FINAL EXAM

MATH 254 Linear Algebra, St. Francis Xavier University

April 16, 2008 **9-11:30 am**

Instructor: Tara Taylor

NAME (PRINT) ____

STUDENT NUMBER __________________________________

SIGNATURE ____

You can use calculators. Please write answers on the question sheets, and use the back sides for scrap paper. There are two sections to the exam. The first section consists of 15 true/false questions, each worth 2 marks, for a total of 30 marks. The second section consists of 8 pages of long-answer questions. Each page is worth 10 marks, with the best 7 pages taken for marks, for a total of 70. The total exam is out of 100. The last page contains the properties of vector spaces. You may remove this page.

*** Good luck! Have a good summer! ***

1 True/False Questions

Each question is worth 2 marks, for a total of thirty (30) marks. No explanation is required, just fill in T for true or F for false in the blank before the statement.

1. ______ If the $m \times n$ matrix A has rank m, then there are more rows than columns.

2. ______ If $T : V \to W$ is an isomorphism with $\dim(V) = 3$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a basis for W.

3. ______ If A and B are both $n \times n$ matrices such that they have the same determinant and rank, then they are similar matrices.

4. ______ If A is 4×3 and B is 3×5, then it is possible for $C = AB$ to have rank 4.

5. ______ If $\mathbf{X}$ and $\mathbf{Y}$ are non-zero, they are orthogonal if and only if
$$||\mathbf{X} + \mathbf{Y}||^2 = ||\mathbf{X}||^2 + ||\mathbf{Y}||^2$$

6. ______ If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then T cannot be onto.

7. ______ Let U be some subspace of $\mathbb{R}^n$. If $X \in U$, then $X \notin U^{\perp}$.

8. ______ The set U of all polynomials that have degree exactly equal to 5 is a vector space.

9. ______ It is possible for $\mathbb{R}^7$ to have a subspace U that has the same dimension as its orthogonal complement.

10. ______ The Gram-Schmidt algorithm would convert the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ into three orthogonal vectors.

11. ______ $\{a_1 + b_1 x, a_2 + b_2 x\}$ is a basis for $\mathbf{P}_1$ if and only if $\{b_1 + a_1 x, b_2 + a_2 x\}$ is a basis for $\mathbf{P}_1$.

12. ______ It is possible for a matrix with all complex entries to have all real eigenvalues.

13. ______ The set of 3×3 skew-symmetric matrices is a subspace of M_{33} with dimension 3.

14. ______ $\mathbb{C}^n$ has dimension n.

15. ______ Tara really likes the Cayley-Hamilton theorem!

2 Long Answer Questions

There are 8 pages of questions, each PAGE worth 10 marks. The best 7 pages will be taken for marks, for a total of 70. Note that questions on one page may be unrelated!

Page 1

1. Let $T : M_{22} \to \mathbb{R}^4$ be defined by [7]

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-c, 2a-c, b-d, 2b-d)$$

   (a) Prove that T is an isomorphism.

   (b) Find the inverse of T.

   (c) Use this information to find a basis for M_{22} that is not the standard basis.

2. Let $\mathbf{Z} = \begin{bmatrix} -i \\ 2+i \\ -4 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 3+i \\ 2-i \\ 1+i \end{bmatrix}$. Find $||\mathbf{Z}||$ and $\langle \mathbf{Z}, \mathbf{W} \rangle$. [3]

3. Let A be some 3×3 matrix. [4]

   (a) Prove that null$A$ is a subspace of $\mathbb{R}^3$.

   (b) If null$A$ is a plane, then what is im$A$ (geometrically)? Explain.

4. Let $T : M_{nn} \rightarrow \mathbb{R}$ be defined by $T(A) = \text{tr}(A)$ (the trace of A- so the sum of the diagonal entries of A). T is a linear transformation (you do NOT need to show that). Find the dimension of the kernel of T. [3]

5. If A is any 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$, find $3A^2 - 18A$. [3]

6. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{bmatrix}$. A can be reduced to $R = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (don't show!).

   (a) Find a basis for the null space of A.

   (b) Find a basis for the row space of A.

   (c) Find a basis for the column space of A.

