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THE ROLE OF THE SHAFT IN THE GOLF SWING

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Abstract—Current marketing of golf clubs places great emphasis on the importance of the correct choice of shaft in relation to the golfer. The design of shafts is based on a body of received wisdom for which there appears to be little in the way of hard evidence, either of a theoretical or experimental nature. In this paper the behaviour of the shaft in the golf swing is investigated using a suitable dynamic computer simulation and by making direct strain gauge measurements on the shaft during actual golf swings. The conclusion is, contrary to popular belief, that shaft bending flexibility plays a minor dynamic role in the golf swing and that the conventional tests associated with shaft specification are peculiarly inappropriate to the swing dynamics; other tests are proposed. A concomitant conclusion is that it should be difficult for the golfer to actually identify shaft flexibility. It is found that if golfers are asked to hit golf balls with sets of clubs having different shafts but identical swingweights the success rate in identifying the shaft is surprisingly low.

INTRODUCTION

'We challenge anyone to prove the shaft is not the most important component of a golf club'. This bold statement accompanied a recent advertisement for golf clubs in a popular golf magazine and is typical of the emphasis that is currently being placed on the role of the shaft in golf club design. Shafts are currently being manufactured to closer tolerances than ever before and in a wider range. What is in doubt is the basis of the design philosophy. Roughly speaking, the received wisdom dictates that the more flexible shaft is better suited to the weaker player, while the professional or good amateur should use a stiffer shaft. Considered as an empirical result arising from the long development of golf, it must be taken seriously; however, within the context of the dynamics of the golf swing, it is not an obvious conclusion. A typical explanation (Tolhurst, 1989) of the effect of shaft flexibility runs along the following lines (for those unfamiliar with golf, a glossary of terms is given in Appendix 1). The shaft is initially bent backwards in the early part of the downswing and then recovers just before impact. If the shaft is too flexible for the golfer, it springs forward too far, thereby increasing the effective loft and closing the clubface; if too stiff, it is still bent backwards, the loft is decreased and the face is open. In this sense the shaft should be matched to the golfer's swing speed and hand action. Claims are sometimes made for increased clubhead speed at impact, assuming that the golfer's timing can utilise the springing back of the shaft. The link between change in loft and closing or opening of the clubface is due to the clubhead being attached to the shaft at the lie angle (Appendix 1). The actual extent to which the loft is changed by shaft bending can be controlled by the location of the so-called kick-point which relates to the shape of the bent shaft at impact; a shaft with a low kick-point shows greater curvature near the tip region than one with a high kick-point.

Shaft design is currently verified by three basic tests. In the first the shaft is clamped at its butt over a standard length and then loaded transversely by a weight near the tip; the tip deflection yields a measure of the shaft bending stiffness, and the location of the maximum deviation of a chord joining the butt and tip from the deflection curve yields the kick-point. The second test measures the tip rotation due to an applied tip torque, so yielding the torsional stiffness. The third test measures the fundamental transverse-vibration frequency of the clamped shaft carrying a tip mass equivalent to a clubhead.

The purpose of this study was to gain an understanding of the flexing behaviour of the shaft during the swing by constructing a suitable dynamic model and to test the validity of the model experimentally. Shaft flex may affect the feel of a golf club during the swing; this difficult area warrants a full investigation on its own but results of a rather limited study are reported here since they add some weight to the conclusions.

METHODS

Computer simulation

Flexibility of the shaft is manifest in bending and twisting. The former is the subject of careful design, so that desired characteristics are obtained, whereas the latter is seen as a necessary evil, which must be minimised. In this study emphasis is on bending flexibility; consequently, the dynamic model is a development of the well-established plane two-link model of the golf swing (Budney and Bellow, 1979; Daish, 1972). This consists of an *arm link* driven by a *shoulder torque* and a *club link* driven by a *wrist torque*; the new element added to the established model is the bending flexibility of the club link. Although the bodily motions of a golfer are quite complicated, numerous

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photographic and kinematic studies have shown that the downswing is, indeed, executed more or less in a plane. One aspect of the real three-dimensional swing, which is imported into the essentially two-dimensional or plane model, derives from the fact that the centre of mass of the clubhead does not lie on the shaft. Due to the large centrifugal forces involved in the swing, this has an important effect on shaft bending and, consequently, for this reason, it is necessary to allow for rotation of the clubhead in the plane of the swing, although this rotation has no direct effect on the momentum balances or impact mechanics in the plane. A description of the full equations of motion together with details of the computation are set out in Appendix 2.

One integral feature of the model is crucial and needs emphasis. The shaft has a bending stiffness, which is measured in the standard cantilever bending test already referred to. However, during the swing, particularly near impact when the centrifugal forces are large, the shaft also acquires an additional bending stiffness due to the fact that the shaft is under considerable tension. In a full drive or similar-distance shot this bending stiffness due to tension is, in the 40 ms or so before impact, large enough to be comparable to the conventional bending stiffness.

