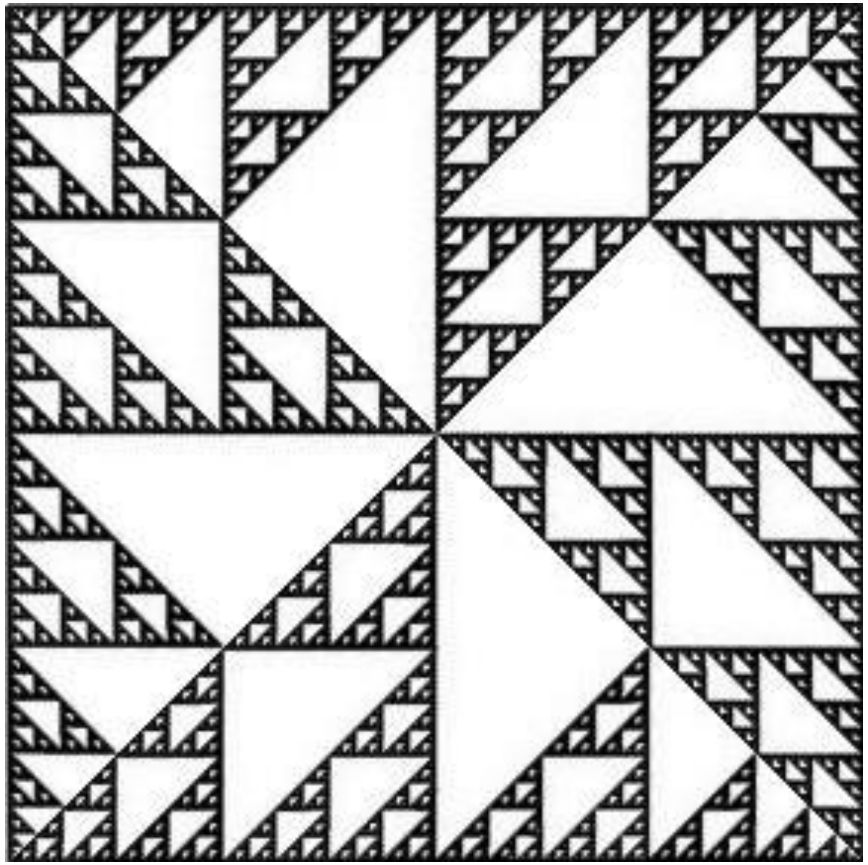
Using Triangle Sierpinski Relatives to Visualize Subgroups of the Symmetries of the Square



Tara Taylor **ttaylor@stfx.ca**

Department of Mathematics and Statistics

St. Francis Xavier University, Antigonish, NS, Canada

Bridges Halifax 2023

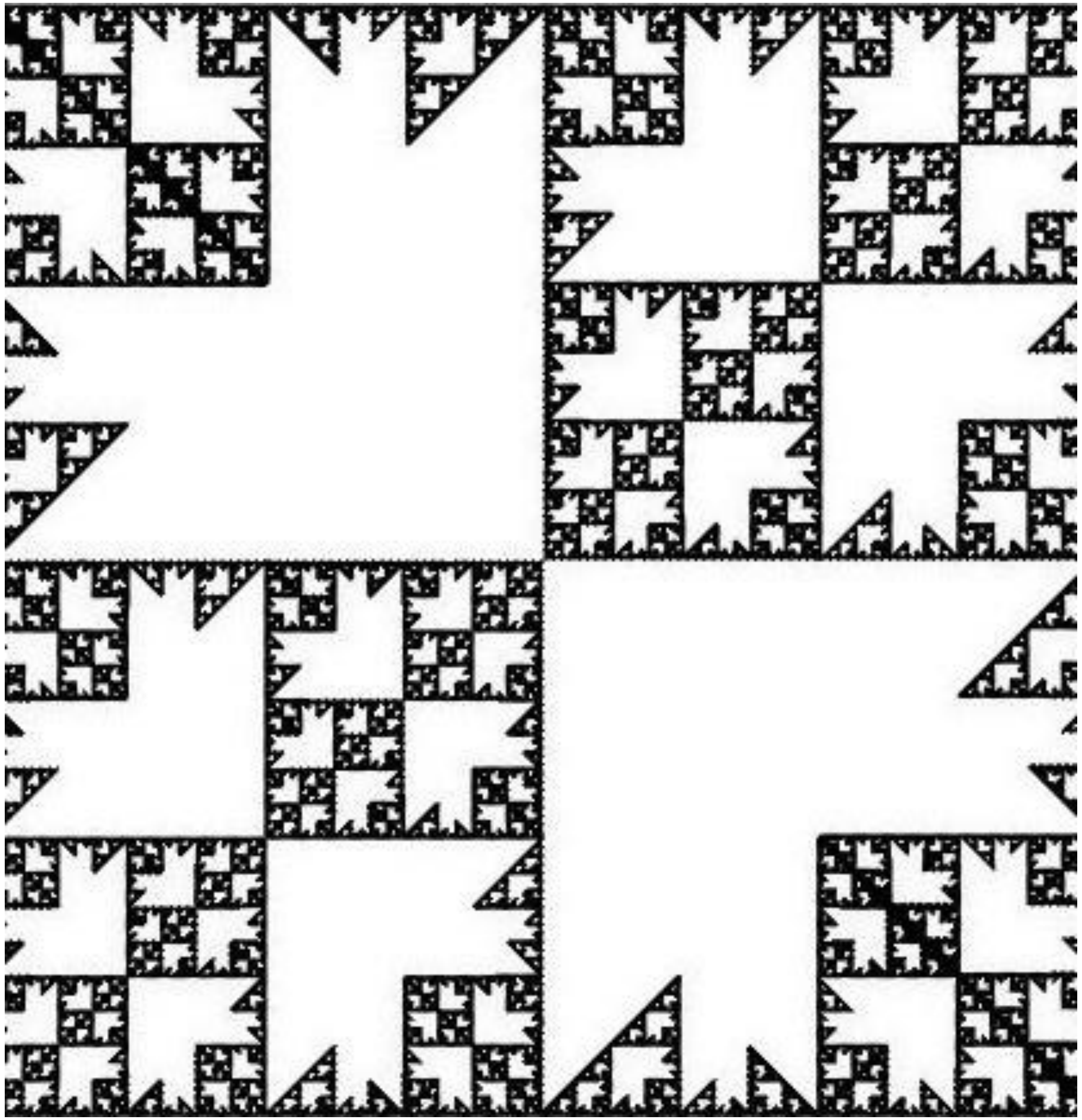
Land Acknowledgement

I *would like to begin by acknowledging* that we are in Mi’kma’ki, the ancestral and unceded territory of the Mi’kmaq People. Antigonish is of Mi'kmaq origin.

- According to Dr. Silas Rand, it is derived from “Nalegitkoonechk”, meaning "where branches are torn off." It is said that there the bears broke down branches to get the beech nuts.
- Second opinion: derived from "Nartigonneich" which translates to "a river of fish with many waters" or "the place where the waters meet."

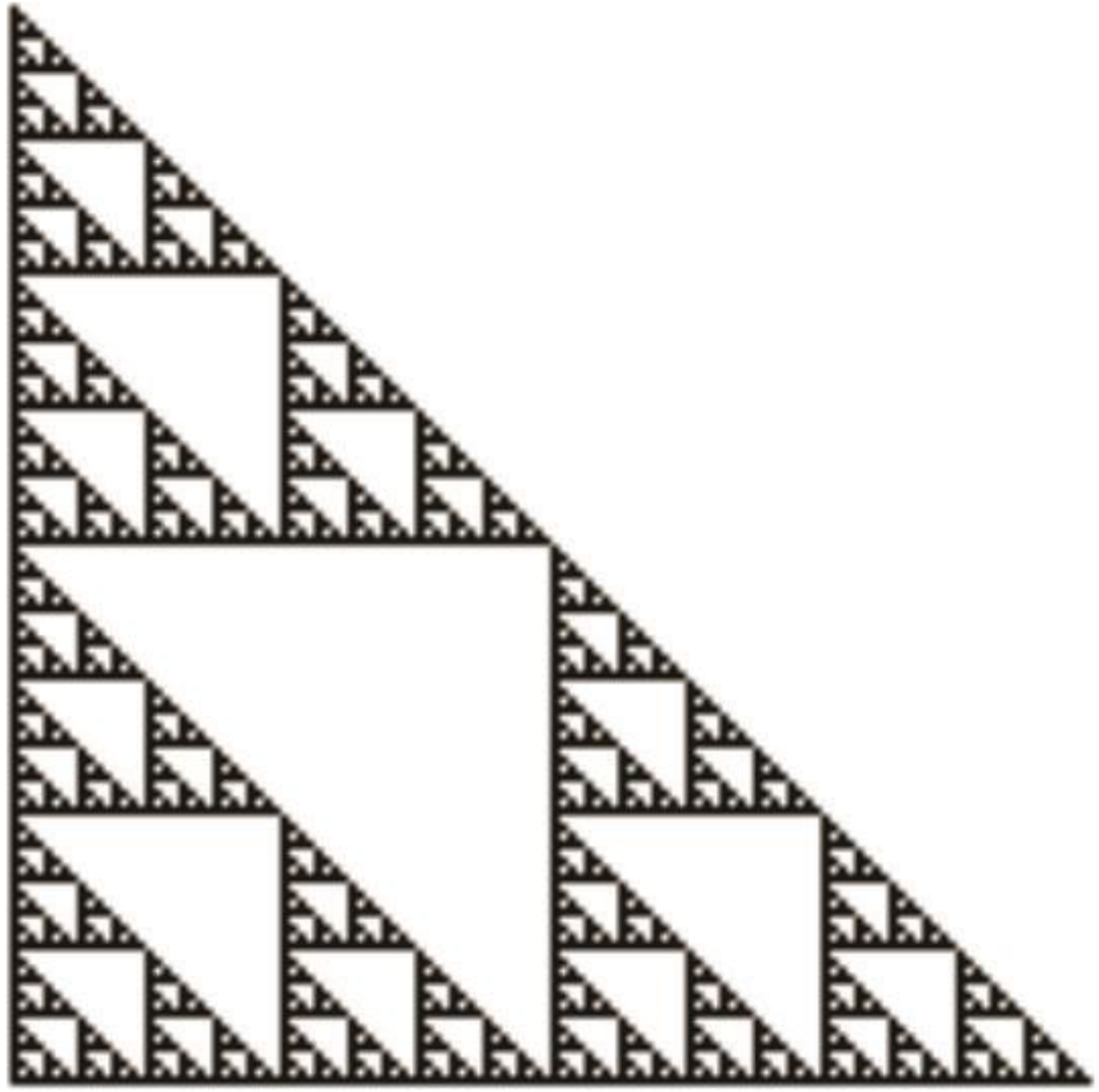
Outline

- Intro
- Math Background: Groups, symmetries of the square, iterated function systems, convex hulls
- Triangle Sierpinski Relatives
- Building new fractals



