Important Definitions and Results for Linear Algebra

Last updated: April 8, 2015

Chapter 5

- A **linear combination** (LC) of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n$ in $\mathbb{R}^m$ is an expression of the form

$$c_i\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

  where the c_i are scalars.

- A set of vectors is **linearly independent** (LI) if the vector equation

$$c_i\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

  has only the trivial solution (all $c_i = 0$). If a set of vectors is not linearly independent, then it is **linearly dependent** (LD). That is, the vector equation has a non-trivial solution where some $c_i \neq 0$.

- **Theorem.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Then S is a linearly dependent set if and only if one of the vectors is a linear combination of the others.

- An **elementary vector in** $\mathbb{R}^m$ is a vector that has one component equal to 1 and all other components equal to 0.

- **Theorem.** Let S be a set of n vectors in $\mathbb{R}^m$ with $n > m$. Then S is linearly dependent.

- **Theorem.** Let S be a set of m vectors in $\mathbb{R}^m$, and let A be the matrix whose columns are the vectors. Then S is linearly independent if and only if $\det A \neq 0$.

- **Theorem.** Let S be a set of m vectors in $\mathbb{R}^m$, and let A be the matrix whose columns are the vectors. Let B be the row-reduced echelon form of A. Then the set S is linearly independent if and only if the columns are the elementary vectors.

- **Theorem.** Let S be a set of m vectors in $\mathbb{R}^m$, and let A be the matrix whose columns are the vectors. Let B be the row-reduced echelon form of A. Suppose the jth column of B is not an elementary vector. Then S is linearly dependent and $\mathbf{v}_j = b_{1j}\mathbf{w}_1 + \cdots + b_k\mathbf{w}_{kj}$, where $\mathbf{w}_i, \cdots, \mathbf{w}_k$ are the vectors in S that have been transformed to elementary vectors that precede the jth column.

- A **subspace** of $\mathbb{R}^m$ is a nonempty set $\mathbf{S}$ of vectors satisfying the following two conditions:

  1. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are both in $\mathbf{S}$, then $\mathbf{v}_1 + \mathbf{v}_2$ is in $\mathbf{S}$
  2. If $\mathbf{v}$ is in $\mathbf{S}$, then $c\mathbf{v}$ is in $\mathbf{S}$ for any scalar c.

- **Theorem.** Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^m$. The set of all linear combinations, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, of the vectors in T, is a subspace of $\mathbb{R}^m$.

- Let $\mathbf{S}$ be a subspace of $\mathbb{R}^m$. A subset T of vectors from $\mathbf{S}$ **spans** $\mathbf{S}$ if every vector in $\mathbf{S}$ can be written as a linear combination of elements of T. T is a **spanning set** for $\mathbf{S}$.

- **Theorem.** Let $\mathbf{S}$ be the subspace of $\mathbb{R}^m$ spanned by the set $T = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. If one of the vectors in T is a linear combination of the others, then the subset of T obtained by deleting that vector still spans $\mathbf{S}$.

- Let $\mathbf{S}$ be a subspace of $\mathbb{R}^m$. A set T of vectors in $\mathbf{S}$ is a **basis** for $\mathbf{S}$ if

