On Providing a Complete Ordering of Non-Combinable Alternative Prospects

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Abstract

Throughout the social sciences, choices frequently have to be made between alternatives that cannot be combined, some examples of which include lumpy investment prospects, distinct policy options, treatment protocols, and partner selection. When facing such choice sets, the decision-maker often encounters a situation where there is no universally dominant prospect given the priorities confronting them, and a second best solution has to be found. The problem is that dominance techniques only provide a partial ordering. By generating a synthetic "first best" prospect from the set of available alternatives which is universally dominant under a given imperative, indices of the proximity of all available alternatives to this dominant prospect can be constructed, providing a complete ordering of prospects under the chosen imperative. The ranking satisfies many of the axioms of choice, and is shown to be independent of irrelevant alternatives. Two empirical examples of the application of the technique are provided: choosing between redistributive policy alternatives, and choosing between child circumstance scenarios to promote better wellbeing outcomes in adulthood within a multi-dimensional context.

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1 Introduction

Stochastic dominance criteria, a cornerstone in the advance of expected utility and prospect based choice theory, provides the set of conditions which, if satisfied, establishes the unambiguous superiority of one outcome distribution over another, predicated upon the nature of the decision-maker's preferences. The technique has a well-developed theory for a wide variety of applications throughout the social sciences (See for example Atkinson (1987), Atkinson and Bourguignon (1982), Baker (2009), Beach and Davidson (1983), Duclos et al. (2005), Lefranc et al. (2008, 2009), Moyes and Shorrocks (1994), Naga (2005), and Rothe (2010)), yet it is seldom used in practice by decision-makers[1] for two reasons. Firstly, the method only provides an incomplete ordering (comparisons are not always conclusive). Secondly, it does not yield an intuitive measure of the relative benefits of alternative choices, in that it does not provide a metric for the degree of advancement toward, or retreat from, the decision-maker's imperative. The absence of a conclusive comparison leaves no basis for ranking alternatives, or tools for assessing progress toward a policy goal. The statistics proposed here fill that void.

Though these practical difficulties associated with employing stochastic dominance techniques have received little attention until recently[2], they have long been recognized in the theoretical finance literature which typically applies dominance techniques to reduce the size of optimal choice sets (See Bawa et al. (1985); Fishburn (1974); Leshno and Levy (2002); Post (2016); Tsetlin et al. (2015)). As long ago as 1970, the conventional second order stochastic dominance criterion appropriate for risk averse actors (Rothschild and Stiglitz 1970) was acknowledged by the same authors (Rothschild and Stiglitz 1971) to have no obvious comparative statics properties[3]. That literature responded to this concern with the introduction of an alternative form of dominance, namely *Central Dominance*, characterizing "greater central riskiness" (Gollier 1996). Central Dominance, which is neither stronger nor weaker than second order stochastic dominance, characterizes the

[1]For example Stochastic Dominance Techniques are not to be found anywhere in the lexicon of a recent report and companion volume on Tax Design for the Institute For Fiscal Studies (Adam et al. (2010), and Mirrlees et al. (2011)).

[2]See unpublished presentation "Stochastic Dominance and Completeness" by James Foster on Advances in Stochastic Dominance for Welfare Analysis Conference website, `http://www.ferdi.fr/en/node/1601` , Clermont-Ferrand, September 17-18 2014"

[3]They demonstrated that an increase in risk characterized by 2^{nd} order dominance does not necessarily induce all agents to reduce holdings of the risky asset.

necessary and sufficient conditions under which a change in risk changes the optimal value of an agent's decision variable in a predictable fashion for all risk-averse agents[4]. An important feature of this analysis is that the decision variable is continuously related to the risk measure[5], so that incremental changes in the decision variable can be contemplated as a consequence of incremental changes in risk. However, the notion of Central Dominance has not yet found expression in other Choice literatures (for an exception see Chuang et al. (2013)), probably because in many situations, policy alternatives are generally not continuously connected in the manner that a convex combination of a risky and risk free asset can be contemplated in the portfolio problem[6]. Rather, alternative policies are usually a collection of distinct, mutually exclusive outcomes, and the choice problem is that of picking one of them. In these circumstances, where a variety of alternative policy outcomes is being contemplated (usually in terms of the income distributions they each imply), a collection of pairwise dominance comparisons will have to be made without recourse to the comparative static feature that the notion of central dominance provides.

While much can be learned about the relative status of alternative outcomes by considering them under different orders of dominance comparison, the partial ordering nature of the technique frequently renders the comparisons inconclusive. The practice has been to seek the order at which dominance of one distribution over the other is achieved, for such a comparison is unambiguous at that order of dominance. Unfortunately this level of dominance may not accord with the decision-maker's priorities. In fact, successive orders of dominance comparison attach increasing importance (weight) to lower values (left tail realizations) of the outcome variable in question, so that as a decision-maker increases the order of dominance comparison, it may be construed as reflecting an imperative of increasing concern for the left tail end realizations of a distribution. Further, in terms of a collection of pairwise comparisons, this can be a lengthy and impractical process, which frequently fails to yield a definitive conclusion.

Here indices are proposed for measuring the extent to which one policy is "better" than another within the context of a specific dominance class or order of dominance, the choice of which reflects the particular priorities confronting the policymaker. An attrac-

[4]Chuang et al. (2013) have developed tests for Central dominance.