Strain gauge measurements

Three golfers, labelled A, B and C, with handicaps of 11, 5 and 2, respectively, were used in these experiments; golfers B and C showed a high degree of consistency in repeating their swings. The club used for the tests was a metal driver with R shaft and having 10.5° of loft. Groups of foil strain gauges (Showa, 2 mm) were bonded to the shaft at three stations to measure two bending moments at each station, parallel and normal to the clubface. Tension and torque were also measured at one station. The gauges were calibrated directly for these moments and forces by appropriate static loadings of the shaft. The strain gauge amplifiers (Fylde mini-bal and mini-amp systems) transmitted data via an analogue/digital convertor (Barr-Brown PCI-20089) directly onto a computer (Toshiba 3100SX) controlled by data acquisition software (Labtech Notebook); data was stored in a form suitable for subsequent analysis using the spreadsheet Lotus 1-2-3. The typical run extended over 10 s beginning at address and ending at follow-through. Sampling was at the rate of 200 readings per second; at this sampling rate it was not possible to read all the gauges simultaneously and successive swings were used. Graphical displays of data could be obtained on the Toshiba screen immediately after a run in order to check for failure or inconsistency. Initial analysis located the region of interest from just before top of swing to shortly after impact by reference to the obvious point of impact with the ball. The swing was also observed by a single high-speed video system running at a rate of 200 frames per second and shutter speed of 1/10,000 of a second placed facing the golfer; the video system was an NAC Model HSV-400 with associated *XY*-coordinator Model V-78-E coupled to an IBM-compatible PC using the motion analysis package MOVIAS (version 3). A video recording was also made along the line-of-flight direction to establish the angle of the golfer's swing plane. The video recordings allowed the gauge data to be correlated with swing position through the common time of impact and also allowed the construction of stick diagrams for the swing and the estimation of clubhead velocity and acceleration. The most important information obtained from the recordings gave the orientation of the clubhead relative to the plane of the swing. This was needed to transform the parallel-to-clubface and normal-to-clubface bending-moment components to in-swing-plane and out-of-swing-plane components since only the in-swing-plane components are relevant to the simulation. This aspect of the experimental method was the least satisfactory. The clubface was marked and the video image was clear; nevertheless, an accurate measurement of the orientation was difficult. Typically, the clubhead rotates little during the downswing until within about 50 ms of impact when it rotates through approximately a right-angle to square the clubface at impact. A simple smoothing routine was used to obtain both in-plane and out-of-plane bending moments. Out-of-plane bending moments were very small except near impact, where the values were consistent with out-of-plane bending of the shaft due to centre-of-mass offset: the behaviour of the resolved bending moments serves to give confidence in the plane-swing model. Were experiments of this type to be repeated, a more accurate and automatic way of measuring orientation, perhaps using laser marker techniques, would be advisable. Questions of cost precluded such an approach in these experiments.

RESULTS

A set of results is shown in full for golfer B only; results for golfers A and C naturally differ in detail but not enough to alter the overall picture as far as the shaft behaviour is concerned. A typical set of results for a swing simulation is shown in Fig. 1. Clubhead speed at impact estimated from the video via the MOVIAS program was $41.4\,\mathrm{m\,s^{-1}}$. The initiation time for the swing simulation was chosen as 400 ms before impact, at which point in the backswing the shaft is straight and the wrist torque is just beginning to reverse: angular velocities at initiation were estimated from the video recording. The diagrams have been extracted from a continuous screen display to illustrate the main features. The club used for the simulation is a driver; a long iron club would show a very similar swing except that the bending forward of the shaft before impact would be much reduced. The stick diagrams in Fig. 1 show the shaft deflection to scale, and it is difficult to perceive detail; so, the shaft