  1. T is linearly independent
     AND
  2. T spans $\mathbf{S}$

- **Theorem.** Let $\mathbf{S}$ be the subspace of $\mathbb{R}^m$ generated by the set $T = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Then there is a subset of T that is a basis for $\mathbf{S}$.
- **Theorem.** Let $\mathbf{S}$ be the subspace of $\mathbb{R}^m$ generated by the set $T = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, and let $T_1 = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_s\}$ be a linearly independent subset of $\mathbf{S}$. Then there is a subset T_2 of T such that the union of T_1 and T_2 is a basis for $\mathbf{S}$.
- **Theorem.** Let $\mathbf{S}$ be a subspace of $\mathbb{R}^m$. Then any basis for $\mathbf{S}$ contains the same number of vectors as any other basis.
- Let $\mathbf{S}$ be a subspace of $\mathbb{R}^m$. The **dimension** of $\mathbf{S}$ is equal to the number of vectors in any basis for $\mathbf{S}$.
- **Theorem.** Let A be an $m \times n$ matrix, and let B be its row-reduced echelon form. Let the rank of A (the number of nonzero rows of B) be equal to r. Then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$.
- **Theorem.** Let $\mathbf{S}$ be the subspace of $\mathbb{R}^m$ with dimension n, and let T be a subset of $\mathbf{S}$.
  1. If the number of vectors in T is greater than n, then T is not linearly independent.
  2. If the number of vectors is less than n, then T does not span $\mathbf{S}$.
  3. If the number of vectors in T is equal to n, then T is linearly independent if and only if T spans $\mathbf{S}$.
- **Theorem.** Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ be a set of vectors in $\mathbb{R}^m$. Then T is a basis for $\mathbb{R}^m$ if and only if $\det A \neq 0$, where A is the corresponding $m \times m$ matrix.
- Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ generated by the columns of A is called the **column space** of A. Its dimension is called the **column rank** of A.
- Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^n$ generated by the rows of A is called the **row space** of A. Its dimension is called the **row rank** of A.
- **Theorem.** Column rank = row rank = rank.
- To find bases the column and row spaces, first row reduce A of B. Find the leading ones of B. The corresponding columns of the original matrix A form a basis for the column space and the rows of B form a basis for the row space. Note that you could alternately find a basis for the row space by looking at A^T.

Chapter 6

- A **vector space** is a non-empty set V of objects called vectors that can be added and multiplied by scalars in special ways. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and any scalars c and d,
  1. (Closure under addition) $\mathbf{u} + \mathbf{v} \in V$.
  2. (Closure under scalar multiplication) $c\mathbf{v} \in V$.
  3. (Commutativity of addition) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
  4. (Associativity of addition) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
  5. (Zero) There is an element denoted $\mathbf{0}$ in V and called the *zero vector* with the property that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in V$.
  6. (Negatives) There exists a vector called the *negative* of $\mathbf{v}$, denoted by $-\mathbf{v}$, with the property that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  7. (Scalar associativity) $c(d\mathbf{v}) = (cd)\mathbf{v}$.
  8. (Distributivity) $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ and $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

9. (One) $1\mathbf{v} = \mathbf{v}$.

- **Theorem.** Let $\mathbf{V}$ be a vector space, let $\mathbf{v} \in \mathbf{V}$, and let $c \in \mathbb{R}$. Then
  1. $0\mathbf{v} = \mathbf{0}$
  2. $(-1)\mathbf{v} = -\mathbf{v}$
  3. $c\mathbf{0} = \mathbf{0}$
- Let $\mathbf{V}$ be a vector space, and let $\mathbf{W}$ be a subset. Then $\mathbf{W}$ is a **subspace** of $\mathbf{V}$ if $\mathbf{W}$ is a vector space using the operations of $\mathbf{V}$.
- **Theorem.** Let $\mathbf{V}$ be a vector space, and let $\mathbf{W}$ be a non-empty subset. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$ if and only if $\mathbf{W}$ is closed under addition and scalar multiplication.
- Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a set of vectors in a vector space $\mathbf{V}$. The set S is **linearly independent** if the equation
$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0}$$
has only the trivial solution (all $c_i = 0$). An infinite set is linearly independent if every finite subset is linearly independent. If S is not linearly independent, then it is **linearly dependent**.
- Let $\mathbf{V}$ be a vector space. A set S of vectors in $\mathbf{V}$ **spans** (or **generates**) $\mathbf{V}$ if every vector in $\mathbf{V}$ can be expressed as a linear combination of finite subset of S.
- Let $\mathbf{V}$ be a vector space. A set S of vectors in $\mathbf{V}$ is a **basis** for $\mathbf{V}$ if
  1. S is linearly independent
  2. S spans $\mathbf{V}$
- **Theorem.** Each of the following is a basis for their respective vector space:
  1. The set $\{1, x, x^2, \cdots, x^n\}$ for $\mathbf{P}_n$
  2. The set $\{1, x, x^2, \cdots\}$ for $\mathbf{P}$
  3. The set $\{E^{ij} | i = 1, .., m, j = 1, .., n\}$ where E^{ij} has a 1 in entry ij and zeroes everywhere else.
- **Theorem.** Let $\mathbf{V}$ be a vector space spanned by the set T. If one of the vectors in T is a linear combination of the others, then the set obtained by deleting that vector still spans.
- **Theorem.** Let $\mathbf{V}$ be a vector space spanned by a set T. Then there is a subset of T that is a basis.
- **Theorem.** Let $\mathbf{V}$ be a vector space spanned by the set T and let T_1 be a linearly independent subset of $\mathbf{V}$. Then there is a subset T_2 of T such that the union of T_1 and T_2 is a basis for $\mathbf{V}$.
- **Theorem.** Let $\mathbf{V}$ be a vector space with bases B and B'. Then B and B' have the same number of elements.
- The **dimension** of a vector space that has a finite spanning set is equal to the number of vectors in a basis.
- **Theorem.**
  1. $\mathbf{P}_n$ has dimension $n + 1$.
  2. $\mathbf{P}$ is infinite dimensional.
  3. $M^{n,m}$ has dimension nm.
- **Theorem.** Let S be a subset of a vector space $\mathbf{V}$ of dimension n.
  1. If S contains fewer than n vectors, then it doesn't span.

