily have to transfer someone else's endowment, rather this could achieved through taxing one agent on account of her endowment and transferring the monies derived to the other more needly agent. That is we can transfer purchasing power. There is no sense in attempting to control prices since the virtues of a competitive equilibrium is achieved through prices attaining its desired role of "information" collection (Further isn't it a more direct approach if we were to just transfer money to someone needy, then through the convoluted mechanism that is the price mechanism?). Therefore as long as taxation is on an agent's level of endowment, there will be no loss in efficiency in the allocation. Only when taxation is dependent on consumer behavior will the agent's marginal cost of her choice be altered, and thereby leading to inefficiencies.

   Although it is true that the taxation of endowment will change behavior, based on the First Welfare Theorem, competitive forces will ensure that the market equilibrium attained given the new endowment allocation will be Pareto efficient.

   However, the actual process is more difficult than we have made it out to be. In truth, most of us are born into this world with nothing but ourselves. If we have good ways to value each other, we could then tax each of our true potential, "endowment". But it is not always possible. Ultimately, what is ultimately observable, such as taxing of labour supply which we may presume is highly dependent on our endowment, will distort our choices, and consequently result in inefficiencies.

   One way out of this problem is to have a lump sum tax. This tax is independent on endowment. However, with the monies received, the social planner could then transfer funding between differing agents, and it would be non-distortionary.

   To repeat the key points again:

   (a) Prices should be used to reflect relative scarcity of goods as opposed to attempting to use it as a distributive mechanism.

(b) Lump Sum transfer of wealth are non-distortionary, and consequently are the best tool to achieve distributive aims.

12 The Calculus of Pareto Efficient Allocations

We will now derive the equilibrium condition more formally. By definition, a Pareto efficient allocation is one where each agent is as well off as possible, given the utility of the other agent.

Let the two agents we are examining be agents A and B. In deriving a optimal choice for herself, agent A know that B would be agreeable if agent B gets at least the level of utility that agent B would have gotten had agent B remained at the level of her endowment. Let that utility level be $\overline{u}$. We can then write agent A's problem in the general equilibrium context as follows;

$$\max_{x_A^1, x_A^2, x_B^1, x_B^2} u_A(x_A^1, x_A^2)$$

such that

$$u_B(x_B^1, x_B^2) \geq \overline{u}$$
$$x_A^1 + x_B^1 \leq \omega^1$$
$$x_A^2 + x_B^2 \leq \omega^2$$

where $\omega^1 = \omega_A^1 + \omega_B^1$ and $\omega^2 = \omega_A^2 + \omega_B^2$. This is a constrained problem which we can change into an unconstrained problem by using the Lagrangian Method;

$$\max_{x_A^1, x_A^2, x_B^1, x_B^2, \lambda_1, \lambda_2, \lambda_3} L = u_A(x_A^1, x_A^2) + \left\{\lambda_1 \left\{\overline{u} - u_B(x_B^1, x_B^2)\right\} + \lambda_2 \left\{\omega^1 - x_A^1 + x_B^1\right\} + \lambda_3 \left\{\omega^2 - x_A^2 + x_B^2\right\}\right\}$$

The first order conditions for the problem are thus

$$\frac{\partial L}{\partial x_A^1} = \frac{\partial u_A}{\partial x_A^1} - \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_A^2} = \frac{\partial u_A}{\partial x_A^2} - \lambda_3 = 0$$
$$\frac{\partial L}{\partial x_B^1} = \lambda_1 \frac{\partial u_B}{\partial x_B^1} - \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_B^2} = \lambda_1 \frac{\partial u_B}{\partial x_B^2} - \lambda_3 = 0$$
$$u_B(x_B^1, x_B^2) = \overline{u}$$
$$x_A^1 + x_B^1 = \omega^1$$
$$x_A^2 + x_B^2 = \omega^2$$

The last three conditions are just a restatement of the constraints. Notice that the constraints has become equality, instead of the inequality. We know that in equilibrium, the feasibility conditions would be fully met, and that it is an optimal choice for agent A to just meet the minimum utility that agent B seeks, rather than a higher level of utility.

The key insights, particularly the equilibrium condition are obtained from the first four first order condition. The first two conditions yield the following;

$$MRS_A = \frac{\frac{\partial u_A}{\partial x_A^1}}{\frac{\partial u_A}{\partial x_A^2}} = \frac{\lambda_2}{\lambda_3}$$

Similarly, the third and fourth equations yield the following;

$$MRS_B = \frac{\frac{\partial u_B}{\partial x_B^1}}{\frac{\partial u_B}{\partial x_B^2}} = \frac{\lambda_2}{\lambda_3}$$

Equating the last two conditions yield what we have found through reasoning,

$$MRS_A = MRS_B$$

Recall that from your Intermediate Microeconomics I, equilibrium is characterized as MRS equating with the price ratio. Since both agents would be consuming up to the boundary of the budget constraint in the sense that the MRS would equate with the price ratio for both consumers, and since both agents face the same price, the full condition can be written thus as;

$$MRS_A = MRS_B = \frac{p_A}{p_B}$$

Note that the Lagrange Multipliers are acting just like the prices. They are in fact typically referred to as the shadow or efficiency prices.