Math 254 Winter 2015 Assignment 6 (Optional) SOLUTIONS (1)

Section 7.7 1, 3 Section 8.1 6, 16 (#8 of 4.4) 29 (#6) Section 8.2 4, 6, 16, 22, 26

Section 7.7

1. Show that whether or not the affine function $S(\underline{x}) = A\underline{x} + \underline{b}$ is a contraction does not depend on $\underline{b}$

need to check $\| S(x) - S(y) \| = \| (A\underline{x} + b) - (A\underline{y} + b) \| = \| A\underline{x} - A\underline{y} \|$

the $\underline{b}$'s cancel out so doesn't depend on $\underline{b}$

3. For each, is S a contraction mapping? If so, find contraction factor

from #1 $\underline{b}$ doesn't matter

a) $A = \begin{bmatrix} 1/3 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Let $\underline{x} = (x_1, y_1)$ $\underline{y} = (x_2, y_2)$

$$\| A\underline{x} - A\underline{y} \| = \left\| \left(\frac{x_1}{3}, \frac{y_1}{4}\right) - \left(\frac{x_2}{3}, \frac{y_2}{3}\right) \right\| = \sqrt{\left(\frac{x_1 - x_2}{3}\right)^2 + \left(\frac{y_1 - y_2}{4}\right)^2}$$

$$\leq \sqrt{\left(\frac{x_1 - x_2}{3}\right)^2 + \left(\frac{y_1 - y_2}{3}\right)^2}$$

since $\frac{1}{16} \leq \frac{1}{9}$

$= \frac{1}{3} \| \underline{x} - \underline{y} \|$ so yes, factor is $\frac{1}{3}$

b) $A = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ $b = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$

$$\| Ax - Ay \| = \left\| \left(\frac{-y_1}{2}, \frac{x_1}{2}\right) - \left(\frac{-y_2}{2}, \frac{x_2}{2}\right) \right\| = \sqrt{\left(\frac{-y_1 + y_2}{2}\right)^2 + \left(\frac{x_1 - x_2}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_1 - x_2}{\ }\right)^2 + \left(\frac{y_1 - y_2}{\ }\right)^2}$$

$= \frac{1}{2}\|x - y\|$ So yes, contraction factor is $\frac{1}{2}$

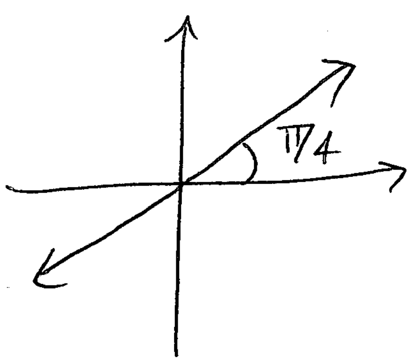
c) $A = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\|Ax - Ay\| = \left\|\left(\left(\frac{4x_1}{3} - \frac{4x_2}{3}\right), \left(\frac{y_1}{4} - \frac{y_2}{4}\right)\right)\right\| = \sqrt{\left(\frac{4}{3}\right)^2 (x_1 - x_2)^2 + \left(\frac{1}{4}\right)^2 (y_1 - y_2)^2}$

$\frac{4}{3} > 1$ so NOT a contraction

Section 8.1

6. The transformation $T^{\pi/4} : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects every vector x in $\mathbb{R}^2$ in the line $y = x$



$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ $\theta = \pi/4$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so $(x, y) \to (y, x)$

$\det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

so $\lambda = 1, \lambda = -1$ are eigenvalues

$\lambda = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ So $-x + y = 0 \Rightarrow y = x$

eigenvectors of form (x, x) vectors along line $y = x$

$\lambda = -1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so $x + y = 0 \to y = -x$

eigenvectors of form $(x, -x)$

vectors on line $y = -x$

16 (for #8 of 4.4)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 1-\lambda & 2 \\ 2 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 2 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 2\right] + 0 - 1\left(1 - 2(1-\lambda)\right)$$

$$= (1-\lambda)\left[1 - 2\lambda + \lambda^2 - 2\right] - (-1 + 2\lambda) = (1-\lambda)(\lambda^2 - 2\lambda - 1) - (2\lambda - 1)$$

