Math 254 Winter 2015 Assignment 4

Section 6.4 2, 8, 18, 26, 28
Section 6.8 3b, d, 5

Section 6.4

2. $(p,g) = p(0)\,g(0)$ p, g in P_1

Inner product?

$p(x) = a_0 + a_1 x$ $p(0) = a_0$ $g(x) = b_0 + b_1 x$ $g(0) = b_0$

2

So $p(0)g(0) = a_0 b_0$ $(p,p) = a_0^2$

Could have $p(0)p(0) = 0$ but $p(x) \neq 0$

ex $p(x) = 2x$ then $(p,p) = 0$ but $p(x) \neq 0$

NOT an innerproduct

8. $\{(0,1,0), (1,0,1), (0,0,1)\}$ orthogonal? orthonormal?

$\quad v_1 \qquad v_2 \qquad v_3$

2

$v_1 \cdot v_2 = 0 + 0 + 0 = 0$ ✓
$v_1 \cdot v_3 = 0 + 0 + 0 = 0$ ✓
$v_2 \cdot v_3 = 0 + 0 + 1 = 1 \neq 0$ Not orthogonal, not orthonormal

18. $x = (-1, 0, 1)$ as LC of $v_1 = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$, $v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, $v_3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

2

$X = c_1 v_1 + c_2 v_2 + c_3 v_3$ $c_3 = x \cdot v_3 = -\frac{2}{3} + \frac{1}{3} = \frac{-1}{3}$

where $c_1 = x \cdot v_1 = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$

$c_2 = x \cdot v_2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

Check: $\frac{1}{3}\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) + \frac{4}{3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = (-1, 0, 1)$ ✓

26. $\{\overset{u_1}{(1,1,1,0)}, \overset{u_2}{(1,-1,1,-1)}, \overset{u_3}{(1,0,1,1)}, \overset{u_4}{(0,1,1,1)}\}$

Use GS to find orthonormal basis

$$v_1 = u_1 = (1,1,1,0)$$

$$(u_2, v_1) = 1 - 1 + 1 = 1$$
$$(v_1, v_1) = 1+1+1 = 3$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1$$

$$= (1,-1,1,-1) - \tfrac{1}{3}(1,1,1,0) = \left(\tfrac{2}{3}, -\tfrac{4}{3}, \tfrac{2}{3}, -1\right)$$

check: $v_1 \circ v_2 = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} + 0 = 0$ ✓

$$v_3 = u_3 - \frac{(u_3, v_1)}{(v_1, v_1)} v_1 - \frac{(u_3, v_2)}{(v_2, v_2)} v_2$$

$$(u_3, v_1) = 1 + 1 = 2$$
$$(u_3, v_2) = \tfrac{2}{3} + \tfrac{2}{3} - 1 = \tfrac{1}{3}$$
$$(v_2, v_2) = \tfrac{4}{9} + \tfrac{16}{9} + \tfrac{4}{9} + \tfrac{9}{9} = \tfrac{33}{9} = \tfrac{11}{3}$$

$$= (1,0,1,1) - \tfrac{2}{3}(1,1,1,0) - \tfrac{1}{11}\left(\tfrac{2}{3}, -\tfrac{4}{3}, \tfrac{2}{3}, -1\right)$$

$$= \left(\tfrac{33}{33}, 0, \tfrac{33}{33}, \tfrac{33}{33}\right) - \left(\tfrac{22}{33}, \tfrac{22}{33}, \tfrac{22}{33}, 0\right) - \left(\tfrac{2}{33}, -\tfrac{4}{33}, \tfrac{2}{33}, -\tfrac{3}{33}\right)$$

$$= \left(\tfrac{9}{33}, -\tfrac{18}{33}, \tfrac{9}{33}, \tfrac{36}{33}\right) = \left(\tfrac{3}{11}, -\tfrac{6}{11}, \tfrac{3}{11}, \tfrac{12}{11}\right)$$

check $v_1 \circ v_3 = (1,1,1,0) \circ \left(\frac{3}{11}, -\frac{6}{11}, \frac{3}{11}, \frac{12}{11}\right) = 0$ ✓