2. If S contains more than n vectors, then it is not linearly independent.
3. If S contains n vectors and is either linearly independent or spans then it is a basis.

- Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ be a basis for a vector space $\mathbf{V}$, and let $\mathbf{v} \in \mathbf{V}$. The unique vector $\mathbf{x} = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ such that
$$\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$$
is called the **coordinate vector** of $\mathbf{v}$ with respect to the basis B.

- **Theorem.** Let $\mathbf{V}$ be a vector space with basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$, let $\mathbf{v}, \mathbf{v}_1, \cdots, \mathbf{v}_m$ be elements of $\mathbf{V}$, and let $\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_m$ be the respective coordinate vectors. Then $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_m$ if and only if $\mathbf{x}$ is a linear combination of $\mathbf{x}_1, \cdots, \mathbf{x}_m$. That is, for scalars $d_1, \cdots, d_m$,
$$\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_m\mathbf{v}_m \Longleftrightarrow \mathbf{x} = d_1\mathbf{x}_1 + \cdots + d_m\mathbf{x}_m$$

- **Corollary.** Following the previous theorem, $\mathbf{v}_1, \cdots, \mathbf{v}_m$ is linearly independent if and only if $\mathbf{x}_1, \cdots, \mathbf{x}_m$ is linearly independent.

- Let $\mathbf{V}$ be a vector space. An **inner product** for $\mathbf{V}$ is a function that associates with every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ a real number $(\mathbf{u}, \mathbf{v})$ satisfying the following properties. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{V}$ and $c \in \mathbb{R}$,
  1. $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
  2. $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w})$
  3. $(c\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v})$
  4. $(\mathbf{u}, \mathbf{u}) \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$

- An **inner product space** is a vector space together with an inner product for the vector space.

- **Theorem.** Let $\mathbf{V}$ be an inner product space. Then for all vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}$, and scalars c_1 and c_2,
  1. $(c_1\mathbf{u}_1 + c_2\mathbf{u}_2, \mathbf{v}) = c_1(\mathbf{u}_1, \mathbf{v}) + c_2(\mathbf{u}_2, \mathbf{v})$
  2. $(\mathbf{0}, \mathbf{v}) = 0$

- Cauchy-Schwarz Inequality:
$$|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space are **orthogonal** if $(\mathbf{u}, \mathbf{v}) = 0$. A set of vectors is orthogonal if every pair of vectors in the set is orthogonal. A set is **orthonormal** if it is orthogonal and every vector has length equal to 1.

- **Theorem.** Any orthogonal/orthonormal set of non-zero vectors is linearly independent.