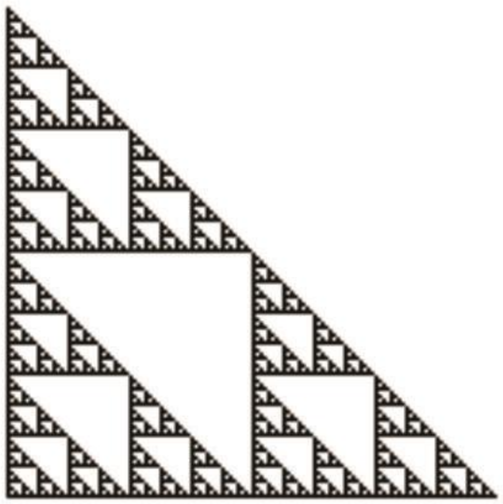
Sierpinski Gasket

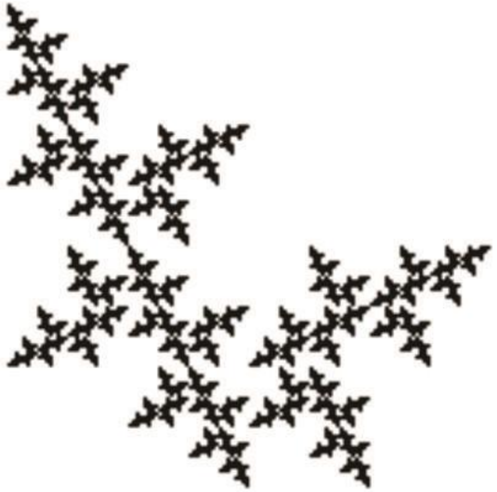
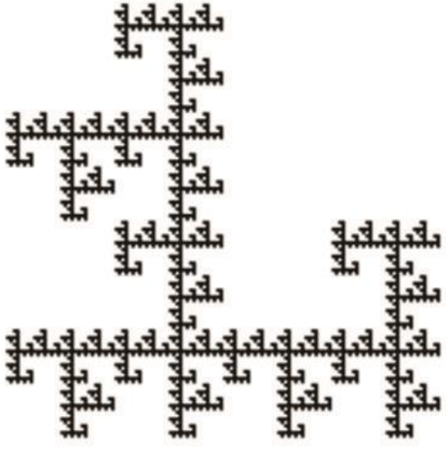
- Well-known fractal that can be expressed as the union of three smaller versions of itself (lengths scaled down by a factor of 2)
- fractal dimension equals $\frac{\ln 3}{\ln 2} \approx 1.585$



Sierpinski Relatives

- Relatives possess the same scaling properties and hence the **same fractal dimension**
- The **topological properties vary**
- The gasket is multiply-connected (possesses holes)
- Possible for a relative to be completely disconnected, disconnected with straight line segments, or simply-connected (no holes)



Groups

A group is a set of elements G with a binary operation $*$ such that $x*y \in G$ for all $x,y \in G$ and

- Associativity: $x*(y*z) = (x*y)*z$ for all $x,y,z \in G$;
- Identity element e such that $x*e = x = e*x$ for all $x \in G$;
- Every element x has an inverse y such that $x*y = e = y*x$.

The set of integers with addition is a group: the identity is 0; the inverse of x is $-x$.

The set of integers with multiplication is not a group: there is an identity but not every element has an inverse

Dihedral Groups

The nth Dihedral Groups, D_n, consist of the set of symmetries of the regular polygon with n sides under composition. D_n has $2n$ elements (n rotations and n reflections).

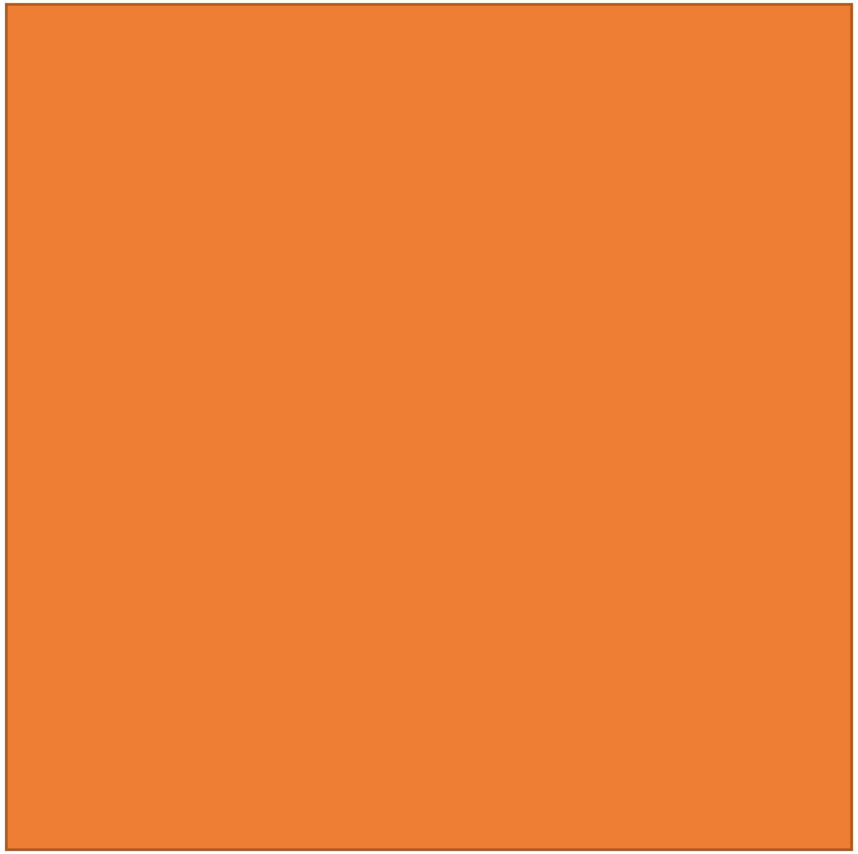
A group is abelian if the operation is commutative: $x*y=y*x$ for all $x,y \in G$

D_n is not abelian

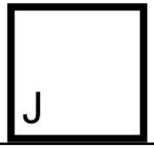
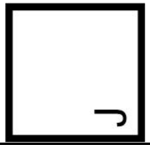
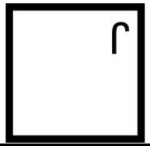
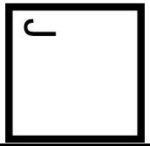
By Jim.belk - Modified from original Image File:Stop_sign_MUTCD.svg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=7502151

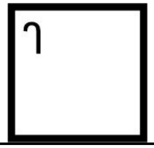
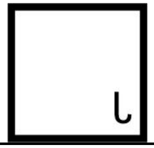
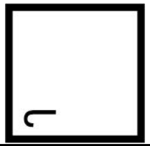
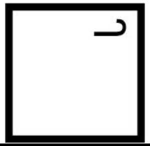
D_4: Symmetries of the Square

What are the symmetries of the square?



Symmetries of the Square

Label	Action	Verbal Description
a (ρ_0)		No change
b (ρ_1)		90° rotation counter-clockwise
c (ρ_2)		180° rotation
d (ρ_3)		270° rotation

Label	Action	Verbal Description
e (μ_2)		Reflection across horizontal line
f (μ_1)		Reflection across vertical line
g (δ_1)		Diagonal Reflection
h (δ_2)		Oher Diagonal Reflection

Binary Operation Table

- $x * y$ means do x first, then do y
- Each is row by column
- Order matters: $f * b = g$ but $b * f = h$
- Identity is a
- Every element has inverse

*	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	c	d	a	g	h	f	e
c	c	d	a	b	f	e	h	g
d	d	a	b	c	h	g	e	f
e	e	h	f	g	a	c	d	b
f	f	g	e	h	c	a	b	d
g	g	e	h	f	b	d	a	c
h	h	f	g	e	d	b	c	a