$v_2 \circ v_3 = \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}, -1\right) \circ \left(\frac{3}{11}, -\frac{6}{11}, \frac{3}{11}, \frac{12}{11}\right) = \frac{6}{33} + \frac{24}{33} + \frac{6}{33} - \frac{36}{33} = 0$ ✓

$$v_4 = u_4 - \frac{(u_4, v_1)}{(v_1, v_1)} v_1 - \frac{(u_4, v_2)}{(v_2, v_2)} v_2 - \frac{(u_4, v_3)}{(v_3, v_3)} v_3$$

$(u_4, v_1) = (0,1,1,1) \cdot (1,1,1,0) = 2$

$(u_4, v_2) = (0,1,1,1) \circ \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}, -1\right) = -\frac{4}{3} + \frac{2}{3} - \frac{3}{3} = -\frac{5}{3}$

$(u_4, v_3) = (0,1,1,1) \circ \left(\frac{3}{11}, -\frac{6}{11}, \frac{3}{11}, \frac{12}{11}\right) = -\frac{6}{11} + \frac{3}{11} + \frac{12}{11} = \frac{9}{11}$

6

$$(v_3, v_3) = \frac{9}{121} + \frac{36}{121} + \frac{9}{121} + \frac{144}{121} = \frac{198}{121} = \frac{18}{11}$$

③

$$v_4 = (0,1,1,1) - \frac{2}{3}(1,1,1,0) + \frac{+5/\cancel{3}}{11/\cancel{3}}\left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}, -1\right) - \frac{9/\cancel{11}}{\cancel{18}/\cancel{11}_{2}}\left(\frac{3}{11}, \frac{-6}{11}, \frac{3}{11}, \frac{12}{11}\right)$$

$$= \cancel{(0,0,0,0)}$$

$$= \left(0, \frac{33}{33}, \frac{33}{33}, \frac{33}{33}\right) - \left(\frac{22}{33}, \frac{22}{33}, \frac{22}{33}, 0\right) + \left(\frac{10}{33}, \frac{-20}{33}, \frac{10}{33}, -\frac{15}{33}\right) - \left(\frac{3}{22}, \frac{-6}{22}, \frac{3}{22}, \frac{12}{22}\right)$$

$$= \left(-\frac{12}{33}, \frac{-9}{33}, \frac{21}{33}, \frac{18}{33}\right) - \left(\frac{3}{22}, \frac{-6}{22}, \frac{3}{22}, \frac{12}{22}\right)$$

$$= \left(-\frac{4}{11}, -\frac{3}{11}, \frac{7}{11}, \frac{6}{11}\right) - \left(\frac{3}{22}, -\frac{6}{22}, \frac{3}{22}, \frac{12}{22}\right)$$

$$= \left(-\frac{8}{22}, \frac{-6}{22}, \frac{14}{22}, \frac{12}{22}\right) - \left(\frac{3}{22}, \frac{-6}{22}, \frac{3}{22}, \frac{12}{22}\right)$$

$$= \left(\frac{-11}{22}, 0, \frac{11}{22}, 0\right) = \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right)$$

check: $v_1 \cdot v_4 = (1,1,1,0) \cdot \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right) = 0$ ✓

$v_2 \cdot v_4 = \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}, -1\right) \cdot \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right) = 0$ ✓

$v_3 \cdot v_4 = \left(\frac{3}{11}, \frac{-6}{11}, \frac{3}{11}, \frac{12}{11}\right) \cdot \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right) = 0$ ✓ phew!

Normalize:

$\|v_1\|^2 = 3 \rightarrow \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$

$\|v_2\|^2 = \frac{11}{3} \rightarrow \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{11}}\left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}, -1\right)$

$\|v_3\|^2 = \frac{18}{11} \rightarrow \frac{v_3}{\|v_3\|} = \frac{3\sqrt{2}}{\sqrt{11}}\left(\frac{3}{11}, \frac{-6}{11}, \frac{3}{11}, \frac{12}{11}\right)$

$\|v_4\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ so $\frac{v_4}{\|v_4\|} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{2}, 0\right)$

28. $S = \{1, x, x^2\}$ $\quad [a,b] = [0,1]$

$\quad\;\; u_1, u_2, u_3$

$(f,g) = \int_0^1 f(x)g(x)\,dx$

$V_1 = u_1 = 1$ $\qquad (u_2, v_1) = (x, 1) = \int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$

$V_2 = u_2 - \frac{(u_2, V_1)}{(V_1, V_1)} V_1$ $\qquad (v_1, v_1) = (1,1) = \int_0^1 1\,dx = x\Big]_0^1 = 1$ $\quad 5$