- **Theorem.** Let $T = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be an orthonormal basis for some vector space $\mathbf{V}$. Then
$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$
where
$$c_i = (\mathbf{x}, \mathbf{v}_i)$$

- **Gram-Schmidt Process** to convert a basis to an orthogonal basis. Let $T = \{\mathbf{u}_1, \cdots \mathbf{u}_n\}$ be a basis. Then $T' = \{\mathbf{v}_1, \cdots \mathbf{v}_n\}$ is an orthogonal basis, where

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{(\mathbf{u}_2, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{(\mathbf{u}_3, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 - \frac{(\mathbf{u}_3, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_n &= \mathbf{u}_n - \frac{(\mathbf{u}_n, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 - \cdots - \frac{(\mathbf{u}_n, \mathbf{v}_{n-1})}{(\mathbf{v}_{n-1}, \mathbf{v}_{n-1})}\mathbf{v}_{n-1}
\end{aligned} \tag{1}$$

- **Theorem.** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^m$. The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathbf{x} = proj_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$$

Chapter 7

- Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces. A **linear transformation** from $\mathbf{V}$ to $\mathbf{W}$ is a function $T : \mathbf{V} \to \mathbf{W}$ that satisfies the following two conditions. For each $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and scalar a,

  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(a\mathbf{u}) = aT(\mathbf{u})$

- The linear transformation $0 : \mathbf{V} \to \mathbf{W}$ defined by $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ is called the **zero transformation**.

- The linear transformation $I : \mathbf{V} \to \mathbf{V}$ defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ is called the **identity transformation**.

- **Theorem.** Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then

  1. $T(\mathbf{0}) = \mathbf{0}$
  2. $T(-\mathbf{u}) = -T(\mathbf{u})$
  3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

  (Note: in general, $-\mathbf{u} = (-1)\mathbf{u}$.)

- **Theorem** Let A be an $m \times n$ matrix. Then $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

- Let T_θ be given by $T_\theta(\mathbf{x}) = A_\theta\mathbf{x}$, where

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  Then $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a **rotation**, it is the map that rotates a vector θ counter-clockwise around the origin.

- Let $A = cI$, where I is the $n \times n$ identity matrix and c is some scalar. Then $T_c(\mathbf{x})$ is a **dilation** by a factor of c if $c > 1$ (or $|c| > 1$) and $T_c(\mathbf{x})$ is a **contraction** by a factor of c if $0 < c < 1$ (or $|c| < 1$).

- Let T^θ be given by $T^\theta(\mathbf{x}) = A^\theta\mathbf{x}$, where

$$A^\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

  Then T^θ is **reflection** across the line l that goes through the origin and forms an angle of θ with the positive x-axis.

- A **similarity** is a map that preserves the shape of geometric figures. An **isometry** is a map that preserves not only shapes but distances as well. Every isometry is a similarity. Rotations and reflections are isometries. Contractions and dilations are similarities.

- **Theorem.** If $S : \mathbf{V} \to \mathbf{W}$ and $T : \mathbf{V} \to \mathbf{W}$ are linear transformations, $B = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is a basis for $\mathbf{V}$ and $S(\mathbf{b}_i) = T(\mathbf{b}_i)$ for all i, then $S = T$.

- Very useful: a linear transformation is determined by its action on a basis (or any spanning set).

- Let S and T be linear transformations from $\mathbf{V}$ to $\mathbf{W}$ and let c be a scalar.

  1. The **sum** $S + T$ is given by
  $$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$$
  2. The **difference** $S - T$ is given by
  $$(S - T)(\mathbf{v}) = S(\mathbf{v}) - T(\mathbf{v})$$
  3. The **negative** $-S$ is given by
  $$(-S)(\mathbf{v}) = -(S(\mathbf{v}))$$
  4. The **scalar multiple** cS is given by
  $$(cS)(\mathbf{v}) = c(S(\mathbf{v}))$$

- **Theorem.** $S + T, S - T, -S$ and cS defined above are all linear transformations.

- Let $S : \mathbf{U} \to \mathbf{V}$ and $T : \mathbf{V} \to \mathbf{W}$ be linear transformations and let $L : \mathbf{V} \to \mathbf{V}$ be a linear operator (a linear transformation of a vector space to itself).