$$= \cancel{\lambda^2} - 2\lambda - 1 - \cancel{\lambda^3} + \cancel{2\lambda^2} + \cancel{\lambda} - 2\cancel{\lambda} + 1 = -\lambda^3 + 3\lambda^2 - 3\lambda$$

$$= -\lambda\underbrace{(+\lambda^2 - 3\lambda + 3)}$$

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(3)}}{2(3)}$$ irreducible

So just $\lambda = 0$ is root

$A - 0I$ $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

let $z = r$ free parameter

$x - z = 0 \to x = z = r$

$y + 3z = 0 \to y = -3z = -3r$

vectors of form $(\cancel{x}, y, z) = (r, -3r, r)$

so basis $\{(1, -3, 1)\}$

27 for #6 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

find characteristic polynomial of A

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 2-\lambda \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix} + 0$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 1\right] + \left(-1(2-\lambda) - 0\right)$$

$$= (2-\lambda)\left[4 - 4\lambda + \lambda^2 - 1\right] + -2 + \lambda$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 3) + \lambda - 2 = 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + \lambda - 2$$

$$= -\lambda^3 + 6\lambda^2 - 10\lambda + 4$$

Now apply to A. $A^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & -14 & 6 \\ -14 & 20 & -14 \\ 6 & -14 & 14 \end{bmatrix}$$

so $-A^3 + 6A^2 - 10A + 4I$

$$= \begin{bmatrix} -14 & 14 & -6 \\ 14 & -20 & 14 \\ -6 & 14 & -14 \end{bmatrix} + \begin{bmatrix} 30 & -24 & 6 \\ -24 & 36 & -24 \\ 6 & -24 & 30 \end{bmatrix} - \begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 20 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

Section 8.2 4, 6, 16, 22, 26

4. $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ d'ble?

find eigenvalues $\begin{vmatrix} 0-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$ $= (-\lambda)(2-\lambda) + 1$

$= -2\lambda + \lambda^2 + 1$

$= \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

$\lambda = 1$

eigenvectors $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ has just $-x + y = 0$

$x = y$ so (x, x)

eigenspace basis $\{(1,1)\}$

only one basis vector so NOT diagonalizable.

6. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ Same as from 8.1 #27 question

characteristic polynomial is $-\lambda^3 + 6\lambda^2 - 10\lambda + 4$

$= \cancel{(\lambda-2)(4\lambda^2 - 10\lambda + 4)} = (\lambda-2)(-\lambda^2 + 4\lambda - 2)$

$= -(\lambda-2)(\lambda^2 - 4\lambda + 2)$ $\lambda = 2, \underbrace{2+\sqrt{2}, 2-\sqrt{2}}$

didn't know eigenvalues would be so complicated!

Don't worry about finding P for these eigenvalues

Diagonal matrix is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$

16. Is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ orthogonal? (6)

No. column 1 $\cdot$ column 3 $= 1$ $\times$ should be 0

(need column vectors to be orthonormal)

22. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ char poly $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$$= -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$= -\lambda\left[-\lambda(1-\lambda) - 0\right] - (1-\lambda-0) = -\lambda(-\lambda+\lambda^2) - 1 + \lambda$$

$$= \lambda^2 - \lambda^3 - 1 + \lambda = -\lambda^3 + \lambda^2 + \lambda - 1. = (\lambda-1)(-\lambda^2+1) = -(\lambda-1)(\lambda-1)(\lambda+$$

$\lambda = 1$: $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ z is free
y is free $\quad -x+y=0 \rightarrow x=y$
So eigenspace basis $\{(1,1,0), (0,0,1)\}$

$\lambda = -1$ $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $z=0 \quad x=-y$
eigenspace basis $\{(-1,1,0)\}$

normalize columns to get $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

then $P^{-1}AP = P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

#26 $\det(A^TA) = \det A^T \det A$
$= \det A \det A = (\det A)^2$
Need $(\det A)^2 = \det I = 1$
So $\det A = 1$ or -1