

Test 2

**MATH 254 Linear Algebra, St. Francis Xavier University**

**March 27, 2015**

**9:15am-10:05am**

Instructor: Tara Taylor

NAME (PRINT) \_\_\_\_\_

SOLUTIONS

STUDENT NUMBER \_\_\_\_\_

SIGNATURE \_\_\_\_\_

---

You can use calculators (but they shouldn't be needed). Please write answers on the question sheets, and use the back sides for scrap paper. There are two sections to the test. The first section consists of 8 true/false questions, each worth 2 marks, for a total of 16 marks. The second section consists written answer questions for a total of 24 marks. The total test is out of 40. If you are stuck on a question, move on to another one. The point-value for each question is a good indication of how much time you should spend on a question. The last page contains info that might be helpful, you can rip it off. You can also write on the backs of pages if you need more room.

---

## 1 True/False Questions

Each question is worth 2 marks, for a total of 16 marks. No explanation is required, just fill in T for true or F for false in the blank before the statement.

1. F If  $T : V \rightarrow W$  is a linear transformation and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $V$ , then  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  is a basis for  $W$ .
2. F If  $T : P_3 \rightarrow \mathbb{R}^3$  is a linear transformation, then  $T$  could be one-to-one.
3. T If  $T : P_3 \rightarrow \mathbb{R}^3$  is a linear transformation, then  $T$  could be onto.
4. T The Koch curve has dimension between 1 and 2 because it has infinite length and no area.
5. F  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(x, y) = (\cos x, 2y)$  is a linear transformation.
6. F Any set of  $n$  nonzero vectors in  $\mathbb{R}^n$  can be converted to an orthonormal basis for  $\mathbb{R}^n$ .
7. T If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has corresponding matrix  $A$  with  $\det(A) \neq 0$ , then  $T$  is an isomorphism.
8. F For the vector space  $M^{2,2}$ , the product  $(A, B)$  defined by  $(A, B) = \det A \times \det B$  is an inner product.

## 2 Long Answer Questions

This section has a total of 24 marks.

1. Let  $B = \{(1, 1, 0), (2, 0, 1), (1, 1, 1)\}$ .  $B$  is a basis for  $\mathbb{R}^3$  (you don't need to show that). [4]

(a) Use the Gram-Schmidt Algorithm (see last page for the algorithm) to convert the basis to an orthogonal basis  $\{v_1, v_2, v_3\}$ .

(b) Check to make sure that the basis is indeed orthogonal.

$$v_1 = (1, 1, 0) \quad u_2 \cdot v_1 = (2, 0, 1) \cdot (1, 1, 0) = 2$$

$$v_2 = u_2 - \left( \frac{u_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1 = (2, 0, 1) - \frac{2}{2} (1, 1, 0) = (1, -1, 1)$$

$$\text{check: } v_1 \cdot v_2 = (1, 1, 0) \cdot (1, -1, 1) = 0 \quad \checkmark$$

$$v_3 = u_3 - \left( \frac{u_3 \cdot v_1}{v_1 \cdot v_1} \right) v_1 - \left( \frac{u_3 \cdot v_2}{v_2 \cdot v_2} \right) v_2$$

$$= (1, 1, 1) - \frac{2}{2} (1, 1, 0) - \frac{1}{3} (1, -1, 1)$$

$$= \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

$$\text{check } v_1 \cdot v_3 = (1, 1, 0) \cdot \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = 0 \quad \checkmark$$

$$u_3 \cdot v_1 = (1, 1, 1) \cdot (1, 1, 0) = 2$$

$$u_3 \cdot v_2 = (1, 1, 1) \cdot (1, -1, 1) = 1$$

$$v_2 \cdot v_2 = (1, -1, 1) \cdot (1, -1, 1) = 3$$

$$v_2 \cdot v_3 = (1, -1, 1) \cdot \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = 0 \quad \checkmark$$

2. Let  $T: \mathbb{R}^2 \rightarrow P_2$  be a linear transformation such that

[4]

$$T(1, 2) = 7 - 3x + x^2, \quad T(-1, 1) = 1 + 2x - 4x^2$$

(a) Find  $T(1, 8)$ .

(b) Find a basis for  $\text{Im} T$  (note: this shouldn't take much effort based on the given info!)

$$\text{a) } (1, 8) = c_1 (1, 2) + c_2 (-1, 1)$$

$$= 3(1, 2) + 2(-1, 1)$$

$$\begin{aligned} c_1 - c_2 &= 1 \\ 2c_1 + c_2 &= 8 \\ \hline 3c_1 &= 9 \rightarrow c_1 = 3 \quad c_2 = c_1 - 1 \\ &= 2 \end{aligned}$$

$$\text{so } T(1, 8) = 3(7 - 3x + x^2) + 2(1 + 2x - 4x^2)$$

$$= 23 - 5x - 5x^2$$

b) Need LI spanning set. All vectors are LC of  $T(1, 2) \vee T(-1, 1)$  and these are LI so

$$\{7 - 3x + x^2, 1 + 2x - 4x^2\} \text{ is a basis for } \text{Im } T$$

3. Recall that in  $\mathbb{R}^2$ , the projection of a vector  $\mathbf{u}$  onto a vector  $\mathbf{v}$  is given by

[4]