  1. The **product** TS is the linear transformation $T \circ S$ defined by
  $$TS(\mathbf{u}) = T(S(\mathbf{u}))$$
  2. The **powers** of L are the linear transformations $L^0 = I$ (the identity), $L^1 = L$, $L^2 = LL$, $L^3 = L^2L$, etc.

- **Theorem.** Let S, T and U be linear transformations and let a and b be scalars. Then the following properties hold provided they are defined (that the vector spaces match up in a suitable way):

  1. $S + T = T + S$
  2. $S + (T + U) = (S + T) + U$
  3. $S + 0 = S$
  4. $S + (-S) = 0$
  5. $S(TU) = (ST)U$
  6. $SI = IS = S$
  7. $S(T + U) = ST + SU$, $(S + T)U = SU + TU$
  8. $a(S + T) = aS = aT$
  9. $(a + b)S = aS + bS$

10. $(ab)S = a(bS)$
11. $1S = S$
12. $S0 = 0S = 0$
13. $a0 = 0$
14. $a(ST) = (aS)T$

- **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ matrix with the ith column equal to $T(e_i)$ (where e_i is from the standard basis for $\mathbb{R}^n$). Then A is the unique matrix for which $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. This matrix is called the **matrix associated to** T.

- **Theorem.** Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ and let A and B be their respective associated matrices. Then the associated matrix of the linear transformation

  1. $S + T$ is $A + B$
  2. $S - T$ is $A - B$
  3. $-S$ is $-A$
  4. cS is cA

- **Theorem.** Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^l$. Let A and B the their respective matrices. Then TS has matrix BA.

- Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. The set of all vectors $\mathbf{x}$ in $\mathbf{V}$ that satisfy $T(\mathbf{x}) = \mathbf{0}$ is called the **kernel** or **nullspace** of T and is denoted by $\ker T$. The kernel is a subspace of $\mathbf{V}$, and its dimension is called the **nullity**.

- A linear transformation is **one-to-one** if whenever $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ then $\mathbf{v}_1 = \mathbf{v}_2$.

- **Theorem.** Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

- Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. The set of all vectors $T(\mathbf{x})$ in $\mathbf{W}$ is called the **image** of T and is denoted by $\mathrm{Im}T$. $\mathrm{Im}T$ is a subspace of $\mathbf{W}$ and its dimension is called the **rank**.

- A linear transformation is **onto** if $\mathrm{Im}T = W$.

- **Very Useful Theorem.** $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then

$$\dim V = \text{nullity} + \text{rank}$$

- An **affine map** S on $\mathbb{R}^n$ has an associated $n \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$ such that

$$S(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

- An affine transformation is a **contraction mapping with contraction factor** $s \in (0, 1)$ if for all vectors $\mathbf{x}$ and $\mathbf{y}$,

$$||S(\mathbf{y} - S(\mathbf{x})|| \leq s||\mathbf{y} - \mathbf{x}||$$

- A collection of contractions $\{S_1, \cdots, S_k\}$ is called an **iterated function system** or **IFS**.

- An IFS has a unique attractor associated with it. No matter what compact (closed and bounded) set you start with, if you apply the IFS, the limit is always the same. Often this limit is a fractal.

Chapter 8

- Let $\mathbf{V}$ be a vector space and let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation. A scalar λ is called an **eigenvalue** of T if there is a nonzero vector $\mathbf{x}$ in $\mathbf{V}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** corresponding to λ.