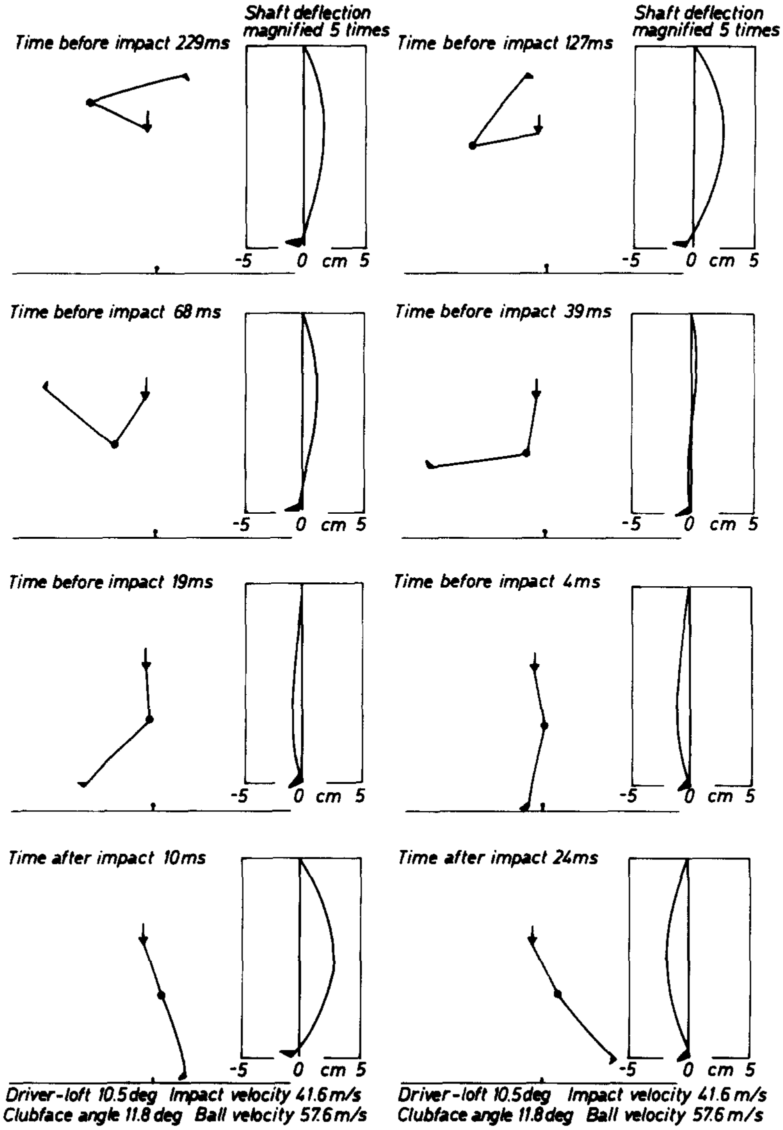

Fig. 1. Swing simulation showing shaft deflection.

deflection is also shown magnified five times. These bending shapes are plotted in such a way that they have the property of being orthogonal (with respect to the club mass distribution) to a rigid-body rotation about the *wrist hinge* (Appendix 2). Other outputs from the simulation, shown in Fig. 2, are clubhead speed, clubhead deflection measured relative to a tangent at the butt end and hand force. The *hand force* is the internal vector force at the wrist hinge acting in opposite directions on the arm and club links; in Fig. 2 the components of this force are shown resolved along and normal to the instantaneous directions of both arm and club links. Figure 2 also shows the tension measured in the shaft at its mid-point: measured mid-point torque is not shown, as it did not exceed 0.5 N m until impact. In Fig. 3 are shown the simulated and measured in-swing-plane bending moments at three stations along the shaft—at 10% (top), 50% (middle) and 90% (bottom) of shaft length from the wrist hinge.

Shoulder and wrist torques for the simulation appropriate to golfer B were synthesised from the strain gauge data in the following way. The shoulder torque can be estimated from the measured shaft tension by utilising the fact that the force exerted on the shaft by the hands is almost wholly axial to the shaft throughout the swing, the tangential component of the hand force being at most about 10% of the total; hence, shaft tension is little different from the total hand force, a result which can readily be demonstrated theoretically. By using a point in the swing at which the angular acceleration of the arm link is zero (about 70 ms before impact) the inertia of the arms/torso is eliminated from the estimate. The agreement between axial (club) hand force and mid-shaft tension (Fig. 2) is very satisfactory when it is noted that the tension at

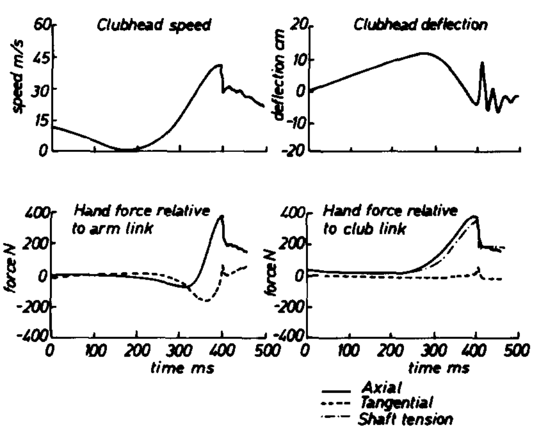

Fig. 2. Swing simulation outputs and measured mid-shaft tension.

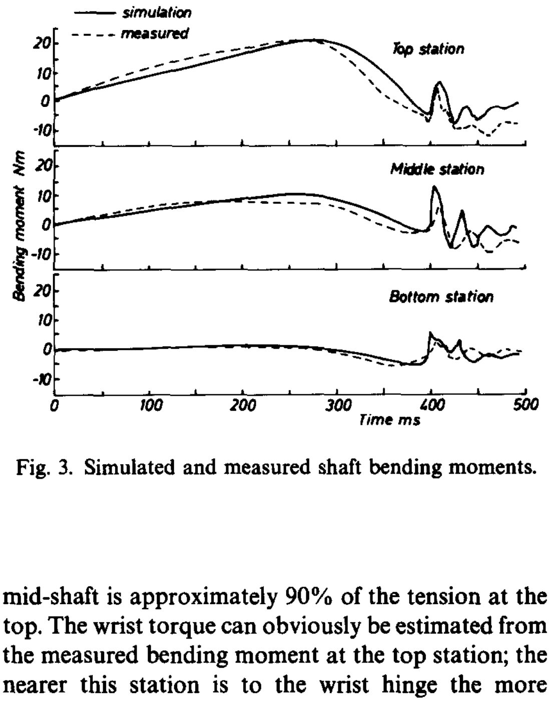

Fig. 3. Simulated and measured shaft bending moments.

mid-shaft is approximately 90% of the tension at the top. The wrist torque can obviously be estimated from the measured bending moment at the top station; the nearer this station is to the wrist hinge the more accurate the estimate. Based on this evidence the shoulder torque was applied as a ramp function with rise time 110 ms and maximum amplitude 83 N m and the wrist torque was applied as a ramp function with rise time 275 ms and maximum amplitude 25 N m. These simplified torques were not intended to match the golfer's inputs exactly. The inputs for golfers A and C were qualitatively similar to those for golfer B and the level of agreement between simulation and measurement much the same; golfer C was a very long hitter generating a clubhead speed of 48 m s^{-1}.