Subgroups

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$\{a, b, c, d\}$

*	a	c	e	f
a	a	c	e	f
c	c	a	f	e
e	e	f	a	c
f	f	e	c	a

$\{a, c, e, f\}$

*	a	c	g	h
a	a	c	g	h
c	c	a	h	g
g	g	h	a	c
h	h	g	c	a

$\{a, c, g, h\}$

*	a	c
a	a	c
c	c	a

$\{a, c\}$

*	a	e
a	a	e
e	e	a

$\{a, e\}$

*	a	f
a	a	f
f	f	a

$\{a, f\}$

*	a	g
a	a	g
g	g	a

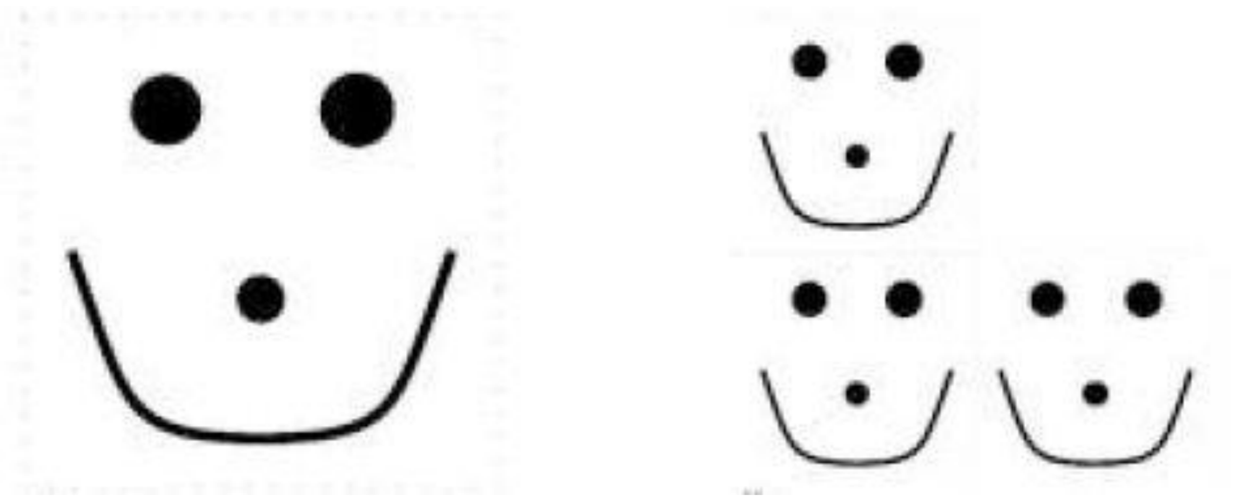
$\{a, g\}$

*	a	h
a	a	h
h	h	a

$\{a, h\}$

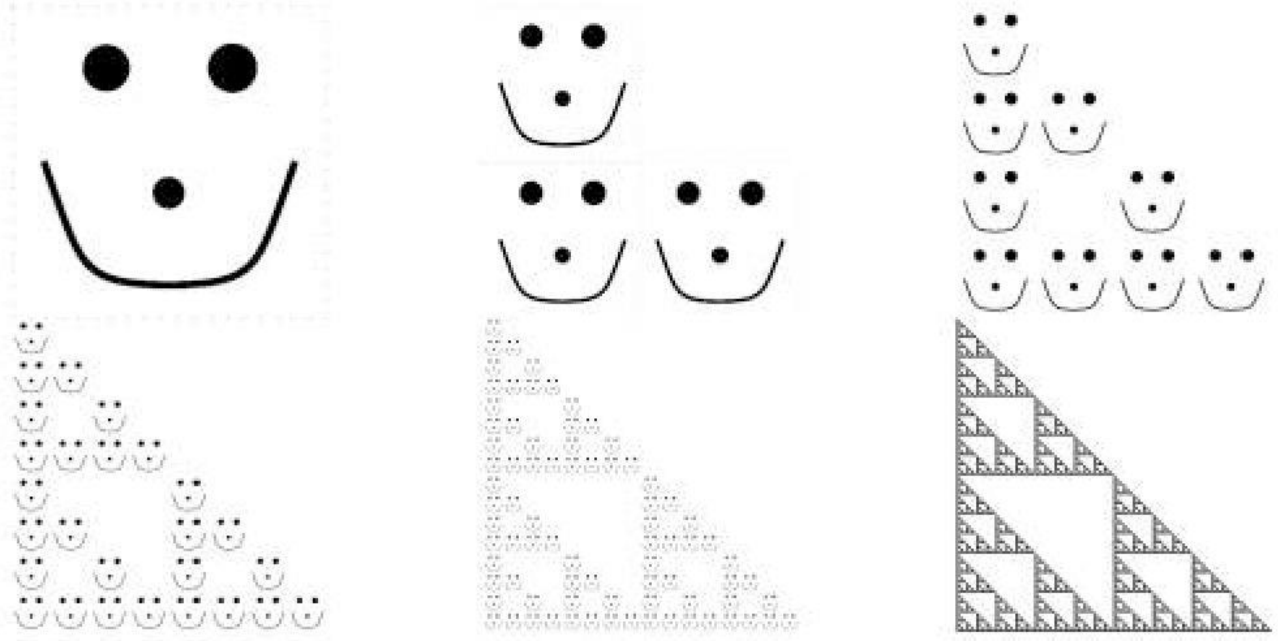
- A *subgroup* of a group is a subset that is also a group under the same operation
- Every group has itself and the set containing only the identity as subgroups

Iterated Function Systems (IFS)



- An *iterated function system* (IFS) is a collection of contractive mappings $\{f_1, f_2, \ldots, f_n\}$.
- A given IFS has a unique attractor A that satisfies
$$A = f_1(A) \cup f_2(A) \ldots f_n(A)$$
- Can start with any compact set and apply maps

IFS

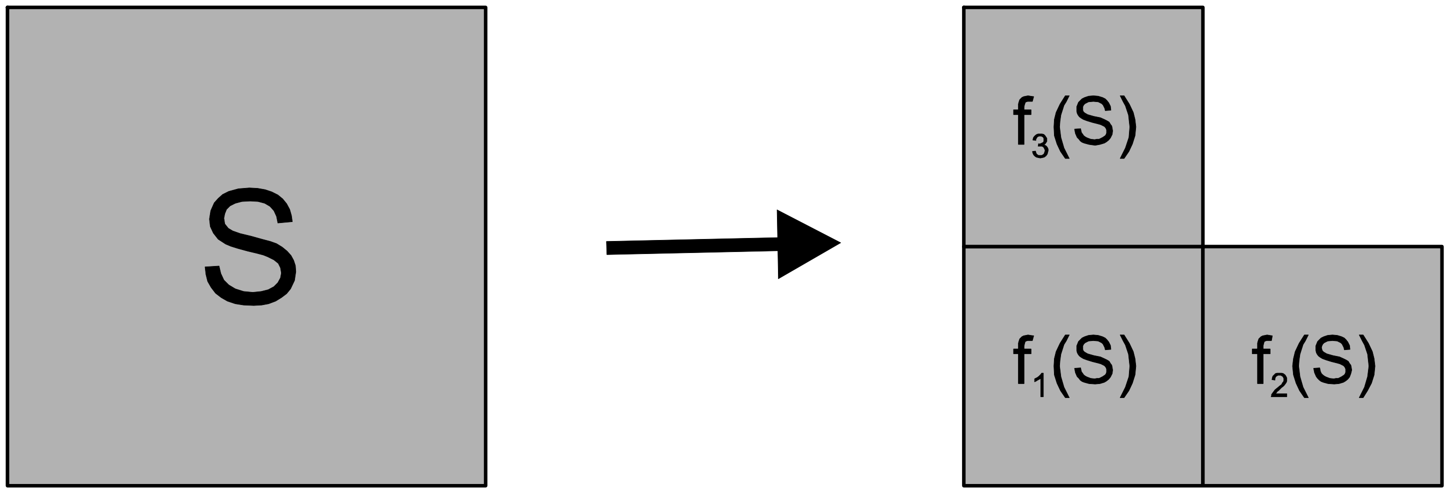


Form a sequence $\{A_n\}$.
$A_0 = X$ and for $n \geq 1$:

$$A_n = \bigcup_{i=1}^{n} f_i(A_{n-1})$$

The limit of the approximations as $n \to \infty$ is A.

Generating the Relatives using Symmetries of the Square

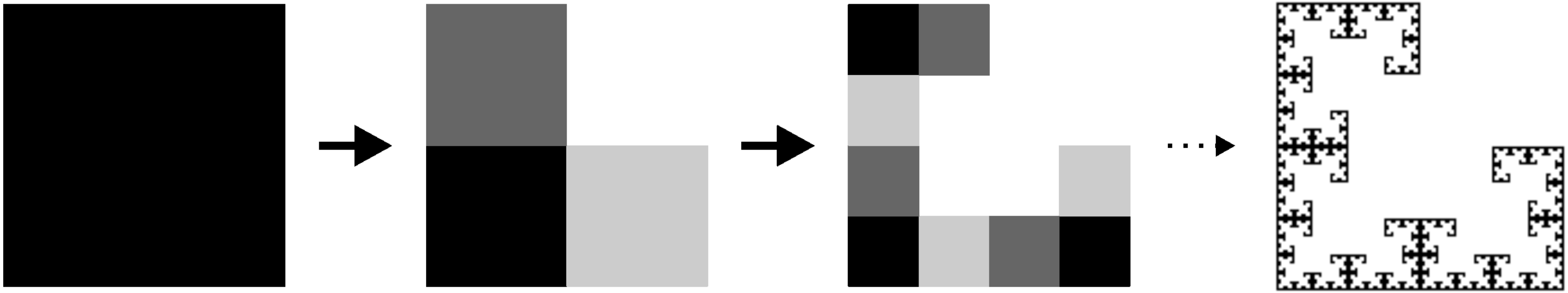


Relative R_{xyz} can be obtained from an IFS $\{f_1, f_2, f_3\}$ acting on the unit square S