   (d) Does A have a right or left inverse? That is, does there exist $C_{3\times 4}$ such that $AC = I_4$ or $D_{3\times 4}$ such that $DA = I_3$? Explain why or why not. [6]

7. If Z is a complex $n \times n$ matrix, it can be shown that $Z = A + iB$, where A and B are both $n \times n$ real matrices (don't show!). Prove that Z is Hermitian if and only if A is symmetric and B is skew-symmetric. (Recall that Z is Hermitian if $Z = Z^*$, the conjugate transpose.) [4]

8. Apply the Gram-Schmidt algorithm to the following set of linearly independent vectors to find an orthogonal basis for $\mathbb{R}^3$ and verify that the set is orthogonal. [6]

$$\mathbf{X}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

9. Let U and W be subspaces of a vector space V such that $\dim(V) = 3$, $\dim(U) = \dim(W) = 2$ and $U \neq W$. Let $U \cap W$ be the subspace defined by

$$U \cap W = \{\mathbf{v} | \mathbf{v} \in U \quad \text{and} \quad \mathbf{v} \in W\}$$

   (a) Show that $\dim(U \cap W) = 1$. Hint: think about bases.

   (b) Interpret this geometrically if $V = \mathbb{R}^3$.

[4]

10. Let $V = \mathbb{R}^3$, let U be the plane given by the equation $x - 3y + 4z = 0$. [6]

(a) Find the orthogonal complement $U^\perp$ and explain your answer in terms of the **definition** of $U^\perp$.

(b) For the vector $\mathbf{X} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, find $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$ such that $\mathbf{X} = \mathbf{u} + \mathbf{w}$.

11. Let f, g be functions on $[a, b]$ and assume that

$$f(a) = g(b) = 3, \quad f(b) = g(a) = 0$$

Show that $\{f, g\}$ are independent in $\mathbf{F}[a, b]$, the vector space of functions on $[a, b]$. [4]

12. Let $U = \{p(x) \in \mathbf{P}_2 | p(3) = 0\}$. [5]

    (a) Show that U is a subspace of $\mathbf{P}_2$.

    (b) Find a basis for U. Hint: think about dimension.

13. Suppose A is $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. [5]

    (a) Prove that $A \sim A^T$.

    (b) Prove that $\det A = \lambda_1 \times \lambda_2 \cdots \times \lambda_n$.

14. Use the properties of vector spaces to show that the zero vector $\mathbf{0}$ is unique for any vector space. [3]

15. Let $T : \mathbb{R}^3 \to \mathbf{P}_3$ and $S : \mathbf{P}_3 \to \mathbb{R}^2$ be linear transformations defined as follows:

$$T(a, b, c) = b + cx + bx^2 + ax^3, \quad S(d + ex + fx^2 + gx^3) = (d, g)$$

(a) Find $ST : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) Determine whether or not each of the three linear transformations T, S, ST are onto.

(c) Based on your answer to (b), form a general conjecture about necessary conditions on T and S for ST to be onto.

16. Let U be a subspace of $\mathbb{R}^n$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(X) = \text{proj}_U(X)$ (the projection of X onto U). [6]

    (a) Show that T is a linear transformation.

    (b) In your own words, what does T do?

    (c) Show that $\ker T = U^\perp$ and $\text{im} T = U$.

17. Let C be a column in $\mathbb{R}^m$ and let R be row in $\mathbb{R}^n$. Then let $A = CR$ be the $m \times n$ matrix formed from multiplying the column and the row. Show that col $A = \text{span}\{C\}$ and that row $A = \text{span}\{R\}$. Hint: think about rank. [4]

Properties of Vector Spaces

A *vector space* is a non-empty set V of objects called vectors that can be added and multiplied by scalars in special ways. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and any scalars c and d,

A1 $\mathbf{u} + \mathbf{v} \in V$.

A2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

A3 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

A4 There is an element denoted $\mathbf{0}$ in V and called the *zero vector* with the property that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in V$.

A5 There exists a vector called the *negative* of $\mathbf{v}$, denoted by $-\mathbf{v}$, with the property that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

S1 $c\mathbf{v} \in V$.

S2 $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$.

S3 $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$.

S4 $c(d\mathbf{v}) = (cd)\mathbf{v}$.

S5 $1\mathbf{v} = \mathbf{v}$.