$\cancel{4}$

$V_2 = x - \frac{1}{2}(1) = x - \frac{1}{2}$

check: $(V_1, V_2) = (1, x - \frac{1}{2}) = \int_0^1 x - \frac{1}{2}\,dx = \frac{x^2}{2} - \frac{1}{2}x\Big]_0^1 = 0$ ✓

$V_3 = u_3 - \frac{(u_3, V_1)}{(V_1, V_1)} V_1 - \frac{(u_3, V_2)}{(V_2, V_2)} V_2$ $\qquad (u_3, v_1) = (x^2, 1) = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$

$(u_3, v_2) = \int_0^1 x^3 - \frac{x^2}{2}\,dx = \frac{x^4}{4} - \frac{x^3}{6}\Big]_0^1 = \frac{1}{12}$

$(V_2, V_2) = (x - \frac{1}{2}, x - \frac{1}{2}) = \int_0^1 x^2 - x + \frac{1}{4}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\Big]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$

$V_3 = x^2 - \frac{1}{3}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}(x - \frac{1}{2}) = x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$

check: $(V_1, V_3) = (1, x^2 - x + \frac{1}{6}) = \int_0^1 x^2 - x + \frac{1}{6}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}\Big]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$ ✓

$(V_2, V_3) = \int_0^1 (x - \frac{1}{2})(x^2 - x + \frac{1}{6})\,dx = \int_0^1 \left(x^3 - \frac{3}{2}x^2 + \frac{2}{3}x - \frac{1}{12}\right)dx = \frac{x^4}{4} - \frac{\cancel{3}x^3}{2\cancel{3}} + \frac{\cancel{2}x^2}{3\cancel{2}} - \frac{1}{12}\Big]_0^1$

$= \frac{1}{4} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12} = \frac{3 - 6 + 4 - 1}{12} = 0$ ✓

& normalize

~~20~~2 (5)

$\|v_1\| = 1$ so $w_1 = 1$

$\|v_2\|^2 = \frac{1}{12}$ so $w_2 = \sqrt{12}(x - \frac{1}{2})$

$$\|v_3\|^2 = \int_0^1 (x^2 - x + \tfrac{1}{6})(x^2 - x + \tfrac{1}{6})\,dx = \int_0^1 x^4 - x^3 + \tfrac{1}{6}x^2 - x^3 + x^2 - \tfrac{1}{6}x + \tfrac{x^2}{6} - \tfrac{x}{6} + \tfrac{1}{36}\,d$$

$$= \int_0^1 x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{x}{3} + \tfrac{1}{36}\,dx = \frac{x^5}{5} - \frac{2x^4}{4} + \frac{4}{3}\frac{x^3}{3} - \frac{x^2}{6} + \frac{x}{36}\Big]_0^1$$

$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{36 - 90 + 80 - 30 + 5}{180} = \frac{1}{180}$$

so $w_3 = \sqrt{180}\left(x^2 - x + \frac{1}{6}\right)$

#5 - easier to distribute resources over branched networks, self similarity helps growth etc

~~0~~1

Section 6.8

3b $\underline{u} = (-2,1)$

$v = (1,2)$ $v' = (-2,1)$

Find orthogonal projection of u onto v & v' 2

$$\text{proj}_v u = \frac{(u,v)}{(v,v)}v = \frac{(-2,1)\cdot(1,2)}{(1,2)\cdot(1,2)}(1,2) = 0$$

makes sense since $u = v'$

$$\text{proj}_{v'} u = \frac{(-2,1)\cdot(-2,1)}{(-2,1)\cdot(-2,1)}(-2,1) = (-2,1)$$

3d $u = (3,-7)$

$(v,v) = 1^2 + 2^2 = 5$

$(v',v') = 4+1 = 5$

$$\text{proj}_v u = \frac{(3,-7)\cdot(1,2)}{5}(1,2) = \left(\frac{3-14}{5}\right)(1,2) = \frac{-11}{5}(1,2) = \left(\frac{-11}{5}, \frac{-22}{5}\right)$$ 2

$$\text{proj}_{v'} u = \frac{(3,-7)(-2,1)}{5}(-2,1) = \left(\frac{-6-7}{5}\right)(-2,1) = \frac{-13}{5}(-2,1) = \left(\frac{+26}{5}, \frac{-13}{5}\right)$$