[5]The risk free−risky asset mix parameter in the case of the portfolio problem or the tax parameter(s) in a public choice problem.

[6]Central Dominance could for example be employed in examining a revenue neutral redistributive tax policy, which is some convex combination of lump sum and progressive tax.

tive feature of the statistic is that it is readily employed in multidimensional contexts. Conceptually the index is based upon an approach found in the statistics literature[7], that of choosing from a finite set of alternative statistical tests on the basis of the visual proximity of each test's power function to the envelope of the set of available power functions which reflects the maximal attainable power, if the most powerful test is employed at each point in the alternative hypothesis. Here alternatives are considered in the context of a dominance class determined by the decision-maker's priorities. The stochastically dominant hull of alternative outcomes (the possibly unattainable "best" outcome frontier) consistent with the imperative is constructed, facilitating calculation of an index based on the proximity to this frontier for each alternative. The alternative with the smallest proximity index is naturally preferred. The result is a complete ordering of alternatives for a given imperative, a partial resolution to the incompleteness problem akin to the classic second best solution (Lipsey and Lancaster 1956-1957) when the first best option (the hull constituted by the alternatives) is not available. In addition, by focusing each alternative's distribution with a common frontier, it reduces the number of comparisons required. The technique may be interpreted as a generalization of the "Almost Stochastic Dominance" comparison technique (Leshno and Levy 2002) to many distributions (as opposed to just two), and to all orders of dominance (as opposed to just the second).

In the following, Section 2 outlines the relationship between the stochastic dominance criteria and wellbeing/objective classes, thereby highlighting the notion that a policymaker may want to make a policy choice in the context of an imperative/priority associated with a particular wellbeing/objective class. In Section 3 the indices appropriate for making such choices are developed. Section 4 exemplifies the technique in two very different scenarios. Firstly, using a sample of weekly pre-tax incomes drawn from the Canadian Labour Force Survey for January 2012, the manner in which the index could be utilized to discern between three mutually exclusive alternative revenue neutral policies is demonstrated, the choice of which is dependent upon the policymaker's priorities. Secondly, demonstrating a multi-dimensional application, panel data drawn from the Inter-University Consortium for Political and Social Research (ICPSR) National Longitudinal Study of Adolescent Health 1994-2008 is used to extend the literature on health gradient to consider adult health, income and education as a triple outcome that is conditioned on a set of observable circumstances and behaviors at youth in the U.S.

[7]See for example Anderson and Leo (2014), Juhl and Xiao (2003), Omelka (2005), and Ramsey (1971).

2 Relationship between Wellbeing Indices & Stochastic Dominance

2.1 Stochastic Dominance & the Agent's Priorities

The notion of stochastic dominance was developed as a criteria for choosing between two potential distributions of a random variable $x \in \mathcal{X}$ (usually income, consumption or portfolio returns) so as to determine the distribution which maximizes $\mathbf{E}[U(\mathcal{X})]$, based upon the properties of the function $U(x)$, where $U(x)$ represents a risk averting felicity function of agents under the income size distribution of x (Levy (1998) provides a summary). Working with $U(x)$, let x be continuously defined over the domain $[a, b]$, and two alternative states described by density functions $f(x)$ and $g(x)$. The family of Stochastic Dominance techniques address the issue: "which state is preferred if the objective is the largest $\mathbf{E}[U(\mathcal{X})]$?". Formally, when the derivatives of $U(x)$ are such that $(-1)^{s+1}\frac{d^sU(x)}{dx^s} > 0$ [8], for $s = 1, \ldots, S$, a sufficient condition for:

$$\mathbf{E}_f[U(\mathcal{X})] - \mathbf{E}_g[U(\mathcal{X})] \;=\; \int_a^b U(x)(dF - dG) \geq 0 \tag{1}$$

is given by the condition for the dominance of distribution G by F at order $s = 1, 2, \ldots, S$:

$$F^s(x) \leq G^s(x) \;\; \forall \;\; x \in [a, b] \quad \text{and} \quad F^s(x) < G^s(x) \;\text{ for some }\; x \in [a, b] \tag{2}$$

$$\text{where}: \; F^s(x) = \int_a^x F^{s-1}(z)dz \quad \text{and} \quad F^0(x) = f(x)$$

Essentially the condition requires that the functions $F^s(x)$ and $G^s(x)$ not cross, so that the dominating distribution is "unambiguously" below the other. It will be useful for the subsequent discussion to note that $F^s(x)$ (or equivalently $G^s(x)$) may be rewritten in incomplete moment form as:

$$F^s(x) \;=\; \frac{1}{(s-1)!}\int_a^x (x - y)^{s-1}dF(y) \tag{3}$$

From equations (1) and (2), the dominating distribution is the preferred distribution, reflecting the desire for greater $\mathbf{E}[U(\mathcal{X})]$. As the order of dominance considered

[8] These standard conditions for risk averting preferences, and the following techniques and statistics, can readily be adapted to accommodate risk loving preferences.