INTERPRETATION OF RESULTS

For the purpose of describing the shaft behaviour the swing may be divided into three phases. In the first phase, which extends from about 230 ms before impact (top of the swing) to about 130 ms before impact, the application of the wrist torque bends the shaft backwards, the maximum deflection occurring near the end of this phase. In the second phase, which consists of the last 130 ms or so to impact, an unfolding of the system due to momentum transfer takes place. The shaft behaves in a quasi-steady manner, gradually straightening and then bending forward due to the anticlockwise centrifugal torque applied at the lower end of the shaft by the offset of the centre of mass of the clubhead behind the shaft centre-line. This torque is closely represented by the bending moment at the bottom station (Fig. 3); near impact the effective clubhead weight is some 150 times its actual weight, so the magnitude of the torque applied is some 3 N m for each centimetre of centre-of-mass offset. The main effect is to give the club an additional dynamic loft at impact with a corresponding closing of the clubface. The nett clubface angle at impact with the ball is recorded by the simulation (Fig. 1). It is important to note that the position of the centre of mass of the clubhead relative to the hands is along the local centrifugal vector, so that if the shaft is bent forward then the club butt-to-arm angle is correspondingly increased and vice versa (see also Fig. 4). In phase three the effect of impact on the shaft is to excite a few cycles of bending vibration at a frequency of about 30 Hz. Some of the energy at impact is spent in exciting the shaft; for example, given the same set of input torques, the simulation shows that a flexible shaft produces an initial ball velocity which is about 4% lower than that produced by an equivalent idealised 'rigid' shaft.

The computer simulation necessarily incorporates a model of the dynamic characteristic or impedance of the wrist joint, and this is an area where information is lacking: experimental work such as that carried out by Brown *et al.* (1982) on other human joints is badly needed. When gripped by a golfer, the club is very far from being clamped, as in the standard bending and vibration tests. The stiffness or the in-phase component of the wrist impedance torque is probably quite small when, near impact, the joint is relaxed and rapidly developing. The damping or the out-of-phase component is large enough near impact to actually apply a retarding torque. The measured top station bending moment in Fig. 3 clearly shows the decrease of the wrist torque approaching impact as the rotation speed of the joint rapidly increases. This behaviour was observed for all the golfers tested and has been observed by others (Budney and Bellow, 1979); details of the impedance used in the model are given in Appendix 2. By varying the impedance-damping level and using the same demand inputs, the simulator can arrive at impact with wrist torques which vary from decelerating (the usual case) through neutral to accelerating. The results from three simulations of this type, based on the swing of golfer B but taken over a rather extreme range for emphasis, are shown in Fig. 4. Note that the clubface angle at impact is the

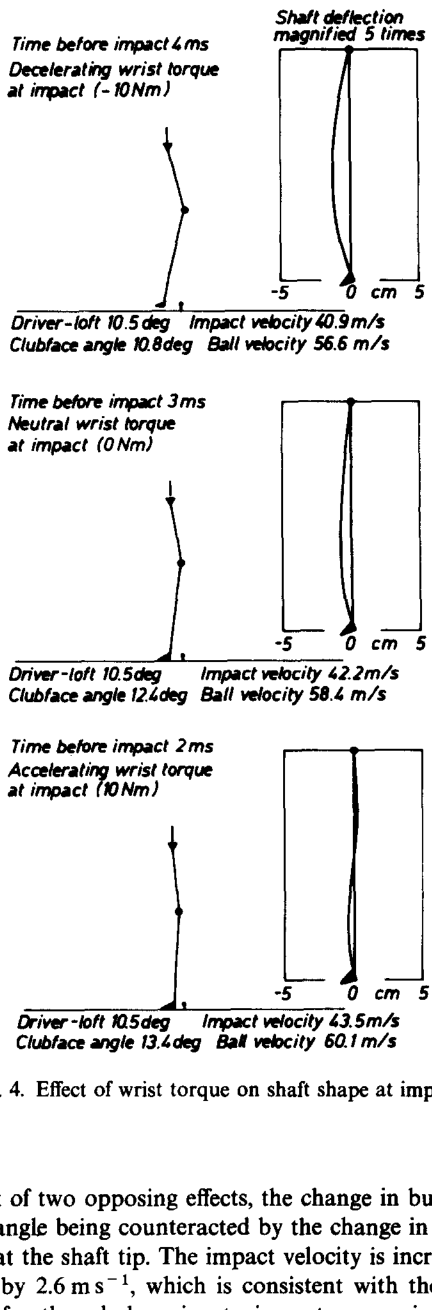

Fig. 4. Effect of wrist torque on shaft shape at impact.

result of two opposing effects, the change in butt-to-arm angle being counteracted by the change in rotation at the shaft tip. The impact velocity is increased only by 2.6 m s^{-1}, which is consistent with the fact that, for the whole swing to impact, approximately 92% of the work done comes from the shoulder torque (via the torso and legs) and only 8% from the wrist torque.

