December Exam

MATH 100:11, St. Francis Xavier University

December 14, 2007 — 9am-12pm
Instructor: Tara Taylor — Location: NH138

NAME (PRINT) SOLUTIONS

STUDENT NUMBER _____

SIGNATURE _____

- You can use a calculator, one index-card sized formula sheet (both sides), and your square for the symmetries of the square.
- Please write answers on the question sheets, and use the back sides for scrap paper.
- There are three sections to this exam. The first section consists of 10 True/False questions each worth 1 mark, for a total of 10 marks. The second section consists long answer questions for a total of 35 marks. There is a bonus question worth 3 bonus marks at the end of the exam.
- The entire exam is out of 45 marks.

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1 True/False Questions

Each question is worth 1 mark, for a total of 10 marks. No explanation is required, just fill in T for true or F for false in the blank before the statement.

1. T If there are twelve people in total to pick from, the number of ways to pick a group of 3 people is the same as the number of ways to pick 9 people. From Pascal's Δ

2. T $5 + 6 + 7 + \cdots + 999 + 1000 = 500490$ $1+2+\ldots 1000 = \frac{1000(1001)}{2} = 500500$

$1+2+3+4 = 10$

3. F The set $A = \{x | x$ is a letter of the word 'snowflake'$\}$ is equivalent to the set $B = \{x | x$ is a letter of the word 'chocolate' $\}$.

4. F For any sets A and B, $(A \cap B)' = A' \cap B'$.

5. T "There exists an integer that is not a rational number" is a statement.

6. T The converse of the statement "If it is Monday, then we have class." is "If we have class, then it is Monday."

7. F The Egyptian algorithm is a method of coming up with new mathematical patterns for drums.

8. F If x and y are both irrational numbers, then $x \cdot y$ is also irrational.

9. F All numbers of the form $2^n - 1$ are prime.

10. F Washing the dishes and drying the dishes are commutative.

2 Long Answer Questions

This section has a total of 35 marks.

1. What does it mean for a set to be 'well-defined'? Give an example of something that is not well-defined, and explain why not. [2]

Given any object, you can determine whether the object is in the set or not

ex $A = \{x \mid x \text{ is a good soccer player}\}$

not well defined because 'good' is subjective

2. If $A = \{1, 2, 2, 2, 3, \{1\}, \{1, 2\}\}$, how many subsets does A have? Explain. [DO NOT list the subsets] [2]

$n(A) = 5$

so $2^5 = 32$ subsets

3. Consider the statement

$$[q \to (r \wedge p)] \vee [\sim s \to p]$$

Suppose that r and s are both false. Find all possible truth values of p and q so that the entire compound statement is true (explain). [3]

$\vee$ - need just one part true

For $q \to (r \wedge p)$ true - r is false, so $r \wedge p$ is false if p is T or F

so need q to be F

For $\sim s \to p$ true s F, $\sim s$ is T so p is T

So if p is T q could be T or F

if p is F q has to be F

4. Find $7234_8 - 4567_8$. [2]

$6_{1} + 8 = 9$, $2 + 8 = 10$, $7 + 8 = 12$

$$\begin{array}{r} 7234_8 \\ -\,4567_8 \\ \hline 2445_8 \end{array}$$

5. The local radio station surveyed a group of 100 students about their musical tastes in terms of Alternative Rock, Hip Hop and Country. The following results were obtained:

- 78 students listen to Alternative Rock ✓
- 65 students do not listen to country
- Every student listens to at least one kind of music
- 12 students listen to all three types of music ✓
- 16 students listen to both hip hop and country ✓
- 25 students listen to Alternative Rock and Country ✓
- 30 students listen to exactly two types of music ✓

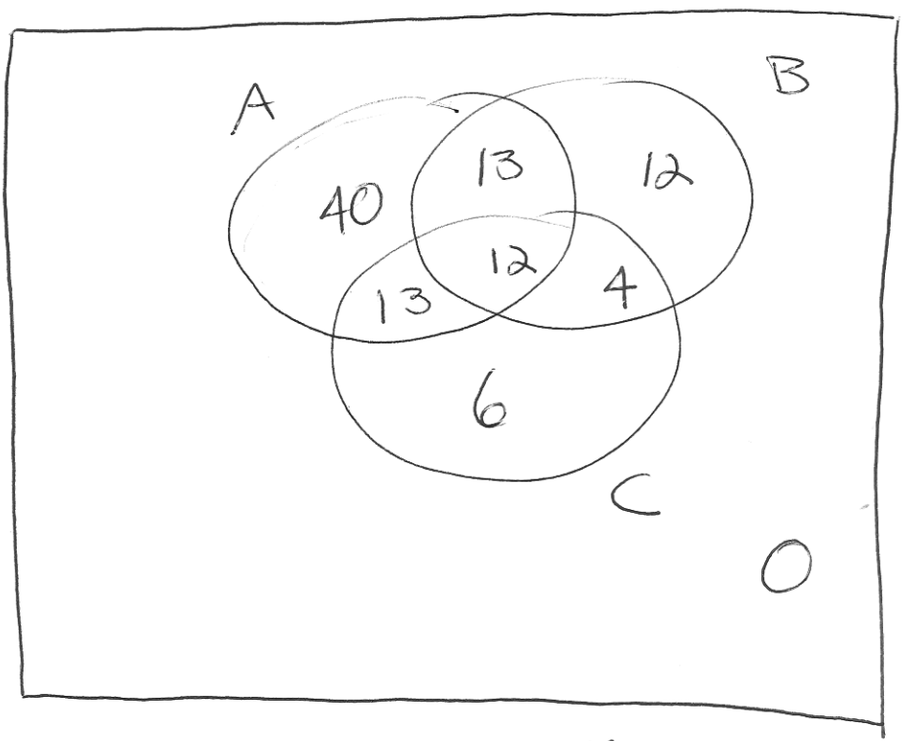
(a) Draw a Venn diagram to represent the situation. Be sure to label the sets and explain where the numbers come from.