increases, it imposes additional constraints on the curvature of the wellbeing function $U(x)$, reflected in the order at which the conditions of equation (2) holds, consequently providing the link between the wellbeing and distribution functions. However, based on equation (3), increasing orders of dominance attaches increasing weight to lower values of x in the population distribution, so that successively higher orders of dominance can be interpreted as reflecting higher orders of concern for the left tail end of the distribution. In consequence, the welfare economics literature (Foster and Shorrocks 1988) commonly draws parallels between a specific orders of dominance (s) and a particular societal preference for income distribution, (for example $s = 1$ is associated with *Utilitarian*; $s = 2$ is associated with *Daltonian*; $\ldots$; $s = \infty$ is associated with *Rawlsian*). On the other hand, for an investor choosing between non-combinable portfolios, increasing s reflects increasing concern over downside risk.[9]

Effective use of the technique requires the decision-maker to choose that order "s" which best reflects the priorities they confront, in terms of the degree of concern for less desirable outcomes. For example, in contemplating alternative income transfer policies, if the the policymaker were indifferent as to where in the income distribution revenue neutral transfers were made, 1st order dominance comparisons are appropriate. On the other hand, particular concern for the poor would demand comparisons be made at values of s greater than 1. The problem addressed here is that frequently empirical distributions cross, and no policy dominates at a given s appropriate for the policymaker's priorities. Furthermore, the pairwise nature of the standard comparison technique renders a complete ordering impractical when the number of alternatives are countably large.

2.2 Wellbeing Index Axioms

To facilitate ease of interpretation, comparability, and the development of the statistical properties, wellbeing indices are expected to obey certain common axioms (Sen 1995). Following common conceptions of social welfare measures, let the variable underlying a measure be $x \in \mathcal{X} \subset \mathbb{R}$. Each individual constituent derives wellbeing $U(x)$, $U : x \in \mathcal{X} \to [0, \infty)$, and $U \in \mathcal{W}$, where $\mathcal{W}$ is the set of continuous wellbeing measures. These realizations have a smooth continuous distribution $G : x \in \mathcal{X} \to [0, 1]$, $G \in \mathcal{P}$,

[9] The comparison procedures have been empirically implemented in several ways. See for example Anderson (1996, 2004), Barrett and Donald (2003), Davidson and Duclos (2000), Knight and Satchell (2008), Linton et al. (2005), and McFadden (1989)

where $\mathcal{P}$ is the set of distribution functions. Then a social welfare index is a functional that maps individual wellbeing onto societal wellbeing, denoted as $\mathcal{I}(G, U)$, where $\mathcal{I}: G(\mathcal{X}) \times U(\mathcal{X}) \longmapsto [0, \infty)$ typically.

Ideally, indices should abide by the following axioms.

Axiom 1: ***Continuity***: *$\mathcal{I}(G, U)$ is a continuous functional on $\mathcal{X}$, with distribution $G \in \mathcal{P}$, and wellbeing from $U \in \mathcal{W}$.*

Axiom 2: ***Wellbeing Function Independence/Scale Independence***: *Let $U_i, U_j \in \mathcal{W}$, $i \neq j$, and $G \in \mathcal{P}$, then $\mathcal{I}(G, U_i) = \mathcal{I}(G, U_j)$.*

Axiom 3: ***Coherence***: *Denote the stochastic dominance relationship at order s by $\succ^s$. If $G_i(x) \succ^s G_j(x) \Rightarrow \mathcal{I}(G_i, U) \geq \mathcal{I}(G_j, U)$, $\forall\ U \in \mathcal{W}$.*

Axiom 4: ***Normalization***: *For any $U \in \mathcal{W}$, and $G \in \mathcal{P}$, $\mathcal{I}(G, U) : G(\mathcal{X}) \times U(\mathcal{X}) \longmapsto [0, 1]$.*

The above four axioms parrallel the standard properties used in inequality measurement. Axiom 1 states that the index needs to be continuous. Axiom 2 implies that the index is independent of scale or functional form of the subjective wellbeing function, so that it would be invariant to monotonic transformations of the wellbeing function. Axiom 3 requires that the ordering be consistent, so that the more preferred distribution yields a larger value of the index. Axiom 4 simply highlights that the index should ideally be normalized, so that the underlying variable is mapped onto the range $[0, 1]$. Taking axioms 3 and 4 together, the mapping of the index must be such that zero is assigned to the least preferred distribution, while 1 is assigned to the most preferred distribution, and all other distributions between the extremes must remain consistent. This then facilitates the ordering of the underlying prospects and/or policies.

An equally important axiom, but often omitted from discussion, is the idea of *independence of irrelevant alternative* as it relates to the development of wellbeing index. Here, we adopt the *relation-theoretic* definition (Sen 1987) as opposed to the *choice-theoretic* definition (Arrow 1951), since a wellbeing index defines an ordering relation.

Axiom 5: ***Independence of Irrelevant Alternatives***: *Let $F \in \mathcal{P}$ be the distribution of an irrelevant alternative. Let Γ be the set of alternatives under consideration. Then for G_i and $G_j \in \Gamma_{-F}$, $\mathcal{I}\left(G_i, U; \Gamma_{-F}\right) \geq \mathcal{I}\left(G_j, U; \Gamma_{-F}\right)$ if and only if $\mathcal{I}\left(G_i, U; \Gamma\right) \geq$*

$\mathcal{I}(G_j, U; \Gamma)$, and $\mathcal{I}(G_j, U; \Gamma_{-F}) \geq \mathcal{I}(G_i, U; \Gamma_{-F})$ if and only if $\mathcal{I}(G_j, U; \Gamma) \geq \mathcal{I}(G_i, U; \Gamma)$, so that the ordering relation between G_i and G_j under $\mathcal{I}(., U; \Gamma_{-F})$ and $\mathcal{I}(., U; \Gamma)$ are exactly the same.