The clamped bending frequency of a driver is typically of the order of 4.5 Hz, whereas the bending frequency observed for a club freely hinged near its butt end as illustrated in Fig. 5(b) is of the order of 25 Hz. The bending frequency for a club, freely rotating about a hinge near its butt end and having a speed of rotation which gives the head a speed of 40 m s^{-1}, is about 32 Hz as observed in the simulation after impact; in this situation the frequency is only weakly dependent on the shaft bending stiffness. The three phases described above are all observable in high-speed photographs of golfers' swings (Maltby, 1982; Cochran and Stobbs, 1969) and confirm an earlier description of shaft behaviour which was based solely on well-founded physical reasoning (Williams, 1969).

DISCUSSION

The main aim of the work reported here was to elucidate, in a qualitative way, the dynamic flexing behaviour of the golf shaft during the swing. In the simulation the precise torques generated by the 'golfer' are open to question but the inertial and imposed forces and torques applied to the shaft itself, as verified by the strain gauge measurements, are undoubtedly typical of the golf swing and, hence, the level of confidence that may be placed in the simulator is high. The simulator could be used as a design tool to explore in detail the interaction between a range of shafts and 'golfers' as represented by their input torques. No attempt is made to do this here, but it is worth remarking that widely different shoulder torques lead to similar simulated swings provided the same amount of work is done: this is not a surprising result if one remembers that the swing is the outcome of a double time integration and is reflected in the observation that, on the golf course, very different looking golf swings can produce similar results.

The burden of the conclusions arising from the study is that shaft flexibility does not play an important *dynamic* role in the swing and the principal effect of flexibility, namely, change in the effective loft at impact, could be estimated from static considerations. A simple test which would serve this purpose is illustrated in Fig. 5(a); this simulates the behaviour of the shaft of a wooden club just before impact when conditions are quasi-steady and the shaft is subject to a large axial load. The outcome of the test is a direct measure of the change in the effective loft (per unit centre-of-headmass offset) which this shaft would give

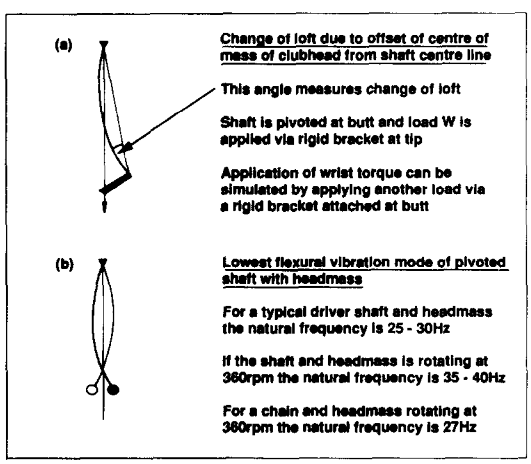

Fig. 5. Proposed shaft tests.

and is related to the player's clubhead speed at impact if the load W is given a value appropriate to the centrifugal force on the head; it automatically takes account of the change in club butt-to-arm angle already referred to earlier. Application of a torque at the butt end can simulate the golfer's wrist torque at impact. This test is reflected in the common practice amongst golfers of resting the toe of a club on the ground and then pushing the butt firmly downwards!

Clearly, another test which should be included in shaft testing is the measurement of the fundamental bending-vibration frequency of a shaft-plus-head suspended freely from a pivot as illustrated in Fig. 5(b). To demonstrate the effect of an increase in the bending stiffness due to centrifugal force, approximate vibration frequencies are shown for a rotating shaft-plus-head and for a rotating chain-plus-head, the chain having the same mass distribution as the shaft but, of course, zero conventional bending stiffness.

If shaft flexibility is not dynamically significant then why is it apparently so important to the golfer? To begin to answer this question in a very preliminary way a series of trials was conducted with a small group of golfers consisting of four amateurs, with handicaps ranging from 13 to 5, and four professionals. Three sets of clubs were specially prepared, each set having identical swing weights but fitted with R, S and X shafts, two sets of five-irons (with solid back and peripherally weighted heads) and a set of drivers. The decals on the shafts were covered by a code letter and the participants were instructed not to bend the shaft or to waggle the club but simply to hit golf balls. The results were somewhat surprising. The amateurs were unable to distinguish one shaft from another either for woods or irons; typically, if pressed as to which shafts they preferred, they would chose the R and X over the S or, if asked for an order of flexibility, would give an almost random selection. Striking of the ball did not seem to be greatly affected, although no measurements were made of ball trajectory. The professionals were not much better at discriminating between the irons but were quite successful with the woods. However, even here success was not complete, although the R shaft could generally be distinguished from the S and X shafts. These very simple and restricted tests do suggest that more extensive and significant work should be undertaken in this area. Recently, Pelz (1990) has conducted 'blind' golfer trials to compare the performance of steel and graphite shafts of differing flexibilities and Van Gheluwe *et al.* (1990) have shown that the kinematics of a golfer's swing appears to be independent of the shaft material.

CONCLUSIONS

Do the results of this study bear out the accepted explanation of the effect of shaft flexibility given in the Introduction? The study confirms that flexibility will produce a change of loft (and close clubface) at impact but shows that these are really quasi-static effects and not dependent on a dynamic 'springing back' of the shaft from its bent position at the start of the swing. It is difficult to give any credence to the suggestion that the shaft can actually be bent backwards at impact and no photographic evidence exists to support this contention. Kick-point should be replaced by a measure of shaft-tip rotation due to an offset axial load. Nor does springing back or 'whipping' of the shaft play a dynamic role in the all-important pre-impact area, although the 'feel' of the violent flexing of the shaft after impact may give the golfer the impression that it does. Indeed, clubhead speed at impact is virtually unchanged if, using the simulator, the shaft is replaced by a heavy chain about 60 ms before impact. Perhaps the major source of misunderstanding about the role of the shaft is associated with the idea that the club is 'gripped' by the golfer, resembling the condition of the built-in butt end of the cantilever test, combined with a lack of appreciation of the magnitude and speed of response of any forces and moments applied by the player, which could significantly affect the course of the swing in the pre-impact area.