(b) How many students listen to Country or Alternative Rock, but not Hip Hop? [5]

A = Alternative Rock

B = Hip Hop

C = Country



a) 12 listen to all 3 - put in middle
16 listen to hiphop + country - 16-12 = 4 left
25 - alt + C 25-12 = 13
30 listen to exactly 2 - so far 13+4=17 so 13 left
78 listen to alt so far 13+13+12 = 38 so 40 left
Every student listens to at least 1 - so none outside
65 not country - so 35 country 13+12+4 = 29, 6 left
100 total so 78+10=88 so far, 12 left

b) C or A but not B? 40+13+6 = 59

6. Consider the following argument:

> If it snows, then James will go skiing. If it doesn't snow, then James will go to a movie. Therefore, if James goes to a movie, then he does not go skiing.

Is the argument valid? Explain why or why not. [3]

Symbols

$$\begin{array}{l} p \to q \\ \sim p \to r \\ \hline \therefore r \to \sim q \end{array}$$

Not valid

Assume premises are true. Does this force conclusion to be true? Only false if r is T, ~q is F, ie q is F. If q is F, then p is F to make p→q T. If p is F, then ~p is T, forces r T ok. So ok if r→~q is F in words - he could go to a movie and go skiing - if he goes to a movie when it snows.

7. (a) Based on the prime factorization for the number 30, explain how you can test whether or not a number is divisible by 30.

(b) Use your divisibility test on the number 24570 to verify that it is a multiple of 30.

[3]

a) $30 = 2 \times 3 \times 5$

needs to pass all 3 divisibility tests

- needs to be even
- sum of digits is multiple of 3
- ends in 0 or 5

$\Rightarrow$ ends in 0 & sum of digits is multiple of 3

b) 24570 ends in 0 ✓

$2+4+5+7 = 18 = 3(6)$ multiple of 3

So yes 24570 is a multiple of 30

8. Convert the decimal $1.533333\cdots = 1.5\overline{3}$ to a fraction in lowest terms. [Hint: split the decimal into a terminal part and a repeating part and convert each part to a fraction]. [2]

$1.533\ldots = 1.5 + 0.0\overline{3}$

$1.5 = \frac{15}{10} = \frac{3}{2}$ $\qquad 0.\overline{3} = \frac{1}{3}$ so $0.0\overline{3} = \frac{1}{30}$

$$\frac{3}{2} + \frac{1}{30} = \frac{45}{30} + \frac{1}{30} = \frac{46}{30} = \frac{23}{15}$$

9. The ancient Egyptian system was base 10 with 7 distinct symbols, so the largest number possible in that system is 9,999,999. What is the largest number possible in a simple grouping system that is base 5 with 8 distinct symbols? Find the number in our usual decimal system. [3]

$9{,}999{,}999 = 9\times10^6 + 9\times10^5 + \ldots + 9\times10^0$

so in the other system

$4\times5^7 + 4\times5^6 + 4\times5^5 + 4\times5^4 + 4\times5^3 + 4\times5^2 + 4\times5^1 + 4\times5^0$

$= 1\times5^8 - 1 = 390624$

(the number that comes right before 390624)

10. Many electronic devices store and communicate information using the ASCII coding system. This system uses 7 digit binary codes to represent characters. For example, the binary equivalents of the numbers 65 to 90 represent the capital letters A to Z. So A is represented by 1000001 (since 1000001 in binary is equal to 65), B is represented by 1000010 (66 in binary), and so on. Strings of binary digits can be put together to make words, with each block of 7 digits representing a certain letter. Translate the following binary string into a word: [4]

1001100 1001111 1010110 1000101

64 32 16 8 4 2 1

1001100 = 64+8+4 = 76 L

1001111 = 64+8+4+2+1 = 79 O

1010110 = 64+16+4+2 = 86 V

1000101 = 64+4+1 = 69 E

LOVE

A = 65 N = 78
B = 66 O = 79
C = 67 P = 80
D = 68 Q = 81
E = 69 R = 82
F = 70 S = 83
G = 71 T = 84
H = 72 U = 85
I = 73 V = 86
J = 74 W = 87
K = 75 X = 88
L = 76 Y = 89
M = 77 Z = 90