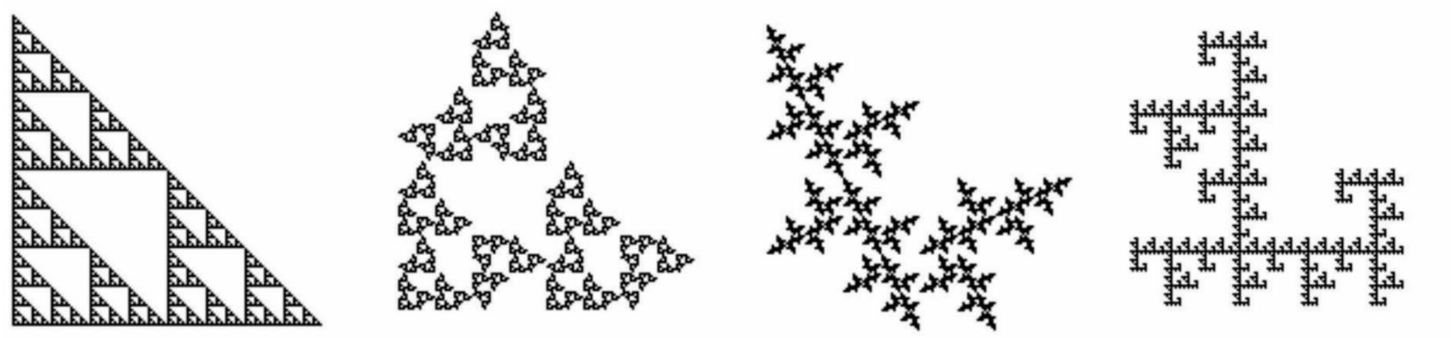
- f_1 involves contraction and the symmetry x
- f_2 involves contraction, the symmetry y and horizontal translation by ½
- f_3 involves contraction, the symmetry z and vertical translation by ½

Example of Sierpinski Relative

R_{abd}



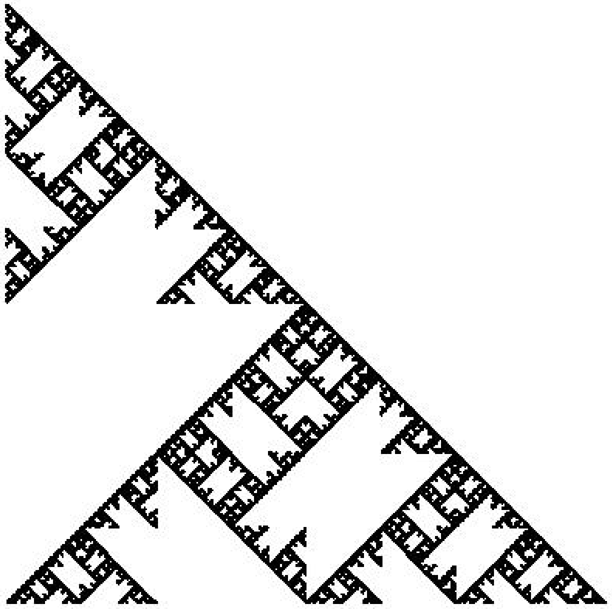
Sierpinski Relatives

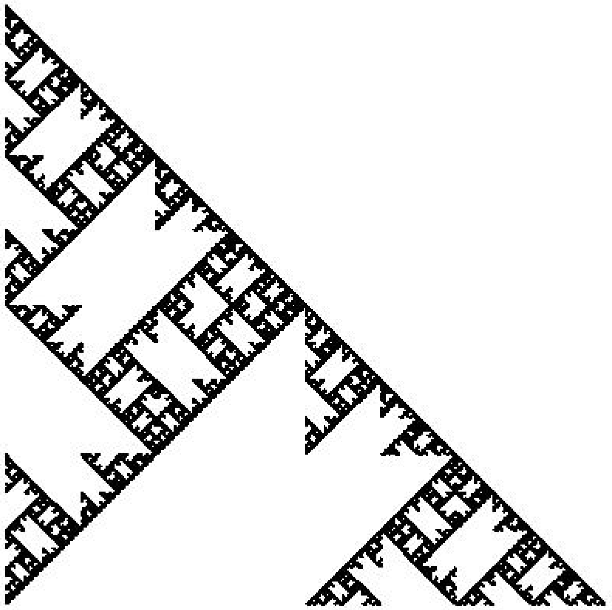


- A relative is self-similar because it is made from smaller versions of itself without gaps or overlaps.
- The similarity dimension D of a self-similar object satisfies $Nr^D = 1,$ where N is the number of smaller versions and r is the scaling ratio.
- $N = 3$ and $r = 1/2$. Thus $D = \ln 3 \,/\, \ln 2 \approx 1.585.$
- There are 8 choices for each symmetry so $8^3 = 512$ possibilities for the IFS.
- The attractors are not all distinct; there are 232 distinct attractors.

Congruent Pairs

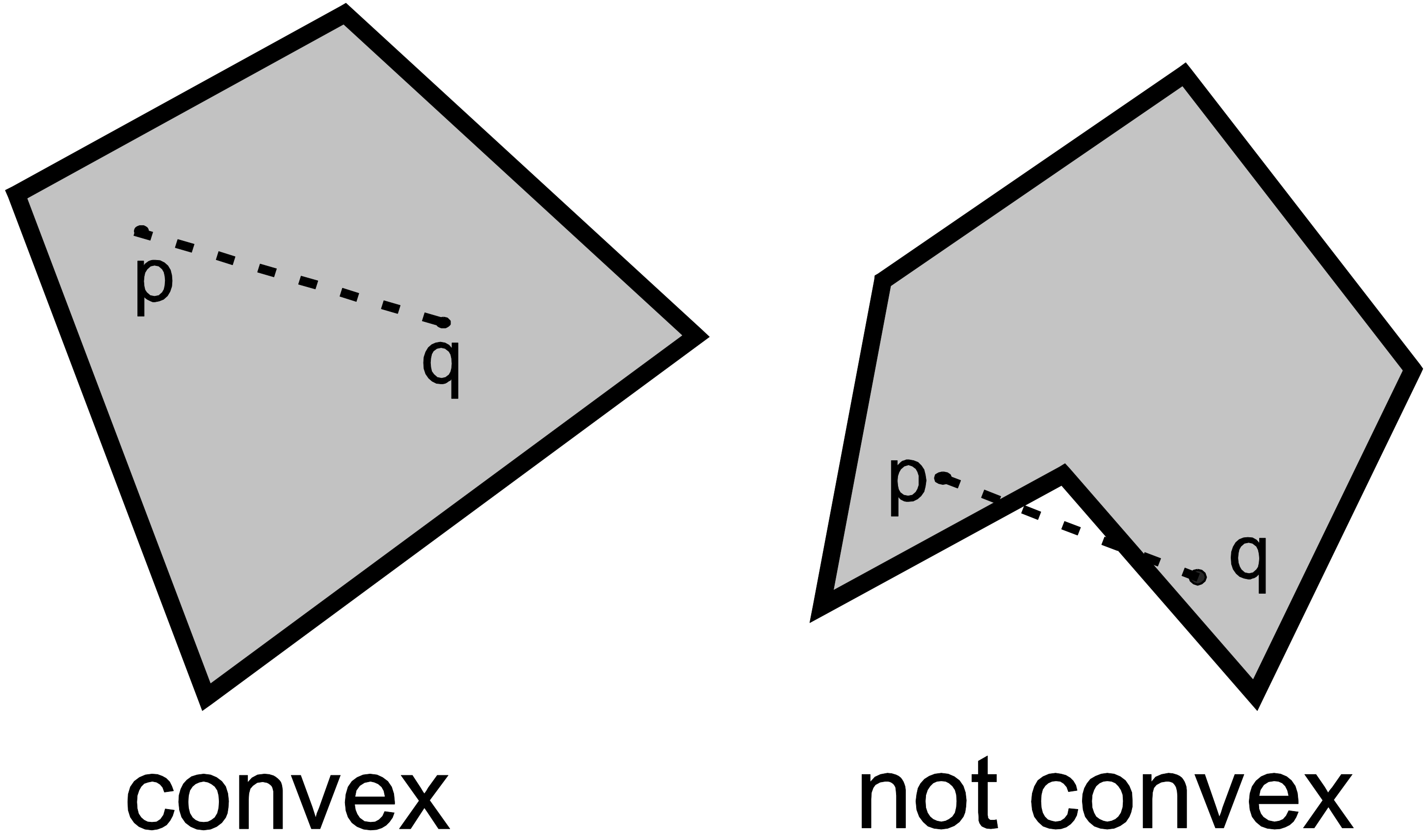
- Some relatives are symmetric via the diagonal symmetry g
- g is the only non-trivial isometry between two relatives
- Non-symmetric relatives come in congruent pairs
- R and R' are congruent if and only if $R' = g(R)$
- Can obtain the unique *congruent match* to $R = R_{xyz}$: $R' = R_{uvw}$, where $u = gxg, v = gzg, w = gyg$
- R_{bag} and its congruent match R_{dga}





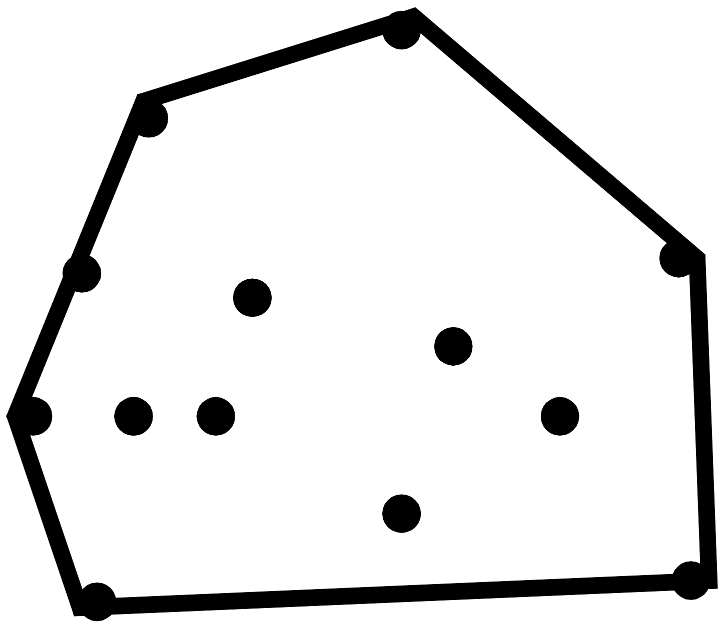
Convex Sets

A set A is convex if for any two points p and q in the set, the line segment $\overline{pq}$ joining them is also in the set