Axiom 5 is not commonly included as a necessary property for wellbeing indices. However, since the objective of this new index is to highlight its practicality, the fulfillment of this axiom allows the practitioner to constrain the policy set to a relevant subset, thereby focusing on the pertinent policy alternatives.

Table 1: Examples of Common Poverty/Inequality Indices

Head Count:	$H = \int_0^z f(x)dx = F(z)$
Normalized Deficit:	$D = \int_0^z \left[1 - \frac{x}{z}\right] f(x)dx$
Lorenz (1905) Curve:	$L = \frac{1}{\mu} \int_0^z x f(x)dx$
Gini (1955) Coefficient:	$G_z = \frac{1}{\mu} \int_0^z F(x)(1 - F(x))dx$
Watts (1968) Measure:	$W = -\int_0^z \ln\left(\frac{x}{z}\right) f(x)dx$
Clark et al. (1981) Measure:	$\frac{1}{c}[1 - (1 - P^*)^c] = \frac{1}{c} \int_0^z \left[1 - \left(\frac{x}{z}\right)^c\right] f(x)dx,\ c \in (-\infty, 1]$
Atkinson (1983) Measure:	$I = 1 - \left[\int_0^z \left(\frac{x}{\mu}\right)^{1-\epsilon} f(x)dx\right]^{\frac{1}{1-\epsilon}}$
Foster et al. (1984) Measure:	$P_a = \int_0^z \left(1 - \frac{x}{z}\right)^a f(x)dx,\ a \in [0, \infty)$
Sen (1987) Measure:	$S = H[D - (1 - D)G_z] = H(G_z) + D(G_z)$

Note: z is the poverty threshold, μ is the mean income, $\epsilon \in (0, 1)$ is the degree of inequality-aversion.

Table 1 enumerates some common inequality and wellbeing indices including those discussed in Atkinson (1987). Denoting the class of s^{th} order wellbeing functions under the decision-maker's consideration as $\mathcal{U}_s$, $s = 1, \ldots, S$, the effect of the axioms can be illustrated by considering some of the measures in the table. Average income, or some monotonic transformation of it, and the headcount ratio provide a complete ordering from the $s = 1$ *Utilitarian* class of wellbeing indices, $\mathcal{U}_{s=1}$. The remainder of the indices in table 1 represent various methods of accounting for societal aversion towards inequality, and can be classified under $\mathcal{U}_s$ class of indices, $s = \{2, 3, \dots\}$. These are achieved through functional choices such as the logarithmic function in the case of the Watts (1968), while the Atkinson (1983) index allows the user to moderate their aversion towards inequality through the parameter $\epsilon \in [0, \infty)$. In lieu of the use of a wellbeing function or inequality aversion function, the indices with the exception of the Head Count index and the Gini coefficient are not *scale independent*, though they meet the requirements of axioms 1, 3 and 5, while the Atkinson (1983), Gini Coefficient, and Head Count indices meet in addition axiom 4. Nonetheless, the remainder of the indices can easily be transformed into an index with a range of $[0, 1]$ to achieve axiom 4. Let $\mathcal{I}_k$ denote the inequality index for the k^{th} alternative, and $k = \{1, \dots, K\}$, where K is the number of policy alternatives under consideration. Then the transformation is simply, $\frac{\mathcal{I}_k - \min\{\mathcal{I}_1, \dots, \mathcal{I}_K\}}{\max\{\mathcal{I}_1, \dots, \mathcal{I}_K\} - \min\{\mathcal{I}_1, \dots, \mathcal{I}_K\}}$.

Stochastic dominance orderings are partial because of the potential for curves to cross, and when they do, an ordering of alternatives cannot be obtained. Here based upon the collection of policies, a complete ordering is achieved by comparing each alternative to a synthetic, piecewise constructed, best outcome over the entire support $\mathcal{X}$. The index in focusing only on the underlying distributions generated by the alternatives, is consequently invariant to the subjective choice of wellbeing function, and is nonparametric in nature. It requires the commitment of the decision-maker to a choice of s, the order of dominance, which is a reflection of their priorities.

3 The Hull Index

3.1 Development of the Hull Index

Consider a set of outcome distributions $\Gamma = \{G_1, G_2, \dots, G_K\}$, which are the consequence of alternative choices being contemplated and, for convenience, let $x \in \mathcal{X} = [0, \infty)$. In a collection of pairwise comparisons within the family of dominance criteria, where the s^{th} order dominance is of the form given in (2) above, Anderson (2004) interpreted dominance between G_i and G_j at a particular order as a function of the degree of separation between the distributions at that order, over the support of the underlying variable(s). In that case, the area between the two curves provide a very natural index of this magnitude. Thus the index when G_i dominates G_j at the s^{th} order is,

$$\mathbb{T}_s(G_j, G_i) = \int_{\mathcal{X}} \left[G_j^s(x) - G_i^s(x) \right] dx \tag{4}$$

so that $\mathbb{T}_s : (G_j(\mathcal{X}), G_i(\mathcal{X})) \longmapsto [0, \infty)$, and it provides an index of such a separation or excess of G_j over G_i derived from the two policies. The metric of the index is related to the units of μ^s, the s^{th} power of the mean of x, so that the division of (4) by μ^s would render $\mathbb{T}_s$ a unit free index. However, such an index only works if G_i dominates G_j at this order. As noted previously, it is not uncommon for there not to be a dominant policy at a given order due to the crossing of the distributions.