11. Recall that in the symmetries of the square we have the following eight elements: M is rotation clockwise by 90, N is rotation by 180, P is rotation by 270, Q is rotation by 360 (or simply do nothing), R is a horizontal flip, S is a vertical flip, T is a flip about the diagonal through the top left corner and bottom right corner, V is a flip about the diagonal through the top right corner and bottom left corner. The set $\{M, N, P, Q, R, S, T, V\}$ with combination as the operation is a group. For example, NP means do N, then do P. For this group, answer the following questions and be sure to explain your answers: [5]

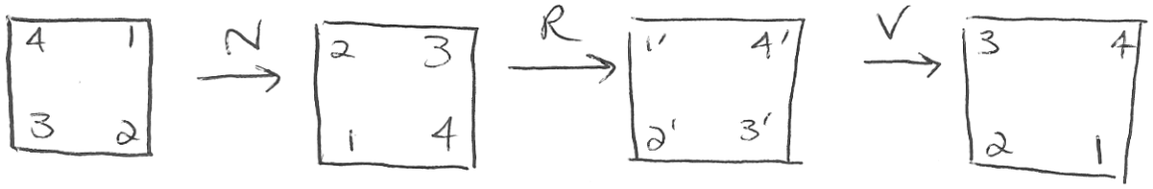
(a) Verify that $(NR)V = N(RV)$.

(b) Find the inverse of T (explain).

(c) Is this system commutative? Why or why not?

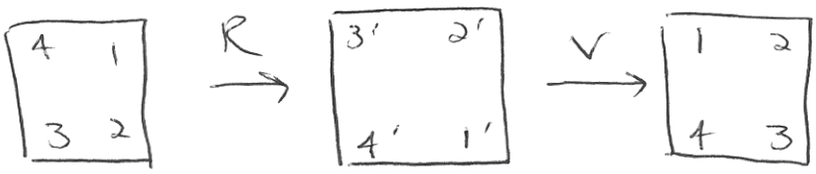
(NR) V

a)



so (NR) V = M

RV:

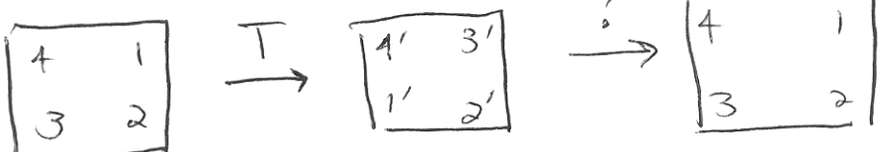


RV = P

So N(RV) = NP = M

(NR)V = M = N(RV)

b)

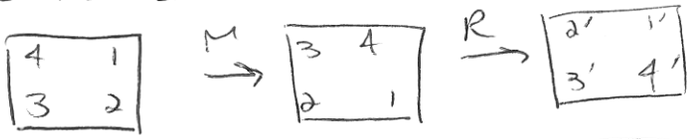


to get back to the start, just do T again

∴ inverse of T is T TT = Q

c) System is not commutative - order matters

MR:



MR = V

RM:

~~MR~~ RM = T

MR ≠ RM

12. And finally- what was your favourite topic in the course so far? [1]

3 Bonus Question

A fractal called the "Koch Snowflake" is made from an equilateral triangle as follows. Starting with an equilateral triangle (top left), replace each side with 4 line segments of equal length as in the top right image. In the next stage (bottom left), each line segment is replace with another four smaller line segments. You continue this process infinitely many times to obtain the snowflake (bottom right). Describe the symmetries of this snowflake. That is, describe the operations (like for the symmetries of the square) that would preserve the shape.

[3 Bonus Points]

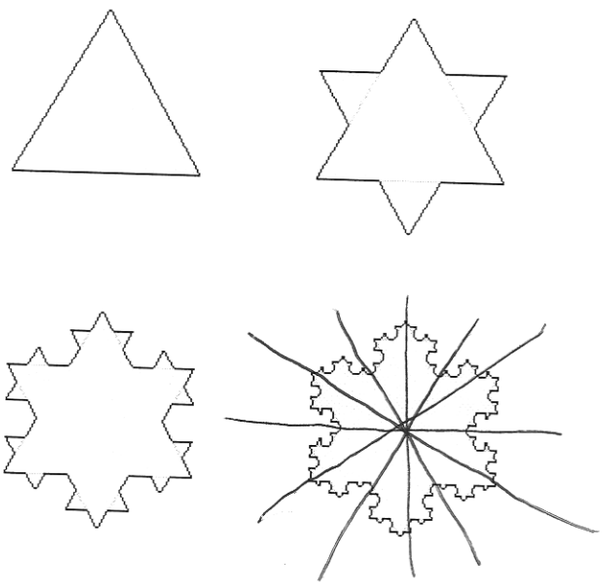


Figure 1: Koch Snowflake starting from Equilateral Triangle

6 rotations: $\frac{360}{6} = 60$

so $60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$

6 flips - through 6 different lines of reflections

so 12 operations in total (like for a hexagon)