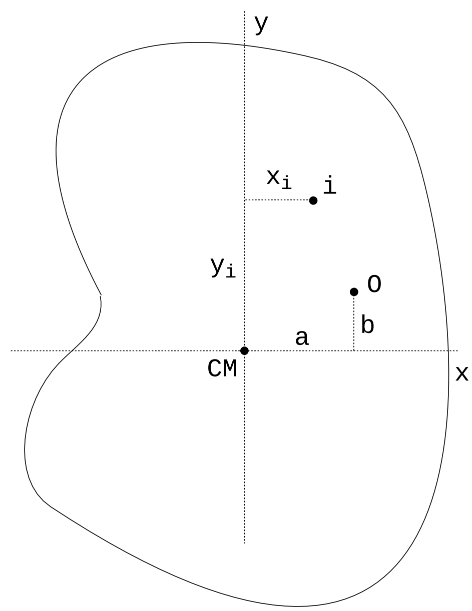
Proof of the Parallel Axis Theorem

Consider a rigid system of particles of mass $M = \Sigma_i m_i$, rotating about a fixed axis O. We place the origin of our coordinate system at the center-of-mass (CM) of our system of particles. The general point $i$ has coordinates $(x_i, y_i)$, the $x$ coordinate of O is $a$, and the $y$ coordinate of O is $b$. Since the CM is at (0,0), we have, by the definition of the center-of-mass, that

$$x_{CM} = \frac{\Sigma_i m_i x_i}{M} = 0 \quad and \quad y_{CM} = \frac{\Sigma_i m_i y_i}{M} = 0$$




By definition, $I_O = \Sigma_i m_i r_{iO}^2$, where $r_{iO}$ is measured from O, not from CM. Thus, $r_{iO}^2 = (x_i - a)^2 + (y_i - b)^2$. Therefore,

$$\begin{aligned} I_O &= \Sigma_i m_i r_{iO}^2 \\ &= \Sigma_i m_i [(x_i - a)^2 + (y_i - b)^2] \\ &= \Sigma_i m_i [x_i^2 - 2ax_i + a^2 + y_i^2 - 2by_i + b^2] \qquad (1) \end{aligned}$$

But since $x_{CM} = \frac{\Sigma_i m_i x_i}{M} = 0$, and similarly for $y$, the cross terms in equation (1) go to zero once the sum over $i$ is done. Therefore,

$$I_O = \Sigma_i m_i (x_i^2 + y_i^2) + \Sigma_i m_i (a^2 + b^2)$$

Since $(x_i^2 + y_i^2)$ is just $r_i^2$, where $r_i$ is now measured from the CM, and $(a^2 + b^2)$ is just $R_O^2$, where $R_O$ is the distance from O to the CM, we have

$$I_O = I_{CM} + MR_O^2$$