Consider now a decision-maker with *utilitarian* priorities faced with the prospect of there being no 1^{st} order dominant policy in Γ. However, the lower frontier or hull of all distributions in the set Γ given by,

$$\mathcal{L}_1(\Gamma) \quad = \quad \min_{G_k \in \Gamma} \{\Gamma\} = \min_{G_k \in \Gamma} \{G_1, G_2, \ldots, G_K\} \tag{5}$$

so that $\mathcal{L}_1 : (G_1(\mathcal{X}), G_2(\mathcal{X}), \ldots, G_K(\mathcal{X})) \longmapsto [0, \infty)$, can be thought of as the best possible synthetic policy on $\mathcal{X}$, if all policies could be so combined. Since by construction $\mathcal{L}_1$ dominates all distributions in Γ at the 1^{st} order, it would thus be the preferred distribution (Note that if one of the distributions in the collection 1^{st} order dominated all of the other distributions, then it would be equal to $\mathcal{L}_1$). Given this fact, the proximity of each of the distributions in the set Γ to $\mathcal{L}_1$, would permit the construction of a complete ranking at the 1^{st} order. Denote $\Phi^1 = \{x \in \mathcal{X} : G_i(x) = G_j(x), i \neq j, G_i, G_j \in \Gamma\}$ as the set of intersection points that separates $\mathcal{X}$ into mutually exclusive closed subsets of $\mathcal{X}$, $\mathcal{X}_n$ $n = \{1, \ldots, N\}$, such that $\mathcal{X} = \bigcup_{n=1}^{N} \mathcal{X}_n$. Then for $G_k \in \Gamma$, the index of proximity used in ranking the policies is,

$$\mathcal{T}_1(G_k; \mathcal{L}_1(\Gamma)) \quad = \quad \int_{\mathcal{X}} [G_k(x) - \mathcal{L}_1(x)] dx \tag{6}$$

$$= \quad \sum_{n=1}^{N} \left[\int_{\mathcal{X}_n} [G_k(x) - \mathcal{L}_{1,n}(x)] dx \right] \tag{7}$$

so that $\mathcal{T}_1 : G_k(\mathcal{X}) \longmapsto [0, \infty)$, and the most preferred policy is that which yields the lowest $\mathcal{T}_1$, in other words the policy closest to the hull over $\mathcal{X}$.

This idea can be generalized to the s^{th} order imperative, where one contemplates the hull constructed from all possible s^{th} order integrals of the candidate distributions, $\Gamma^s = \{G_1^s(x), G_2^s(x), \ldots, G_K^s(x)\}$, so that:

$$\mathcal{L}_s(\Gamma^s) \quad = \quad \min_{G_k^s \in \Gamma^s} \{\Gamma^s\} = \min_{G_k^s \in \Gamma^s} \{G_1^s, G_2^s, \ldots, G_K^s\} \tag{8}$$

and $\mathcal{L}_s : (G_1^s(\mathcal{X}), G_2^s(\mathcal{X}), \ldots, G_K^s(\mathcal{X})) \longmapsto [0.\infty)$, so that as in the case of $s = 1$, $\mathcal{L}_s(\Gamma^s)$ would stochastically dominate all distributions in Γ^s at the s^{th} order, and would thus, if practicable, be the preferred distribution at that order[10]. The general index for ranking all prospective policies $G_k^s \in \Gamma^s$, would then be:

$$\mathcal{T}_s(G_k^s; \mathcal{L}_s(\Gamma^s)) = \int_{\mathcal{X}} [G_k^s(x) - \mathcal{L}_s(x)]\, dx \tag{9}$$

$$= \sum_{n=1}^{N} \left[\int_{\mathcal{X}_n} [G_k^s(x) - \mathcal{L}_{s,n}(x)] dx \right] \tag{10}$$

and $\mathcal{T}_s : G_k^s(\mathcal{X}) \longmapsto [0, \infty)$ [11].

It is worth noting that Leshno and Levy (2002) compared the area of the dominance relationship violation to the full transvariation measure of two distributions under consideration, and if the violation area is proportionately small, then *Almost Stochastic Dominance* is claimed. This is observationally equivalent to measuring the area between the lower envelope of the two alternatives against each of the alternatives in turn, comparing magnitudes and choosing the smallest. Here the envelope of a collection of distributions is employed, and a complete ordering established by the size of the areas bounded by each alternative distribution and the lower envelope.

A possible concern with the $\mathcal{T}_s$ index is that it has no upper bound (i.e. it does not comply with axiom (4), so that a policymaker is unable to discern how far a policy is from the best choice in Γ^s. The following provides an index whose support is on $[0, 1]$. Consider the upper frontier or envelope of all possible s^{th} order integrals amongst the candidate distributions in Γ^s:

$$\mathcal{M}_s(\Gamma^s) = \max_{G_k^s \in \Gamma^s} \{\Gamma^s\} = \max_{G_k^s \in \Gamma^s} \{G_1^s, G_2^s, \ldots, G_K^s\} \tag{11}$$

[10] Again note that if one of the distributions s^{th} order dominated all of the other distributions, it would be equal to $\mathcal{L}_s(\Gamma^s)$. It is of interest to note that for $s = 2$ and support $x \in \mathbb{R}_+ + \{0\}$:

$$\int_0^\infty (G(x) - H(x))dx = \int_0^\infty (1 - H(x)) - (1 - G(x))dx = \mu_{H,x} - \mu_{G,x}$$

which can readily be seen to be a difference in means test.