The implications of this study for club design are somewhat problematical. It does appear that it may not be necessary to provide a range of shafts for iron clubs and possibly not even for wooden clubs. It is possible to argue that the 'feel' imparted to a club through shaft flexibility may actually affect the torque inputs of the player but the golfer tests gave no indication of this: there is room here for further study. If one takes the view that shaft flexibility is not of importance *dynamically*, it then represents an unwanted swing variable and one is led to adopt a stiff shaft, a conclusion also reached by Pelz (1990). The effective loft that a golfer may have enjoyed with a flexible shaft could then be reinstated (without the associated change in clubface angle) by an appropriate adjustment of nominal club loft. Do current stiff shafts represent an upper limit or could stiffer shafts be used? If the latter, then it would be possible to use a light, thin-walled shaft tapering to about twice the tip diameter of the currently available shafts coupled with a return to hosel-in-shaft fabrication; such a shaft would have considerably higher torsional stiffness than the currently available shafts.

Finally, the point should be made that the fixity conditions of a shaft held in a golf machine are very different from those for a golfer and, therefore, the shaft performance data derived from this source do not necessarily apply to the 'real' golf swing.

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APPENDIX 1
GLOSSARY OF GOLFING TERMS

The golf club consists of three parts: the *head*, the *shaft* and the *grip*. The head is either a fairly narrow metal blade (*iron*) or a more extended, bulbous shape with a flattish front face (wood) made traditionally of wood but now also made in metal. The *loft* of the club is the angle which the front face of the head (*clubface*) makes with the vertical. The shaft is attached to the head by the *hosel* or *neck*, which is a cylindrical extension of the head situated near one end of the clubface. The hosel is angled to the head in such a way that when the base of the head is resting flat on the ground the shaft is at an angle to the horizontal called the *lie angle*. The centre of mass of an iron club is situated about halfway along the blade while the centre of mass of a wooden club is offset from the neck roughly along a line at 45° to the clubface. The shaft is either a steel tube or a solid composite tapering down from the grip end (*butt*) to the *tip*. The distribution of the taper controls the bending and torsional properties of the shaft. Shafts are commonly manufactured in three main categories R (regular), S (stiff) and X (extra stiff): within each category different weights and kick-point positions are offered. Typically, the ratio of clamped transverse vibration frequencies for R, S and X shafts is 1.00:1.04:1.08. The grip is simply a rubber tube which is glued to the butt of the shaft.

At impact the clubface is said to be *open* if it is pointing to the right of the target line and *closed* if it is pointing to the left: the *effective loft* is the actual angle of the clubface to the vertical at impact as opposed to the nominal loft of the club. Bending of the shaft in the swing plane near impact not only changes the loft but also opens or closes the clubface; for example, for a driver with lie angle of 55° the clubface will open (close) by an angle which is 70% of the change in the loft.

The *swing plane* is the imaginary plane in which the player swings the clubhead and will generally be inclined somewhat steeper than the lie angle of the club.

Due to the arrangement of bones and joints in the lower arms and wrists, the golfer is forced to roll or rotate his hands during the golf swing if he is to allow the wrist hinge to move freely: the face of the club lies roughly parallel to the swing plane at the top of the swing and is then rotated so as to be at right-angles to the swing plane at impact.

APPENDIX 2
THE MATHEMATICAL MODEL

Equations of motion

Figure A1 illustrates the applied torques and rotation angles for the dynamic model. The relevant equations of motion, listed in the following, consist of

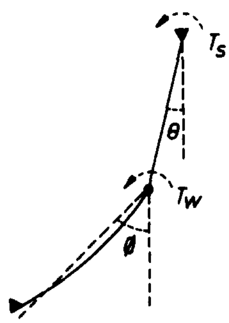

Fig. A1. Applied torques and rotation angles for the dynamic model.

(a) equations (1) and (2) representing the rate of change of angular momentum (including terms due to shaft flexibility) about the upper (fixed) pivot and about the wrist hinge, respectively, and

(b) a variational equation (principle of virtual work) for shaft bending based on the small-deflection theory for slender beams including the effect on bending of tension in the shaft due to centrifugal forces. This integro-differential equation is replaced by a number of ordinary differential equations in time by representing the shaft bending displacement as a series of shape functions with time-dependent amplitudes (the Galerkin method) leading to equations (3), . . . , $(n+2)$.