Convex Hulls

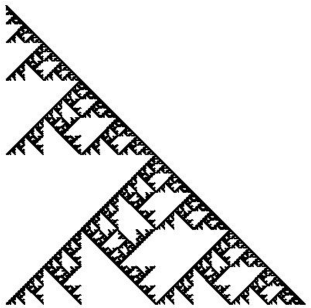
- The convex hull of a set is the smallest convex set that contains the set
- Can visualize the convex hull of a set of points is that it includes its boundary which is like an elastic band around the points and it has everything inside the elastic band
- Many real-world applications to convex hulls (ex. tumour analysis)



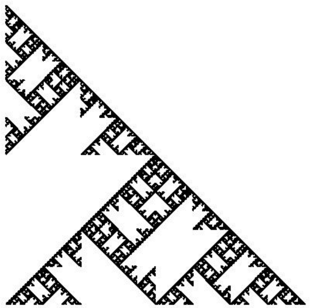
convex hull boundary

Triangle Relatives

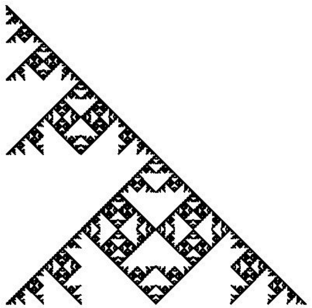
These 8 relatives and their congruent matches are the only non-symmetric relatives that have the same triangle convex hull as the Sierpinski gasket



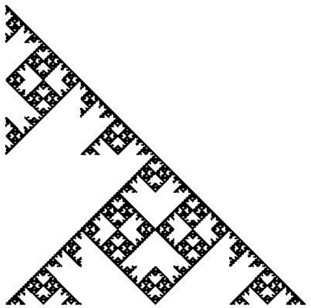
R_{baa}



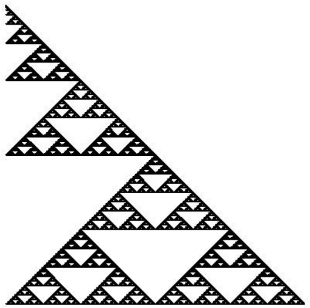
R_{bag}



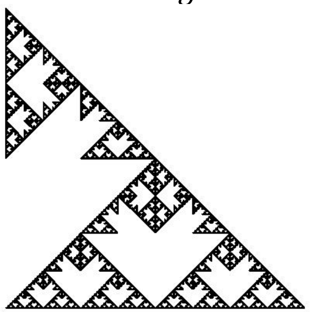
R_{bga}



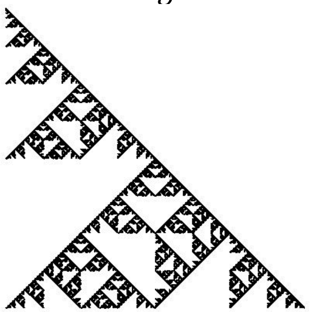
R_{bgg}



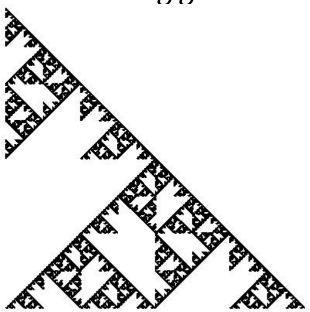
R_{faa}



R_{fag}



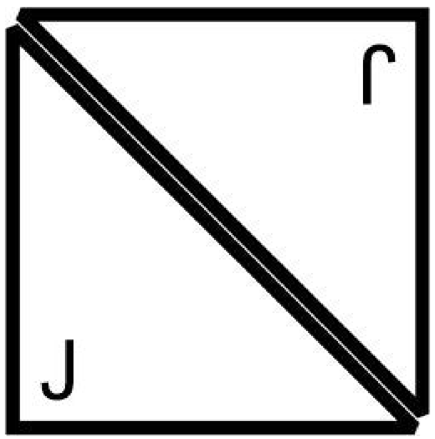
R_{fga}



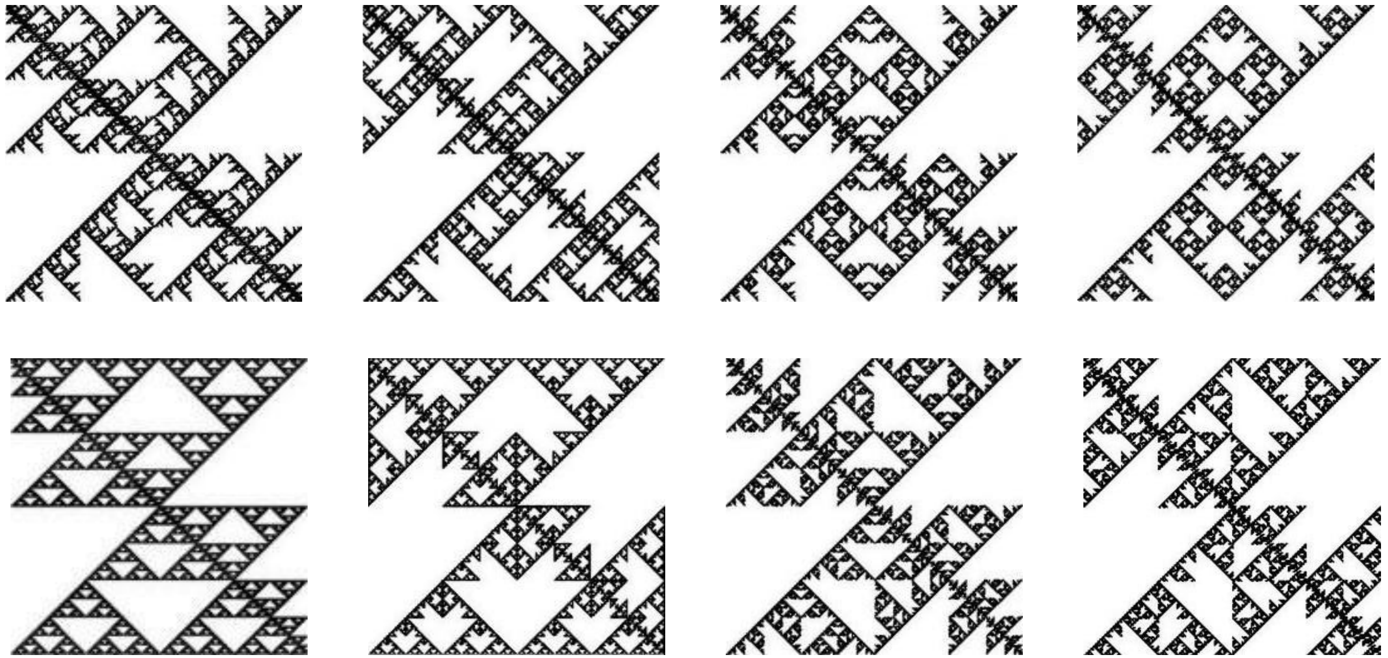
R_{fgg}

Square Fractals made with Triangle Relatives

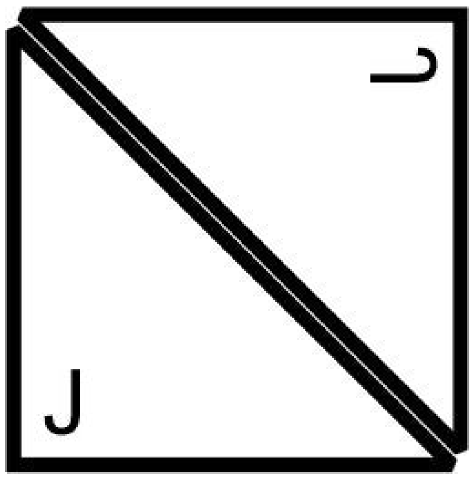
Idea: find all non-trivial subgroups with the fewest number of triangle relatives