[11] Note the similarity with Kolmagorov-Smirnov comparisons which consider the maximal/minimal vertical distance between functions over the support, whereas this comparison is of proximity over the entire support, so that the rankings based on the 2 comparators may differ

such that $\mathcal{M}_s : (G_1^s(\mathcal{X}), G_2^s(\mathcal{X}), \ldots, G_K^s(\mathcal{X})) \longmapsto [0.\infty)$, which constitutes the worst that the policymaker could do if she combined the policies in the worst possible manner. Proximity to $\mathcal{M}_s(\Gamma^s)$ is an index of how bad the chosen policy is, so that the area between this worst synthetic outcome (the upper frontier or envelope) and the best synthetic outcome (the lower frontier or hull) is given by:

$$\mathcal{S}_s(\Gamma^s) = \int\limits_{\mathcal{X}} (\mathcal{M}_s(x) - \mathcal{L}_s(x))\, dx \tag{12}$$

such that $S_s : (\mathcal{M}_s(\mathcal{X}), \mathcal{L}_s(\mathcal{X})) \longmapsto [0, \infty)$, and constitutes a measure of the range of possibilities available to the policymaker for a given policy set Γ^s at the s^{th} order imperative, and is a constant. In effect S_s is a many distribution analogue of Gini's Transvariation (Anderson et al. 2017), which reflects the degree of variation in a collection of distributions. If S_s were close to 0, it would suggest that all policies are very similar in outcome. More importantly, the index of the relative merit of policy $G_m^s \in \Gamma^s$ is thus,

$$\mathcal{H}_s(G_m^s) = 1 - \frac{\mathcal{T}_s(G_m^s)}{\mathcal{S}_s} \tag{13}$$

Since $\mathcal{T}_s(G_m^s) \in [0, \mathcal{S}_s]$, so that $\frac{\mathcal{T}_s(G_m^s)}{\mathcal{S}_s} \in [0,1]$, then $\mathcal{H}_s(G_m^s) \in [0,1]$. Thus the more desirable a policy at order s is, the closer it would be to 1, while the less desirable it is, the closer it would be to 0, thus providing a complete ordering of policies at the s^{th} order of integration. The properties of $\mathcal{H}_s$ will be formalized in the following section.

3.2 Properties of the Hull Index, $\mathcal{H}_s$

Unlike common indices associated with wellbeing functions, $\mathcal{T}_s$ and $\mathcal{H}_s$ are only piecewise continuous, with kinks at the points of intersection between the distribution functions that constitute $\mathcal{L}_s$, so that the discontinuities may increase as the number of distribution functions in Γ^s increases. In other words, since $\mathcal{L}_s$ is the sum of a sequence of continuous functions over mutually exclusive partitions over the support, both $\mathcal{T}_s$ and $\mathcal{H}_s$ are piecewise continuous.

Formally, denote $\Phi^s = \{x \in \mathcal{X} : G_i^s(x) = G_j^s(x), i \neq j, G_i^s, G_j^s \in \Gamma^s\}$ as the set of intersection points that separates $\mathcal{X}$ into mutually exclusive closed subsets of $\mathcal{X}$, $\mathcal{X}_n$ $n = \{1, \ldots, N\}$, such that $\mathcal{X} = \bigcup\limits_{n=1}^{N} \mathcal{X}_n$.

Definition 1: *A function f is* ***piecewise continuous*** *if there is a sequence $\{\mathcal{X}_n\}_{i=1}^{N}$ of closed subsets of $\mathcal{X}$ such that $\mathcal{X} = \bigcup\limits_{n=1}^{N} \mathcal{X}_n$, and $f(x \in \mathcal{X}_n)$ is continuous $\forall \mathcal{X}_n \subset \mathcal{X}$.*

Further, denote $\Psi^s = \{G_i^s \in \Gamma^s : G_i^s = \mathcal{L}_s\}$ as the set of distributions at order s that constitute the hull $\mathcal{L}_s$.

Proposition 1: *1. If $\Phi^s = \mathcal{X}$, and Ψ^s is a singleton, then $\mathcal{T}_s$ and $\mathcal{H}_s$ are continuous.*

2. If $\Phi^s = \mathcal{X}$, and Ψ^s is not a singleton, and is finite, then $\mathcal{T}_s$ and $\mathcal{H}_s$ are continuous.

3. If Φ^s is nonempty and finite, then $\mathcal{T}_s$ and $\mathcal{H}_s$ are piecewise continuous.

Proof. For part 1, when $\Phi^s = \mathcal{X}$, and Ψ^s is a singleton, then there exists a unique stochastically dominant distribution, $G_k^s = \min\{\Gamma^s\}$, so that:

$$\begin{aligned} \mathcal{T}_s(G_j^s) &= \int_{\mathcal{X}} [G_j^s(x) - \mathcal{L}_s(x)]dx \\ &= \int_{\mathcal{X}} [G_j^s(x) - G_k^s(x)]dx \end{aligned}$$

and $\mathcal{T}_s$ is continuous over $\mathcal{X}$, $\forall\ G_j^s \in \Gamma^s$, $j = \{1, \ldots, K\}$, and so too is $\mathcal{H}_s$.