Equation (1):

$$
\begin{aligned}
&(I_{\mathrm{T}}+m_{\mathrm{c}}l_{\mathrm{a}}^{2})\ddot{\theta}+\Bigg(m_{\mathrm{c}}l_{\mathrm{a}}\bar{l}_{\mathrm{s}}\cos(\phi-\theta)\\
&\quad-\left(\sum_{i=1}^{n}E_i\xi_i+\bar{C}\right)\sin(\phi-\theta)\Bigg)\ddot{\phi}\\
&\quad+\sum_{i=1}^{n}\left(E_i\cos(\phi-\theta)-\left(\sum_{j=1}^{n}G_{ij}\xi_j+H_i\bar{C}\right)\sin(\phi-\theta)\right)\ddot{\xi}_i\\
&\quad-\left(m_{\mathrm{c}}l_{\mathrm{a}}\bar{l}_{\mathrm{s}}\sin(\phi-\theta)+\left(\sum_{i=1}^{n}E_i\xi_i+\bar{C}\right)\cos(\phi-\theta)\right)\dot{\phi}^2\\
&\quad-2\sum_{i=1}^{n}E_i\dot{\xi}_i\sin(\phi-\theta)\dot{\phi}\\
&\quad+\bar{g}\sin\theta(l_{\mathrm{a}}m_{\mathrm{c}}+m_{\mathrm{a}}\bar{l}_{\mathrm{a}})+T_{\mathrm{s}}-T_{\mathrm{w}}=0,
\end{aligned}
$$

Equation (2):

$$
\begin{aligned}
&\left(m_{\mathrm{c}}l_{\mathrm{a}}\bar{l}_{\mathrm{s}}\cos(\phi-\theta)-\left(\sum_{i=1}^{n}E_i\xi_i+\bar{C}\right)\sin(\phi-\theta)\right)\ddot{\theta}+I_{\mathrm{c}}\ddot{\phi}\\
&\quad+\left(m_{\mathrm{c}}l_{\mathrm{a}}\bar{l}_{\mathrm{s}}\sin(\phi-\theta)+\left(\sum_{i=1}^{n}E_i\xi_i+\bar{C}\right)\cos(\phi-\theta)\right)\dot{\theta}^2\\
&\quad+\sum_{i=1}^{n}F_i\ddot{\xi}_i+(\bar{g}/l_{\mathrm{a}})\Bigg(m_{\mathrm{c}}l_{\mathrm{a}}\bar{l}_{\mathrm{s}}\sin\phi\\
&\quad+\left(\sum_{i=1}^{n}E_i\xi_i+\bar{C}\right)\cos\phi\Bigg)+T_{\mathrm{w}}=0.
\end{aligned}
$$

Equations $(2+i)$, $i=1, \ldots, n$

$$
\begin{aligned}
&E_i(\sin(\phi-\theta)\dot{\theta}^2+\cos(\phi-\theta)\ddot{\theta})+\sum_{j=1}^{n}k_{ij}(\xi_j+\alpha\dot{\xi}_j)\\
&\quad+(\cos(\phi-\theta)\dot{\theta}^2-\sin(\phi-\theta)\ddot{\theta})\left(\sum_{j=1}^{n}G_{ij}\xi_j+H_i\bar{C}\right)
\end{aligned}
$$

$$+F_i\ddot{\phi}+\sum_{j=1}^{n} M_{ij}\ddot{\xi}_j-\left(\sum_{j=1}^{n}(M_{ij}-K_{ij})\xi_j\right.$$
$$\left.+(\bar{C}/l_s)[s_i(1)-s_i'(1)]\right)\dot{\phi}^2$$
$$+(\bar{g}/l_s)\left(E_i\sin\phi-\left(\sum_{j=1}^{n} G_{ij}\xi_j+H_i\bar{C}\right)\cos\phi\right)$$
$$-T_w s_i'(0)/l_s=0,$$

where

$$s(\sigma)=\sum_{i=1}^{n} s_i(\sigma)\xi_i(t) \quad \text{(bending deflection)},$$

$$m_t(\sigma)=\rho_0 l_s\int_\sigma^1 \rho_s(\tau)\mathrm{d}\tau,\ \bar{m}_t(\sigma)=\rho_0 l_s\int_\sigma^1 \tau\rho_s(\tau)\mathrm{d}\tau,$$

$$E_i=l_s\left(\rho_0 l_s\int_0^1 \rho_s(\sigma)s_i(\sigma)\mathrm{d}\sigma+m_h s_i(1)\right),$$

$$F_i=l_s\left(\rho_0 l_s\int_0^1 \sigma\rho_s(\sigma)s_i(\sigma)\mathrm{d}\sigma+m_h s_i(1)\right),$$

$$G_{ij}=\frac{l_a}{l_s}\int_0^1 [m_t(\sigma)+m_h]s_i'(\sigma)s_j'(\sigma)\mathrm{d}\sigma,$$

$$H_i=\frac{s_i'(1)}{l_s},$$

$$M_{ij}=\rho_0 l_s\int_0^1 \rho_s(\sigma)s_i(\sigma)s_j(\sigma)\,\mathrm{d}\sigma+m_h s_i(1)s_j(1),$$

$$K_{ij}=\int_0^1 [\bar{m}_t(\sigma)+m_h]s_i'(\sigma)s_j'(\sigma)\mathrm{d}\sigma,\quad \bar{C}=l_s m_h\bar{c},$$

$$k_{ij}=\frac{\mathrm{EI}}{l_s^3}\int_0^1 \mathrm{ei}(\sigma)s_i''(\sigma)s_j''(\sigma)\mathrm{d}\sigma,$$

$$T_w=T_w^d+\mathrm{imp}(\theta-\phi,\dot{\theta}-\dot{\phi});$$

dot denotes differentiation with respect to time, t, whereas prime denotes differentiation with respect to σ,

