FINAL EXAM

MATH 254 Linear Algebra, St. Francis Xavier University

April 20, 2009 **9-11:30 am**

Instructor: Tara Taylor

NAME (PRINT) ____

STUDENT NUMBER ______________________________

SIGNATURE ____

- You can use calculators.
- Please write answers on the question sheets, and use the back sides if needed for extra room or for scrap paper.
- There are two sections to the exam. The first section consists of 15 true/false questions, each worth 2 marks, for a total of 30 marks. The second section consists of 8 pages of long-answer questions. Each page is worth 10 marks, with the best 7 pages taken for marks, for a total of 70. The total exam is out of 100.
- The last page contains the properties of vector spaces and the Gram-Schmidt algorithm; you may remove this page.
- Try to read every question carefully- many of them can be done more than one way, with some ways much easier than other ways!

*** Good luck! Have a good summer! ***

1 True/False Questions

Each question is worth 2 marks, for a total of thirty (30) marks. No explanation is required, just fill in T for true or F for false in the blank before the statement.

1. ______ If the $m \times n$ matrix A has rank m, then the columns are linearly independent.

2. ______ The standard basis for $\mathbb{R}^m$ is also a basis for $\mathbb{C}^m$.

3. ______ It is possible to find a linear transformation $T : P^5 \to \mathbb{R}^5$ that is one-to-one.

4. ______ If U is a subspace of some vector space V, and $\mathbf{v} + \mathbf{x} \in U$ for some $\mathbf{v}$ and $\mathbf{x}$ in V, then both $\mathbf{x}$ and $\mathbf{v}$ are also in U.

5. ______ Let $T : V \to W$ be a linear transformation. If $\ker T = V$, then $W = \{\mathbf{0}\}$.

6. ______ If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then T cannot be onto.

7. ______ Let P be an invertible square matrix. Then P is an orthogonal matrix if and only if the columns of P are orthogonal.

8. ______ In $\mathbb{R}^2$, if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other, then $\text{proj}_{\mathbf{u}}\mathbf{v} = \mathbf{0}$.

9. ______ Any affine transformation is a contractive mapping.

10. ______ The Gram-Schmidt algorithm would convert the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ into three orthogonal vectors.

11. ______ $\{a_1 + b_1 x, a_2 + b_2 x\}$ is a basis for $\mathbf{P}_1$ if and only if $\{b_1 + a_1 x, b_2 + a_2 x\}$ is a basis for $\mathbf{P}_1$.

12. ______ The transformation $T_{\pi/4} : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates every vector $\mathbf{x}$ by $\pi/4$ has no eigenvalues.

13. ______ The QR algorithm is the basis for a practical, efficient way to find the eigenvalues of a square matrix.

14. ______ A square $n \times n$ matrix A is diagonalizable if and only if it has n distinct eigenvalues.

15. ______ Tara often makes mistakes at the chalkboard :)

2 Long Answer Questions

There are 8 pages of questions, each PAGE worth 10 marks. The best 7 pages will be taken for marks, for a total of 70. Note that questions on one page may be unrelated!

Page 1

1. Let $T : M^{2,2} \to \mathbb{R}^4$ be defined by [6]

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+c, b+d, b-d, a-c)$$

   (a) Prove that T is one-to-one and onto.
   (b) Find the inverse of T.
   (c) Use this information to find a basis for M_{22} that is not the standard basis.

2. Suppose A is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. [4]

   (a) Prove that A is similar to A^T.
   (b) Prove that $\det A = \lambda_1 \times \lambda_2 \cdots \times \lambda_n$.

3. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{bmatrix}$. A can be reduced to $R = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (don't show!). [3]

   (a) Find a basis for the null space of A.

   (b) Find a basis for the row space of A.

   (c) Find a basis for the column space of A.

4. If Z is a complex $n \times n$ matrix, it can be shown that $Z = A + iB$, where A and B are both $n \times n$ real matrices (don't show!). A matrix M is **skew-symmetric if** $M^T = -M$. Prove that Z is Hermitian (equal to its conjugate transpose) if and only if A is symmetric and B is skew-symmetric. [4]

5. For any vector space V, show that for any $\mathbf{u}, \mathbf{v} \in V$, [3]

$$\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$$

6. Apply the Gram-Schmidt algorithm to the following set of linearly independent vectors to find an orthogonal basis for $\mathbb{R}^3$, and verify that the set is orthogonal. [4]

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 2+i \\ 2-i & 3 \end{bmatrix}$. Verify that A is Hermitian and that the eigenvalues of A are real. [3]

8. Let $T : M_{nn} \to \mathbb{R}$ be defined by $T(A) = \text{tr}(A)$ (the trace of A- so the sum of the diagonal entries of A). T is a linear transformation (you do NOT need to show that). Find the dimension of the kernel of T. Hint: find the image first. [3]