For part 2, when Ψ^s is not a singleton, then at least two distributions completely overlap one another. Then for $G_k^s \in \Psi^s$, $G_k^s = \mathcal{L}_s$, and $\mathcal{L}_s$ is continuous, and in consequence so too is $\mathcal{T}_s$ and $\mathcal{H}_s$.

For part 3, the intersections partition the support $\mathcal{X}$ into N mutually exclusive segments $\mathcal{X}_n$, such that $\bigcup_{n=1}^{N} \mathcal{X}_n = \mathcal{X}$. Denote the typical element of Ψ^s as $\mathcal{L}_{s,k}$ that constitutes the k^{th} segment of $\mathcal{L}_s$. Then,

$$\begin{aligned} \mathcal{T}_s(G_j^s) &= \int_{\mathcal{X}} [G_j^s(x) - \mathcal{L}_s(x)]dx \\ &= \sum_{k=1}^{N} \left(\int_{\mathcal{X}_k} [G_j^s(x) - \mathcal{L}_{s,k}(x)]dx \right) \end{aligned}$$

and $\mathcal{T}_s$ and $\mathcal{H}_s$ are thus piecewise continuous, since the distribution functions that constitute the elements of Ψ^s are continuous.

When instead a subset of Φ^s, denoted as ϕ_n, are continuous segments of $\mathcal{X}$, $\phi_n = \mathcal{X}_n \subset \mathcal{X}$, then the segments has at least two distributions completely overlapping each other, which means that any of the distributions can be set as $\mathcal{L}_{s,k}$, so that the hull index inherits the continuity of that distribution, and $\mathcal{T}_s$ and $\mathcal{H}_s$ remains piecewise continuous.

■

The principal benefit in the use of stochastic dominance as an index of inequality rests in it being derived from the sufficiency conditions discussed above, making it invariant to the policy-maker's choice of underlying wellbeing function. Insofar as $\mathcal{T}_s$ is a function of the underlying distributions, in examining the expected wellbeing across differing states, it inherits these same features and is free of the underlying wellbeing function, and in consequence is *wellbeing invariant and scale independent.*

Proposition 2: *$\mathcal{T}_s$ and $\mathcal{H}_s$ are independent of wellbeing function or scale.*

Proof. Without lost of generality, let $U_A(x)$ and $U_B(x)$ be two differing wellbeing measures. Let $G_1^s \succ_i G_2^s$ for all $x \in \mathcal{X}$, $s = \{1, 2, \dots\}$, and $i = \{A, B\}$. Then $G_1(x) \succ_i G_2(x)$ for $i = \{A, B\}$, implies that,

$$\mathbf{E}_1[U_i(\mathcal{X})] - \mathbf{E}_2[U_i(\mathcal{X})] \;\; = \;\; \int\limits_{\mathcal{X}} U_i(x)(dG_1^s - dG_2^s) \geq 0$$

The sufficient condition for the above inequality is,

$$G_1^s(\mathcal{X}) \;\; \leq \;\; G_2^s(\mathcal{X})$$

for both wellbeing measures U_A and U_B, $\forall\ s = \{1, 2 \dots\}$, which implies that $\mathcal{T}_s$ and $\mathcal{H}_s$ are the same under both wellbeing measures. ■

To ensure ease of interpretation, *coherence* and *normalization* are necessary. To that end, since $\mathcal{T}_s$ is increasing in distance from $\mathcal{L}_s$, then after transformation, $\mathcal{H}_s$ is both coherent, and normalized. This is demonstrated in the following two propositions.

Proposition 3: *$\mathcal{H}_s$ are coherent indices.*

Proof. Let $G_j^s \succ^s G_i^s$ for order $s = \{1, 2, \dots\}$, where $i \neq j$, $i\ j = \{1, 2, \dots, K\}$, under wellbeing function $U \in \mathcal{W}$. Then,

$$\int\limits_{\mathcal{X}} G_j^s(x)dx \;\; \leq \;\; \int\limits_{\mathcal{X}} G_i^s(x)dx$$

$$\Rightarrow \mathcal{T}_s(G_j^s) = \int\limits_{\mathcal{X}} \left[G_j^s(x) - \mathcal{L}_s(x)\right] dx \;\; \leq \;\; \int\limits_{\mathcal{X}} \left[G_i^s(x) - \mathcal{L}_s(x)\right] dx = \mathcal{T}_s(G_i^s)$$

So that,

$$\mathcal{H}_s(G_j^s) \;\; \geq \;\; \mathcal{H}_s(G_i^s)$$

for all $s = \{1, \dots\}$. ■

Proposition 4: *The range of $\mathcal{H}_s$ is closed and bounded, $\mathcal{H}_s \in [0, 1]$.*

Insofar as policy sets evolve, a natural question arises as to whether the statistic possesses an *Independence of Irrelevant Alternatives* (IIA) property. An alternative is irrelevant here, if given the set of choices under examination, the introduction into the choice set of the new alternative does not alter the order generated by $\mathcal{T}_s$ and $\mathcal{H}_s$ for the original choice set (Sen 1987). To formalize this, let the full choice set at the s^{th} order of dominance be:

$$\Gamma^s = \{G_1^s, G_2^s, \ldots, G_K^s\}$$

Without loss of generality, let the irrelevant alternative be $G_i^s(x)$ so that the set of alternatives excluding it is:

$$\Gamma^s_{-G_i^s} = \left\{G_1^s, G_2^s, \ldots, G_{i-1}^s, G_{i+1}^s, \ldots G_K^s\right\}$$

for $i \in \{1, \ldots, K\}$. In addition, define the ordering achieved under $\Gamma^s_{-G_i^s}$ as $\mathcal{O}\left(\Gamma^s_{-G_i^s}\right)$, and the ordering under Γ^s excluding G_i^s in the order as $\mathcal{O}_{-G_i^s}(\Gamma^s)$.