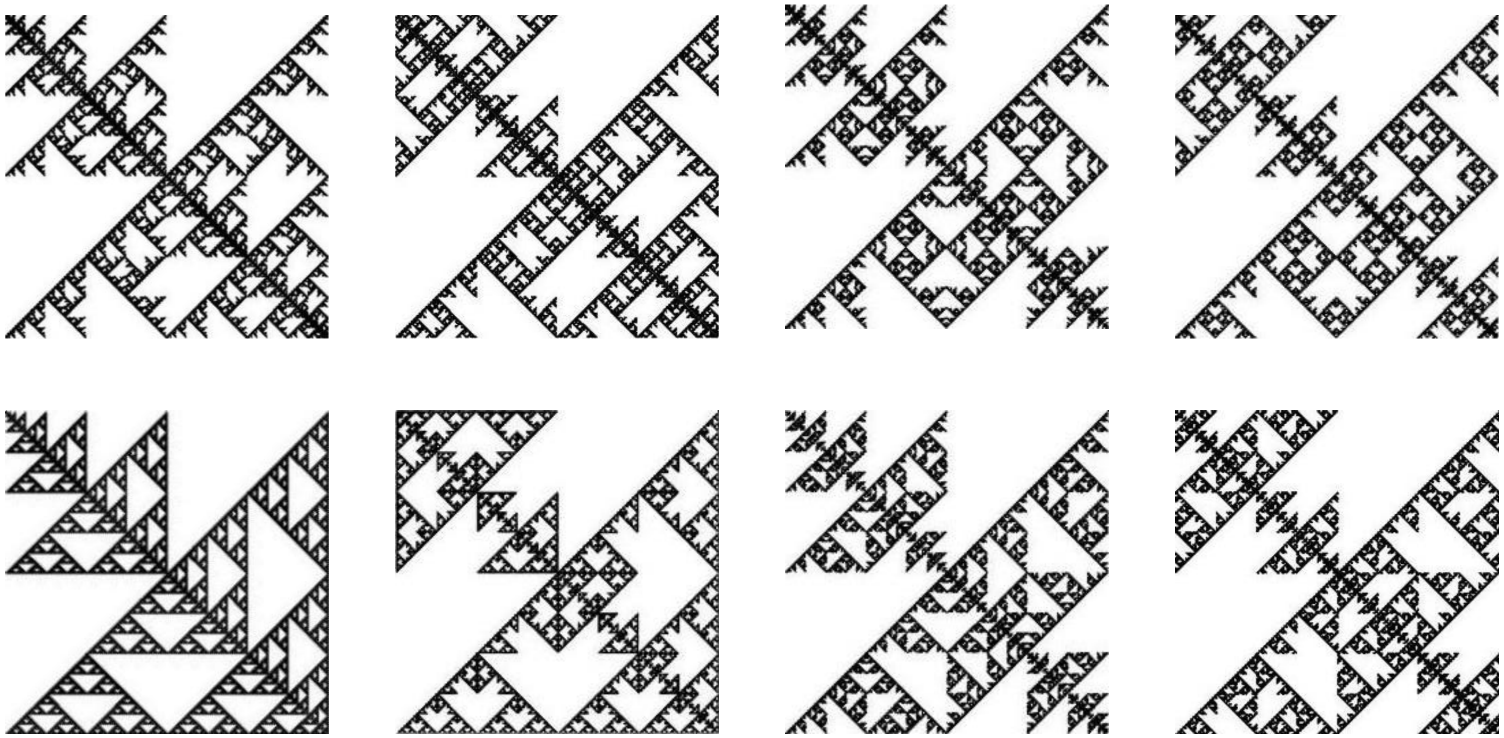
$\{a, c\}$



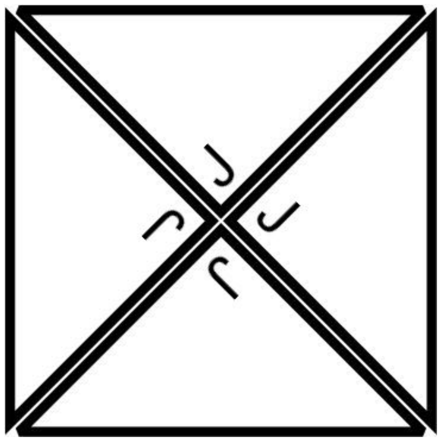
Square Fractals made with Triangle Relatives



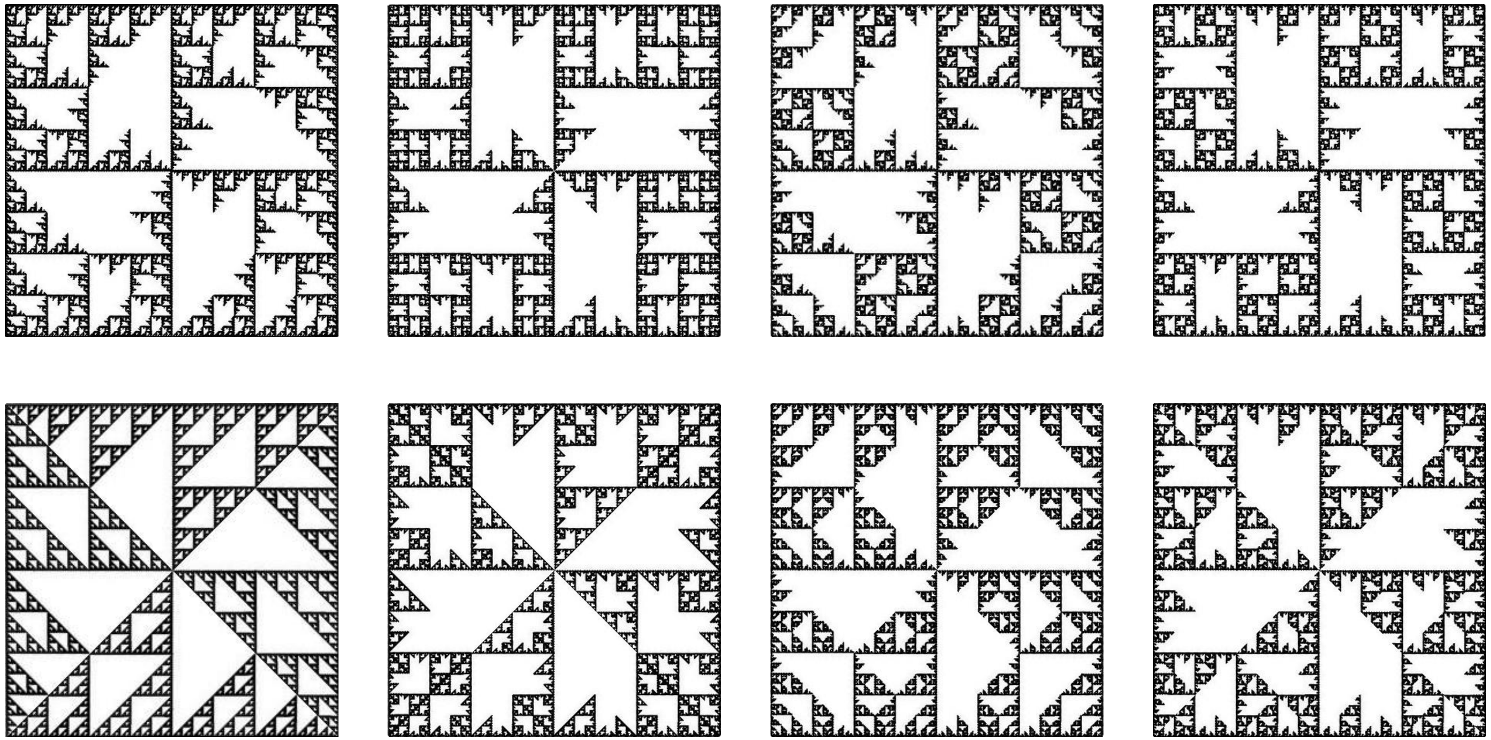
$\{a, h\}$



Square Fractals made with Triangle Relatives



$\{a, b, c, d\}$



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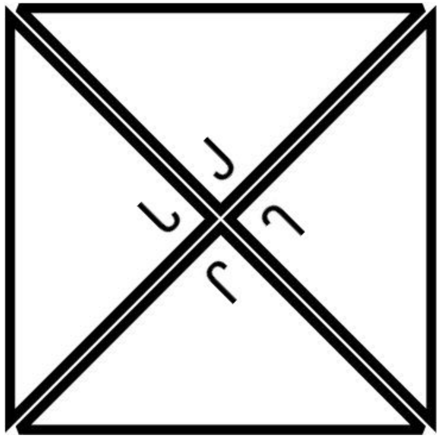
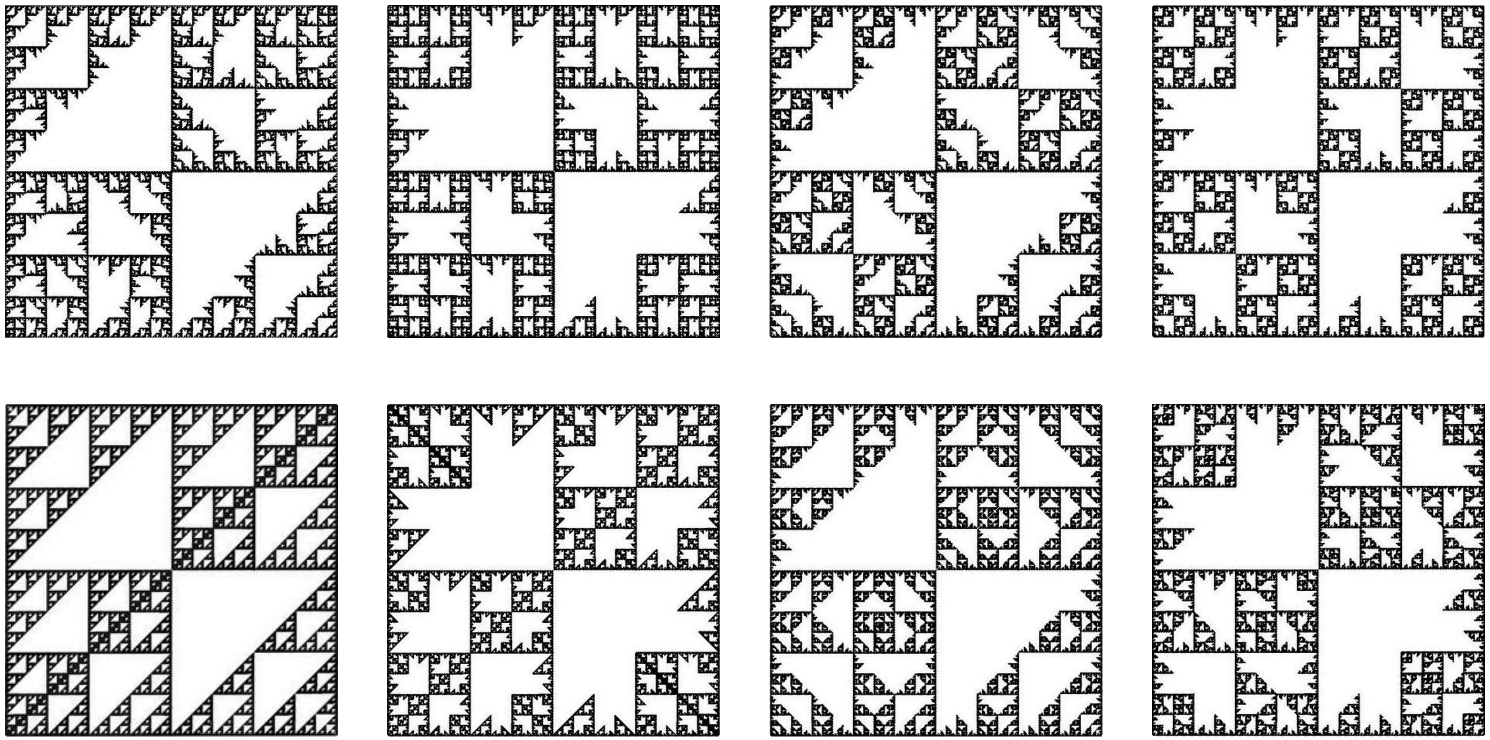
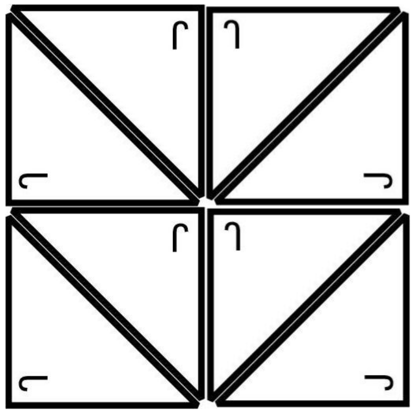