$\bar{c}(\phi)$ = distance of c of m of clubhead from shaft in swing plane,
$\mathrm{ei}(\sigma)$ = bending stiffness ratio of shaft, $\mathrm{ei}(0)=1$,
EI = reference bending stiffness of shaft (at $\sigma=0$),
$\bar{g}=g\cos(\mathrm{swp})$
{ g = gravitational acceleration,
swp = angle of swing plane to vertical,
I_T = moment of inertia of arms/torso about upper pivot,
I_c = moment of inertia of club about wrist pivot,
imp( ) = wrist impedance function,
l_a = length of arm link,
l_s = length of club link,
$\bar{l}_a$ = distance of c of m of arm link from upper pivot,
$\bar{l}_s$ = distance of c of m of club link from wrist pivot,
m_c = mass of club,
m_h = mass of clubhead,
n = number of shape functions,
$s_i(\sigma)$ = ith shape function,
t = time,
T_s = torque at upper pivot ('shoulder torque'),
T_w = torque at wrist pivot ('wrist torque'),
T_w^d = 'demand wrist torque',
α = material damping constant for shaft,
$\theta(t)$ = angle of arm link to vertical,
$\xi_i(t)$ = amplitude of ith shape function,
$\rho_s(\sigma)$ = mass density ratio of shaft per unit length, $\rho(0)=1$,
ρ_0 = reference mass density of shaft (at $\sigma=0$),
σ = dimensionless distance along shaft from wrist pivot,
$\phi(t)$ = angle of club link axis to vertical (Fig. A1).

First-order form of equations of motion

$$\left.\begin{aligned}\dot{x}_i&=z_i\\ \sum_{j=1}^{n+2} m_{ij}(\{x_k\})\dot{z}_j&=F_i(\{x_k\},\{z_k\})\end{aligned}\right\}i,k=1,\ldots,n+2,$$

where

$$\{x_k\}=(\theta,\phi,\xi_1,\xi_2,\ldots,\xi_n).$$

In this form the equations of motion are integrated forward in time using a fourth-order Runge–Kutta scheme.

Momentum balance equations at impact

During forward integration of the equations of motion, impact with the ball is detected by an interpolation of the clubhead path between time steps: the program enters a routine which solves the following algebraic equations representing conservation of momentum of the whole system and then re-enters the integration program with a new set of initial values, the first time step being adjusted to lock onto the pre-impact time markers;

$$\left.\begin{aligned}\sum_{j=1}^{n+2} m_{ij}(\{x_k^{(\mathrm{im})}\})(z_j^{(\mathrm{f})}-z_j^{(\mathrm{i})})&=m_b v u_i,\\ \sum_{i=1}^{n+2} u_i(z_i^{(\mathrm{f})}+e z_i^{(\mathrm{i})})&=v\end{aligned}\right\}i,k=1,\ldots,n+2,$$

where

$$\{u_i\}=(l_a\cos\hat{\theta}^{(\mathrm{im})},l_s\cos\hat{\phi}^{(\mathrm{im})},s_1(1)\cos\hat{\phi}^{(\mathrm{im})},\ldots,s_n(1)\cos\hat{\phi}^{(\mathrm{im})}),$$
$$\hat{\theta}^{(\mathrm{im})}=\theta^{(\mathrm{im})}+\gamma,\ \hat{\phi}^{(\mathrm{im})}=\phi^{(\mathrm{im})}+\gamma,$$

m_b = mass of golf ball,
v = initial ball velocity,
e = coefficient of restitution,
γ = initial angle of ball flight path above horizontal,
$^{(\mathrm{im}),(\mathrm{i}),(\mathrm{f})}$denote impact, initial and final values.

Shape functions—pre-processing program

The n shape functions are defined by $s_i(\sigma)=\sum_{j=1}^{n+1} c_{ij}\sigma^j$, where the coefficients c_{ij} are chosen such that, for all $i,j=1,\ldots,n$,

$$\int_0^1 \rho(\sigma)s_i(\sigma)\sigma\,\mathrm{d}\sigma+m_h s_i(1)=0,$$

$$\int_0^1 \rho(\sigma)s_i(\sigma)s_j(\sigma)\mathrm{d}\sigma+m_h s_i(1)s_j(1)=0,\quad i\neq j.$$

That is to say, with respect to the mass distribution of the club, the shape functions are mutually orthogonal and are also orthogonal to the rigid-body rotation σ. As a result, the left-hand side matrix $[m_{ij}]$ has non-zero entries on the diagonal and in the first row and first column only. Given the club mass and bending stiffness distributions, a pre-processing program generates the shape functions recursively, then computes the stiffness matrix $[k_{ij}]$. The program also calculates the fundamental pivoted vibration frequency of the club [Fig. 5(b)]. It is found that three shape functions are sufficient to give good accuracy for the frequency calculation and to deal with the changes in shape of the shaft which occur throughout the swing simulation.

Wrist impedance function

The hypothetical impedance function consists solely of the damping term const. $(\dot{\theta}-\dot{\phi})^2$, with the same constant used for the three golfers, force demand being entirely embodied in T_w^d. An alternative model using a kinematic demand on club/arm angle and embodying also a non-linear stiffness term weighted towards the limit of wrist rotation gives very similar results in the simulation.