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

(a) Prove that this is a linear transformation.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $T(\mathbf{u}) = \text{proj}_{\mathbf{v}} \mathbf{u}$

(b) Let  $\mathbf{v}_1 = (1, -2)$  and  $\mathbf{u} = (6, -3)$ . Express  $\mathbf{u}$  in terms of its projection onto  $\mathbf{v}_1$  and  $\mathbf{v}_2$  (where  $\mathbf{v}_2$  is a nonzero vector that is orthogonal to  $\mathbf{v}_1$ ).

a) Let  $u_1, u_2 \in \mathbb{R}^2$

$$\text{proj}_{\mathbf{v}} (u_1 + u_2) = \left( \frac{(u_1 + u_2) \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

$$= \left( \frac{u_1 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} + \left( \frac{u_2 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \text{proj}_{\mathbf{v}} u_1 + \text{proj}_{\mathbf{v}} u_2$$

preserves addition

Let  $\underline{u} \in \mathbb{R}^2$ ,  $c \in \mathbb{R}$

$$\text{proj}_{\mathbf{v}} (c\underline{u}) = \left( \frac{c\underline{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \left( \frac{c(\underline{u} \cdot \mathbf{v})}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = c \left[ \left( \frac{\underline{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \right] = c \text{proj}_{\mathbf{v}} \underline{u}$$

preserves scalar mult.

b)  $\mathbf{v}_1 = \begin{matrix} a & b \\ (1, -2) \end{matrix}$   $\mathbf{u} = (6, -3)$   
 $\mathbf{v}_2 = (-b, a) = (2, 1)$

So  $(6, -3) = \frac{12}{5}(1, -2) + \frac{9}{5}(2, 1)$

$$\mathbf{u} \cdot \mathbf{v}_1 = 6 + 6 = 12$$

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 1 + 4 = 5$$

$$\mathbf{u} \cdot \mathbf{v}_2 = 9$$

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 5$$

4. Prove that for a linear transformation  $T$ ,  $\text{Ker}(T) = \{0\}$  if and only if  $T$  is one-to-one. [4]

IF  $\text{Ker } T = \{0\}$ . Suppose  $T(v) = T(u)$ . Then  $T(v) - T(u) = 0$   
 $\Rightarrow T(v-u) = 0$ . Thus  $v-u \in \text{Ker } T$ ,  $v-u = 0$ .  
 $\Rightarrow v = u \quad \therefore 1-1$ .

IF  $T$  is 1-1. We have  $T(0) = 0$  already.

IF  $v \in \text{Ker } T$  then  $T(v) = 0 = T(0)$  so  $v = 0$

$\therefore \text{Ker } T = \{0\}$ .

5. The product

$$(f, g) = \int_{-1}^1 f(x)g(x)dx$$

is an inner product for the vector space  $C[-1, 1]$  of continuous functions on the interval  $[-1, 1]$  (don't show). [4]

(a) Show that the set  $\{1, x\}$  is orthogonal.

(b) Convert the orthogonal set to an orthonormal set.

$$a) (1, x) = \int_{-1}^1 (1)(x)dx = \left. \frac{x^2}{2} \right|_{-1}^1 = \frac{1}{2} - \frac{1}{2} = 0 \quad \therefore \text{orthogonal}$$

$$b) \|1\|^2 = \int_{-1}^1 (1)^2 dx = \left. x \right|_{-1}^1 = 2 \quad \text{so} \quad \|1\| = \sqrt{2}$$

$$\|x\|^2 = \int_{-1}^1 (x)^2 dx = \left. \frac{x^3}{3} \right|_{-1}^1 = \frac{2}{3} \quad \text{so} \quad \|x\| = \sqrt{\frac{2}{3}}$$

orthonormal set  $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x \right\}$

6. The linear transformation  $T : M^{2,2} \rightarrow M^{2,2}$  is given by

[5]

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & c \\ d & a \end{bmatrix}$$

(a) Find  $T^2$ ,  $T^3$  and  $T^4$  (think about what  $T$  does to the elements in a matrix).

(b) Find  $T^{162}$ . (Note- this shouldn't be difficult!)

(c) Prove that  $T$  is invertible and find its inverse.

a)  $T^2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$T^3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & a \\ b & c \end{bmatrix}$$

$$T^4\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \leftarrow \text{back to start so } T^4 = \text{identity}$$

b)  $T^{162} = ?$

$$162 = 4(40) + 2$$

$$T^{162} = (T^4)^{40} T^2 = (I)^{40} T^2 = T^2$$

$$\text{So } T^{162}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

c)  $\text{Ker } T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \begin{bmatrix} b & c \\ d & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$  So  $a=b=c=d=0$   
 $\text{Ker } T = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

nullity = 0,  $T$  is 1-1

$$T : M^{2,2} \rightarrow M^{2,2}$$

so  $\dim M^{2,2} = \text{nullity} + \text{rank}$   
 $4 = 0 + \text{rank}$

$\rightarrow \text{rank} = 4$   $T$  is onto.

$T$  is 1-1 & onto so is invertible

$$T^{-1} : M^{2,2} \rightarrow M^{2,2} \text{ is } T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & a \\ b & c \end{bmatrix} = T^3$$