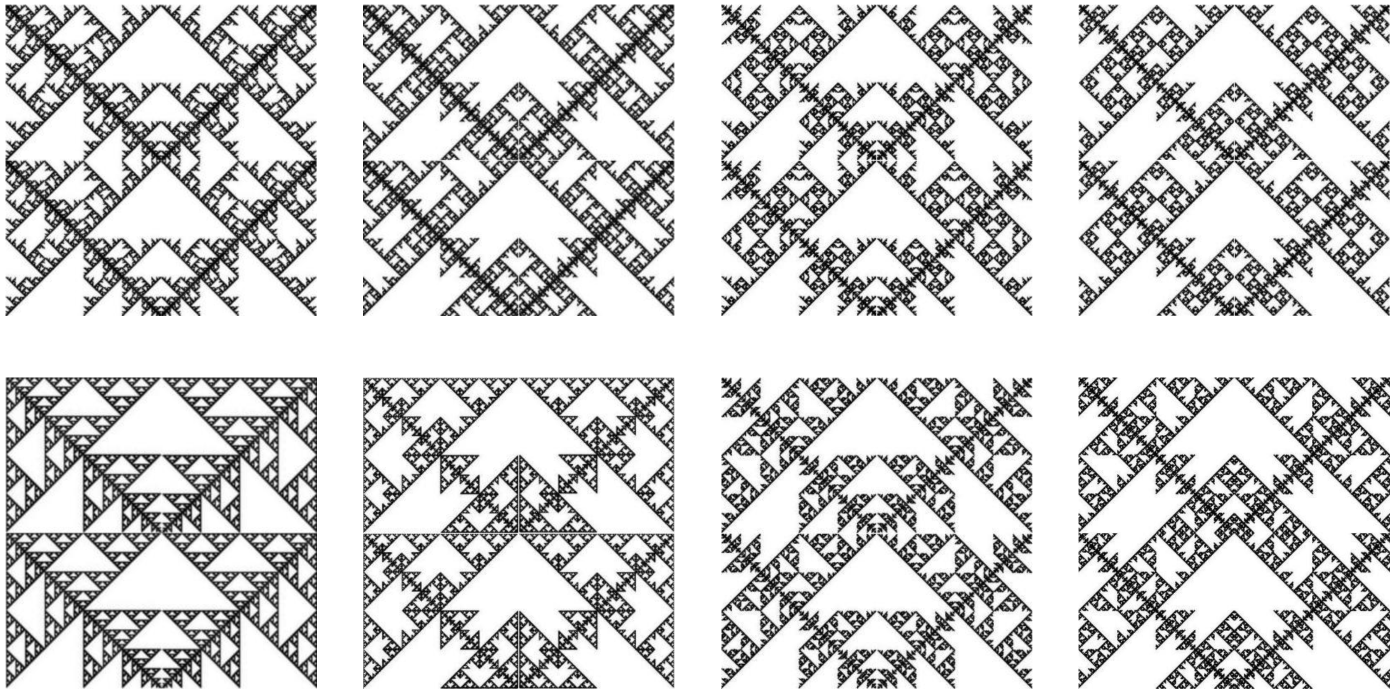
$\{a, c, g, h\}$



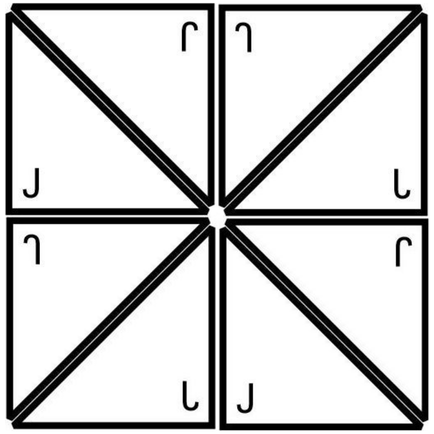
Square Fractals made with Triangle Relatives



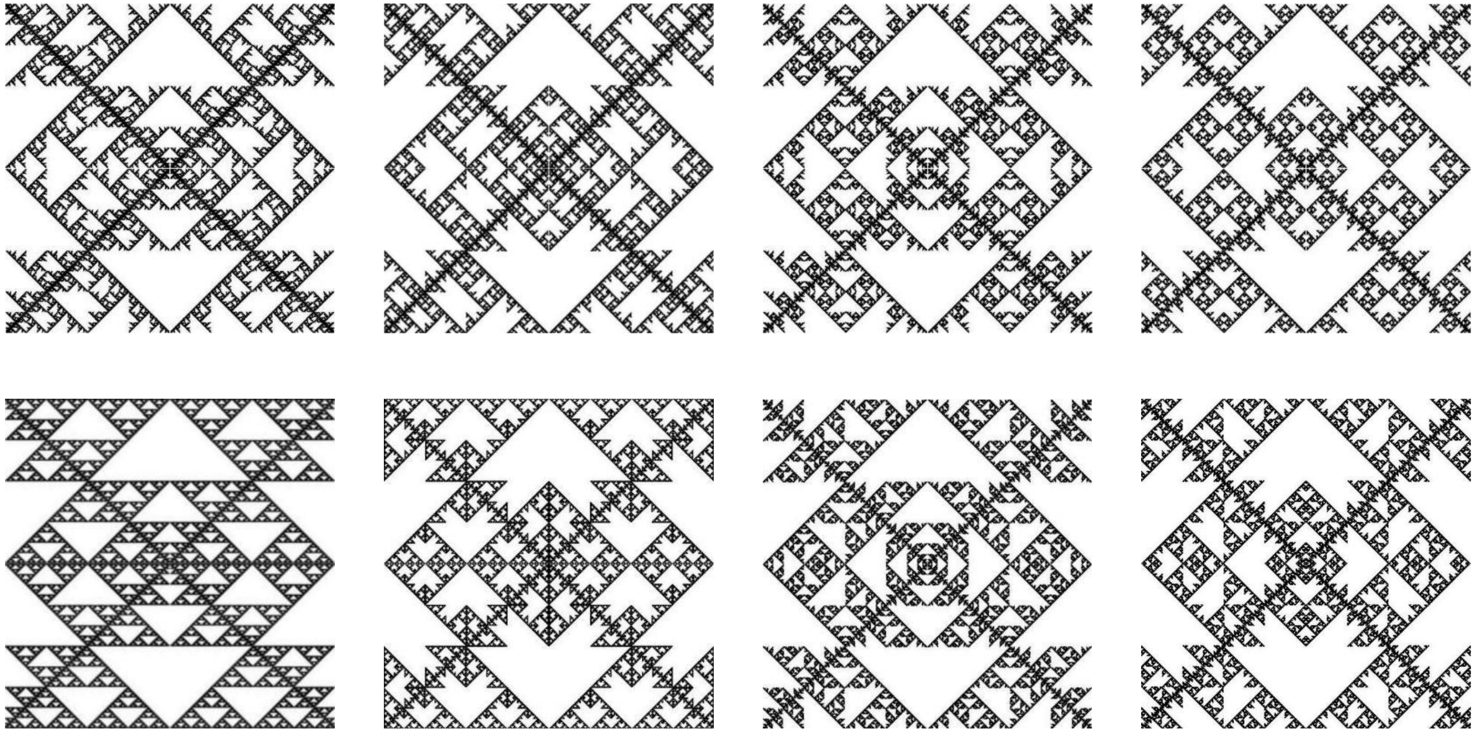
$\{a, f\}$



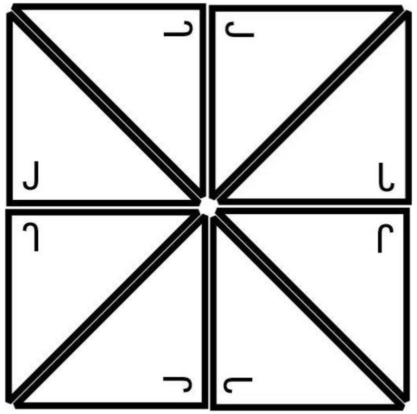
Square Fractals made with Triangle Relatives