- The **eigenspace** of an eigenvalue λ of a matrix A is the set of all solutions of $A - \lambda I)\mathbf{x} = \mathbf{0}$ (including the zero vector). If A is $n \times n$, then the eigenspace is a subspace of $\mathbb{R}^n$.
- If A and B are square $n \times n$ matrices, we say that B is **similar** to A if there exists an invertible matrix P such that $B = P^{-1}AP$. This is denoted by $A \sim B$.
- If $A \sim B$, then
  1. $B \sim A$
  2. $A^T \sim B^T$
  3. If A and B are invertible then $A^{-1} \sim B^{-1}$
  4. $detA = detB$
  5. rank $A =$ rank B
  6. The trace (sum of diagonal entries) of A is equal to the trace of B
  7. A and B have the same characteristic polynomial and hence the same eigenvalues
- A square matrix is said to be **diagonalizable** if it is similar to a diagonal matrix.
- Procedure for diagonalization of A:
  1. Calculate eigenvalues
  2. Calculate bases for eigenspaces
  3. Put bases together. If there are n vectors, then A is diagonalizable and P is the matrix with columns equal to the basis vectors. A is similar to the diagonal matrix whose entries are the eigenvalues of A. If there are less than n vectors, then A is not diagonalizable.
- **Theorem.** Eigenvectors that correspond to different eigenvalues are linearly independent.
- **Theorem.** A square matrix A is diagonalizable if and only if A has n independent eigenvectors.
- **Theorem.** Let A be $n \times n$. If A has n distinct eigenvalues then A is diagonalizable.
- A square matrix A is symmetric if $A = A^T$.
- **Theorem.** Eigenvectors of a symmetric matrix that correspond to distinct eigenvalues are orthogonal.
- An invertible matrix P is orthogonal if $P^{-1} = P^T$.
- **Theorem.** A matrix P is orthogonal if and only if the columns of P are orthonormal.
- A square matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^{-1}AP = P^TAP$ is diagonal.
- **Theorem.** A matrix A is orthogonally diagonalizable if and only if it is symmetric.

***** End of material on exam *****

- A complex number is of the form $z = a + bi$ where a and b are real numbers and $i^2 = -1$. The **absolute value/modulus** of z, denoted by $|z|$, is $\sqrt{a^2 + b^2}$. The **complex conjugate** of z, denoted by $\overline{z}$, is $a - bi$.
- $\mathbb{C}^m$ is a vector space with dimension n. The standard basis is the same as for $\mathbb{R}^m$, but keep in mind that scalars can now be complex numbers.
- Let $\mathbf{z} = (z_1, z_2, \cdots, z_m)$ and $\mathbf{w} = (w_1, w_2, \cdots, w_m)$. Then
  1. $\mathbf{z} \cdot \mathbf{w} = z_1\overline{w_1} + \cdots + z_m\overline{w_m}$

2. $\|\mathbf{z}\| = \sqrt{|z_1|^2 + \cdots + |z_m|^2}$

- **Theorem.** Let $\mathbf{z}$ and $\mathbf{w}$ be vectors in $\mathbb{C}^m$, and let $c \in \mathbb{C}$. Then
  1. $\mathbf{z} \cdot \mathbf{z} \geq 0$ and $\mathbf{z} \cdot \mathbf{z} = 0$ if and only if $\mathbf{z} = \mathbf{0}$
  2. $\mathbf{z} \cdot \mathbf{w} = \overline{\mathbf{w} \cdot \mathbf{z}}$
  3. $(c\mathbf{z}) \cdot \mathbf{w} = c(\mathbf{z} \cdot \mathbf{w})$, $\mathbf{z} \cdot (c\mathbf{w}) = \overline{c}(\mathbf{z} \cdot \mathbf{w})$
  4. $\mathbf{z} \cdot \mathbf{z} = \|\mathbf{z}\|^2$
- **Theorem.** Every $n \times n$ matrix has at least one eigenvalue (it may be complex).
- The **conjugate transpose** of a matrix A, denoted by A^H, is the matrix obtained by replacing every entry by its complex conjugate and then doing the transpose.
- A matrix A is **hermitian** if $A = A^H$.
- **Theorem.** Let A be hermitian. Then all eigenvalues of A are real.
- **Theorem.** Eigenvectors of a hermitian matrix that correspond to distinct eigenvalues are orthogonal.
- An invertible matrix P is **unitary** if $P^{-1} = P^H$.
- An invertible matrix P is unitary if and only if its columns are orthonormal.
- **Spectral Theorem.** Let A be a square matrix with complex entries. Then there exists a unitary matrix P such that $P^{-1}AP$ is diagonal with real entries if and only if A is hermitian.
- **Cayley-Hamilton Theorem.** If $p(x)$ is the characteristic polynomial of a matrix A, then $p(A) = 0$. A matrix is a root of its own characteristic polynomial!