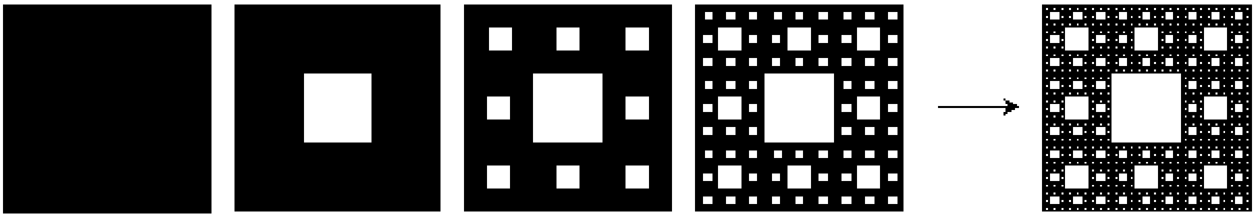

Figure 1: Sierpinski Carpet

9. The Sierpinski Carpet is a fractal that is the attractor of an iterated function system (IFS) with 8 contractive mappings. Start with the unit square S, so the square with corner points $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Divide the square into 9 equal sized squares (so all have length $1/3$). Keep all squares except for the middle one. Then the 8 mappings $\{C_1, C_2, \cdots, C_8\}$ are the mappings that take the unit square to each of the 8 smaller squares as in the diagram. [5]

   (a) Find the explicit maps for C_1, C_3 and C_6.

   (b) Prove that C_3 is a contractive mapping.

   (c) Draw the corresponding images for $C_4C_6(S)$ and $C_3C_4(S)$ (so start with the unit square and see what happens after two iterations).

10. Let U and W be subspaces of a vector space V. Let $U \cap W$ be defined by

$$U \cap W = \{\mathbf{v} | \mathbf{v} \in U \quad \text{and} \quad \mathbf{v} \in W\}$$

   (a) Show that $U \cap W$ is a subspace of V.

   (b) Suppose $\dim(V) = 3$, $\dim(U) = \dim(W) = 2$ and $U \neq W$. Show that $\dim(U \cap W) = 1$. Hint: think about bases. Interpret this geometrically if $V = \mathbb{R}^3$.

[5]

11. Let A be a square matrix. Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Prove that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. [3]

12. Let $U = \{p(x) \in \mathbf{P}_2 | p(2) = 0\}$. [4]

    (a) Show that U is a subspace of $\mathbf{P}_2$.

    (b) Find a basis for U. Hint: think about dimension.

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,4) = (3,0,1,2), \quad T(-2,3) = (-1,4,1,0).$$

    Use that information to find $T(-2,14)$. [3]

14. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in some vector space V. Use the properties of vector spaces to show that if $\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. Be sure to explain what property you are using in each step. [3]

15. A square matrix E is called a **projection** matrix if $E^2 = E = E^T$. [4]

    (a) If $P = I - 2E$ for any projection matrix E and the identity matrix I of the same size, show that P is orthogonal and symmetric.

    (b) Conversely, if P is an orthogonal matrix then show that the matrix $E = \frac{1}{2}(I - P)$ is a projection matrix.

16. Show that any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the form [3]

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

for some constants $a, b, c, d \in \mathbb{R}$. Hint: use the matrix representation of a linear transformation.

17. If $\{A_1, A_2, \cdots, A_{n^2}\}$ is a basis for $M^{n,n}$, prove that $\{A_1^T, A_2^T, \cdots, A_{n^2}^T\}$ (where T denotes transpose) is also a basis for $M^{n,n}$. [4]

18. Let $T : \mathbb{R}^3 \to \mathbf{P}_3$ and $S : \mathbf{P}_3 \to \mathbb{R}^2$ be linear transformations defined as follows: [6]

$$T(a, b, c) = b + cx + bx^2 + ax^3, \quad S(d + ex + fx^2 + gx^3) = (d, g)$$

(a) Find $ST : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) Determine whether or not each of the three linear transformations T, S, ST are onto.

(c) Based on your answer to (b), form a general conjecture about necessary conditions on T and S for ST to be onto.

19. Let $\mathbf{z} = (2 - 4i, 1 + 3i, 1 - i)$, $\quad \mathbf{w} = (3 + i, 2 - i, 5)$. Find $||\mathbf{z}||$ and $\mathbf{z} \cdot \mathbf{w}$. [2.5]

20. Recall that the Cayley-Hamilton Theorem states that any matrix is a root of its characteristic polynomial. Verify this for the matrix [2.5]

$$\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$$

21. A polynomial is **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$. Let E_n and O_n denote the sets of even and odd polynomials in P_n. It can be shown that E_n and O_n are subspaces of P_n. [5]

    (a) Find a basis for E_n (hint: think about which elements from the standard basis for P_n would be in E_n).

    (b) Define a linear transformation $T : P_n \to P_n$ by $T(p(x)) = p(x) - p(-x)$. Show that the kernel of T is E_n and the image of T is O_n.

Properties of Vector Spaces

A *vector space* is a non-empty set V of objects called vectors that can be added and multiplied by scalars in special ways. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and any scalars c and d,

A1 $\mathbf{u}+\mathbf{v} \in V$.

A2 $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.

A3 $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

A4 There is an element denoted $\mathbf{0}$ in V and called the *zero vector* with the property that $\mathbf{0}+\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in V$.

A5 There exists a vector called the *negative* of $\mathbf{v}$, denoted by $-\mathbf{v}$, with the property that $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$.

S1 $c\mathbf{v} \in V$.

S2 $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v}+c\mathbf{w}$.

S3 $(c+d)\mathbf{v} = c\mathbf{v}+d\mathbf{v}$.

S4 $c(d\mathbf{v}) = (cd)\mathbf{v}$.

S5 $1\mathbf{v} = \mathbf{v}$.

Gram-Schmidt Process

Let $T = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$. You can get an orthogonal basis $T' = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ as follows:

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\cdots
\end{aligned}
$$