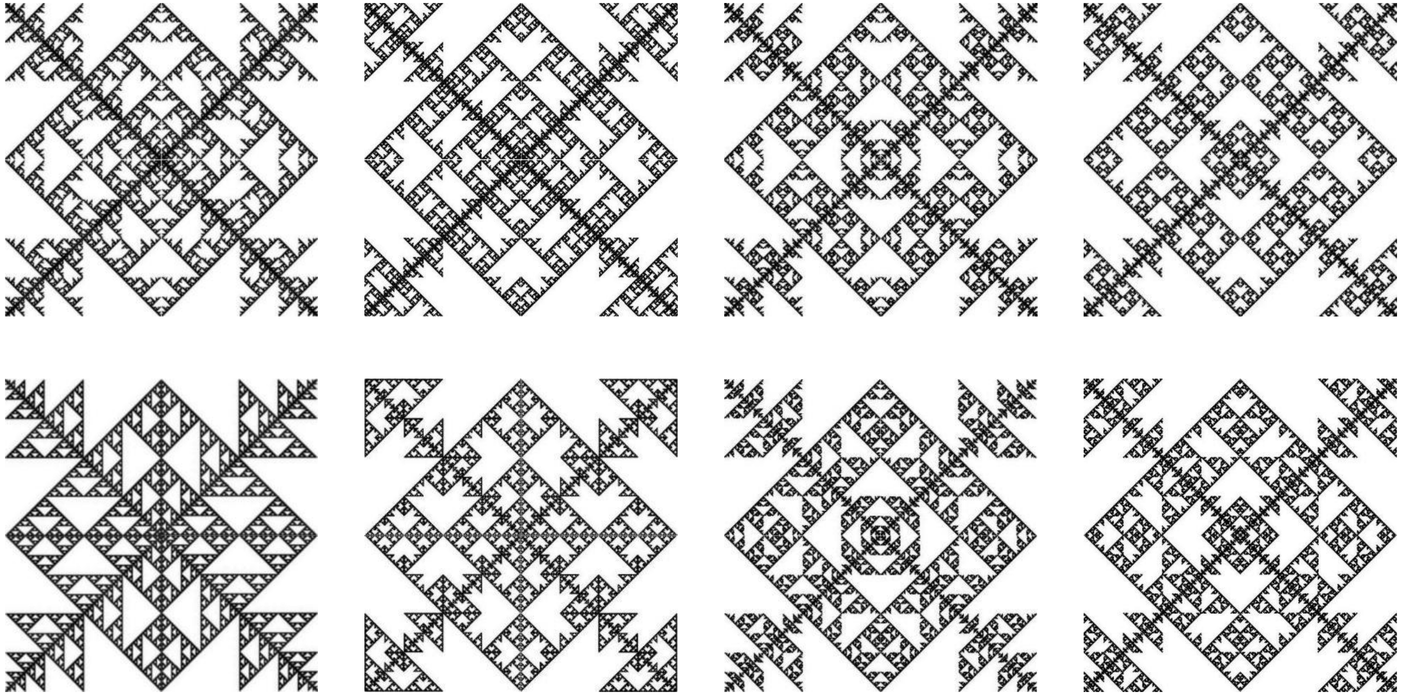
$\{a, c, e, f\}$



Square Fractals made with Triangle Relatives



$\{a, b, c, d, e, f, g, h\}$



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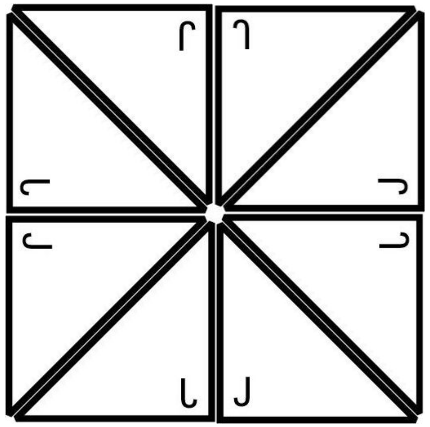
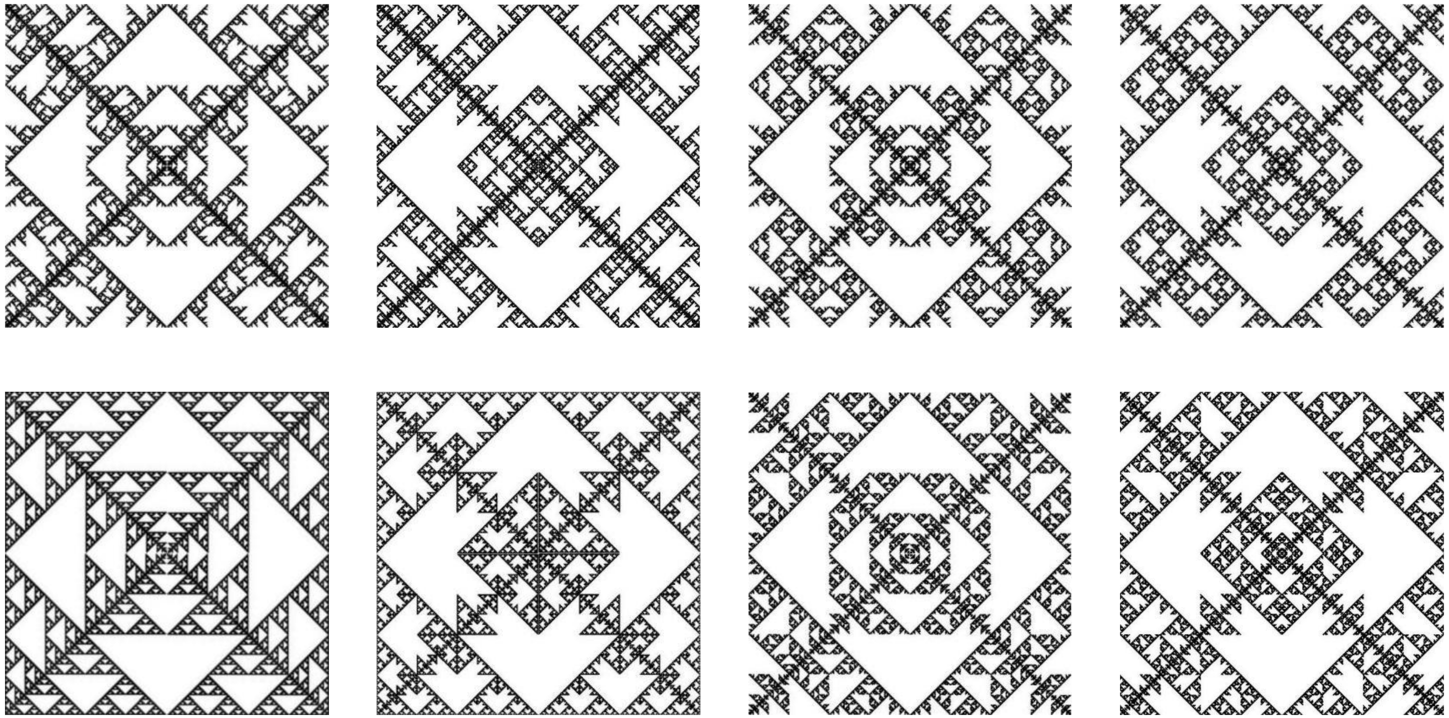
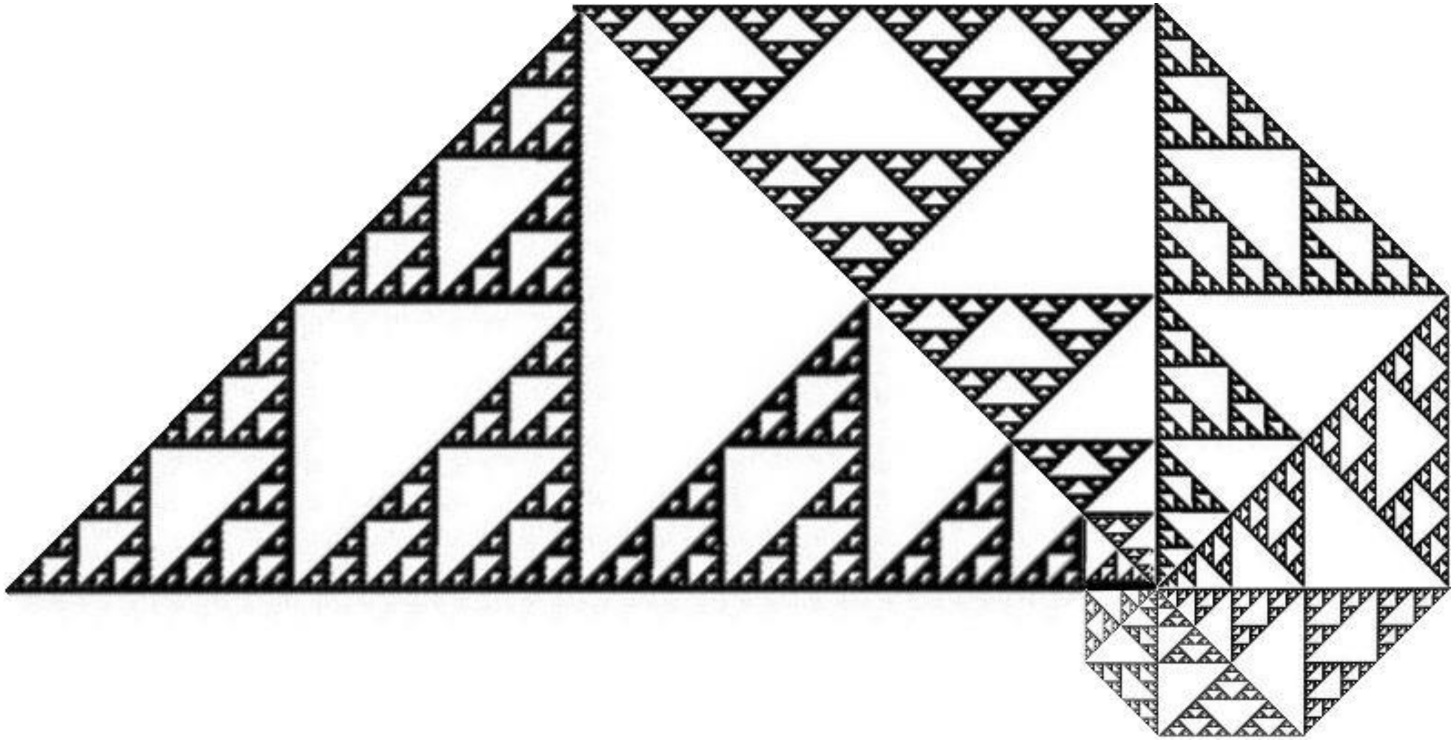


$\{a, b, c, d, e, f, g, h\}$



Squares are just the beginning…



Thank you!

ttaylor@stfx.ca