Test 1

MATH 254 Linear Algebra, St. Francis Xavier University

February 11, 2015 **10:15am-11:05am**

Instructor: Tara Taylor

NAME (PRINT) SOLUTIONS

STUDENT NUMBER ______________________

SIGNATURE ______________________

You can use calculators (but they shouldn't be needed). Please write answers on the question sheets, and use the back sides for scrap paper. The last page contains some terminology and the axioms for a vector space- you can remove this page. There are two sections to the test. The first section consists of 8 true/false questions, each worth 2 marks, for a total of 16 marks. The second section consists of written answer questions for a total of 24 marks. The total test is out of 40. Look through all the questions and start with the ones that you feel confident about, and if you get stuck on one then move on to a different one.

---

1 True/False Questions

Each question is worth 2 marks, for a total of 16 marks. No explanation is required, just fill in T for true or F for false in the blank before the statement.

1. __T__ Any set of non-zero vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbb{R}^2$ such that neither vector is a multiple of the other forms a basis for $\mathbb{R}^2$.

2. __F__ Any line is a subspace of $\mathbb{R}^2$.

3. __F__ The set of polynomials of degree exactly 4 forms a subspace of the vector space of all polynomials.

4. __T__ In any vector space the zero vector is unique.

5. __T__ If the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span a three-dimensional vector space V, then their respective coordinate vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly independent in $\mathbb{R}^3$.

6. __F__ Any set of 6 matrices in $M^{2,2}$ can be reduced to a set that is a basis.

7. __F__ It is possible for a 6×4 matrix A to have a row space with dimension equal to 5.

8. __T__ If $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a set of linearly independent vectors in a vector space V that has dimension equal to n, then for any vector $\mathbf{v}$ there is exactly one solution to

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n$$

2 Long Answer Questions

This section has a total of 24 marks.

1. Let $U = \{A | A$ is in $M^{2,2}$, A is invertible$\}$. Explain why U is not a subspace of $M^{2,2}$. (3)

- Doesn't contain 0
- Not closed under addition ex $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. Consider the set of vectors $\{x+1, x-1, x^2\}$ in P_2. (6)
   (a) Explain why this set is a basis.
   (b) Find the coordinate vector of $-5+5x-4x^2$ with respect to this basis.

a) $\dim P_2 = 3$

vectors are LI: $c_1(x+1) + c_2(x-1) + c_3(x^2) = 0 + 0x + 0x^2$

$\Rightarrow (c_1 - c_2) + (c_1 + c_2)x + c_3 x^2 = 0 + 0x + 0x^2$

$$\left. \begin{array}{l} c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{array} \right\} \Rightarrow c_1 = c_2 = 0$$

$c_3 = 0 \Rightarrow \boxed{c_3 = 0}$

Only trivial solution. Right # of LI vectors so also spans

$\therefore \{x+1, x-1, x^2\}$ is a basis for P_2

b) $-5 + 5x - 4x^2 = c_1(x+1) + c_2(x-1) + c_3(x^2)$

$\Rightarrow -5 + 5x - 4x^2 = (c_1 - c_2) + (c_1 + c_2)x + c_3 x^2$

$c_1 - c_2 = -5$ $\qquad$ $c_3 = -4$

$c_1 + c_2 = 5$

$2c_1 = 0 \rightarrow c_1 = 0$

$\checkmark\ c_2 = 5$

Coordinate vector is $(0, 5, -4)$

3. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$. A can be reduced to $R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (don't show!). (7)

(a) Find a basis for and the dimension of the solution space of A.

(b) Find a basis for the column space of A.

(c) Use the fact that column rank equals row rank to find a basis for the row space of A.

a) Let $z = t$ free parameter

$y + 2z = 0 \rightarrow y = -2z = -2t$

$x + 3z = 0 \rightarrow x = -3z = -3t$

vectors in solution space are of the form $(-3t, -2t, t) = t(-3, -2, 1)$

basis $\{(-3, -2, 1)\}$

dim $= 1$

b) 2 leading ones $\rightarrow$ 2 first columns of original matrix

basis for column space $= \{(1, -1, 0), (-2, 1, 2)\}$

c) Row rank $= 2$ So need 2 LI rows

$\{(1, -2, 1), (-1, 1, -1)\}$ works

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a linearly dependent set of vectors in some vector space V. Let $\mathbf{v}$ be any vector in V. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{v}\}$ is also linearly dependent. (4)

$v_1, \ldots v_n$ is LD so WLOG $v_1 = c_2 v_2 + \ldots c_n v_n$
(LC of others)

then $v_1 = c_2 v_2 + \ldots c_n v_n + 0v$
still LC of others so $\{v_1, \ldots v_n, v\}$ is LD

5. Using just the properties of vector spaces, prove that for any vector space V, if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$. Be sure to state which axiom you are using at each step. (4)

$u + w = v + w$

$\Rightarrow (u+w) + (-w) = (v + w) + (-w)$ Axiom 6 for negatives

$\Rightarrow u + (w + -w) = v + (w + -w)$ Axiom 4 Associativity

$\Rightarrow u + 0 = v + 0$ Axiom 6

$\Rightarrow u = v$ Axiom 5 for zero