Definition 2: *$G_i^s(x)$ is an irrelevant alternative if and only if:*

$$G_i^s(\mathcal{X}) \geq \mathcal{L}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) = \min_{G_k^s \in \Gamma^s_{-G_i^s}} \left\{\Gamma^s_{-G_i^s}\right\} \quad (14)$$

$$G_i^s(\mathcal{X}) \leq \mathcal{M}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) = \max_{G_k^s \in \Gamma^s_{-G_i^s}} \left\{\Gamma^s_{-G_i^s}\right\} \quad (15)$$

with strict inequality holding somewhere.

Essentially, this requires that policy G_i^s be dominated by and dominates respectively, at the s^{th} order, the lower and upper envelopes of all other policies so that they can be formed without resort to distribution G_i^s. It follows then that the ordering generated by $\mathcal{T}_s$ and $\mathcal{H}_s$ has the *Independence of Irrelevant Alternatives* property. Formally,

Proposition 5: *For irrelevant alternative G_i^s,*

$$\mathcal{T}_s(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}) = \mathcal{T}_s(G_k^s; G_k^s \in \Gamma^s)$$

so that,

$$\mathcal{H}_s(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}) = \mathcal{H}_s(G_k^s; G_k^s \in \Gamma^s)$$

$\forall\ G_k^s \in \Gamma^s$, *where $k \neq i$ and $k = \{1, \ldots, K\}$, so that $\mathcal{O}\left(\Gamma^s_{-G_i^s}\right)$ is the same as $\mathcal{O}_{-G_i^s}(\Gamma^s)$.*

Proof. By inequality (14) of the definition 2, for an irrelevant alternative G_i^s:

$$\begin{aligned} \mathcal{L}_s(\Gamma^s) &= \min_{G_k^s \in \Gamma^s} \{\Gamma^s\} \\ &= \min_{G_k^s \in \Gamma^s_{-G_i^s}} \left\{\Gamma^s_{-G_i^s}\right\} = \mathcal{L}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) \end{aligned}$$

In turn,

$$\begin{aligned} \mathcal{T}_s(G_k^s; G_k^s \in \Gamma^s) &= \int_{\mathcal{X}} (G_k^s(x) - \mathcal{L}_s(x))\, dx \\ &= \int_{\mathcal{X}} \left(G_k^s(x) - \mathcal{L}_{s,-G_i^s}(x)\right) dx = \mathcal{T}(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}) \end{aligned}$$

By a similar argument, equation (15) implies that:

$$\begin{aligned} \mathcal{M}_s(\Gamma^s) &= \max_{G_k^s \in \Gamma^s} \{\Gamma^s\} \\ &= \max_{G_k^s \in \Gamma^s_{-G_i^s}} \left\{\Gamma^s_{-G_i^s}\right\} = \mathcal{M}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) \end{aligned}$$

so that,

$$\begin{aligned} \mathcal{S}_s(\Gamma^s) &= \int_{\mathcal{X}} (\mathcal{M}_s(x) - \mathcal{L}_s(x)) dx \\ &= \int_{\mathcal{X}} \left(\mathcal{M}_{s,-G_i^s}(x) - \mathcal{L}_{s,,-G_i^s}(x)\right) dx = \mathcal{S}_s\left(\Gamma^s_{-G_i^s}\right) \end{aligned}$$

Therefore,

$$\mathcal{H}_s(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}) \quad = \quad \mathcal{H}_s(G_k^s; G_k^s \in \Gamma^s)$$

Therefore the ordering $\mathcal{O}\left(\Gamma^s_{-G_i^s}\right)$, and that under $\mathcal{O}_{-G_i^s}(\Gamma^s)$ are the same. ■

What proposition 5 says is that the introduction of an irrelevant alternative, will not affect the ordering of the original choice set as long as it does not alter the evaluation boundaries. Put another way, the benefit of the *Independence of Irrelevant Alternatives* feature is that when facing any new alternative, all the researcher needs to verify is that both $\mathcal{L}_s(\Gamma^s_{-G_i^s})$ and $\mathcal{M}_s(\Gamma^s_{-G_i^s})$ remain intact, eliminating the repeated comparisons each time a new alternative surfaces. It should be noted that it does not preclude the possibility that the irrelevant alternative could be the highest or lowest ranked alternative in the full alternatives set, Γ^s.

In addition, if a policy G_i^s is strictly dominated by a subset of $\Gamma_{-G_i^s}$ and strictly dominates the complement of that set, then it is an irrelevant alternative. Formally, denote the subset of $\Gamma^s_{-G_i^s}$ with strictly dominant policies as $\Gamma^s_{-G_i^s,\succ G_i^s} = \{G_k^s : G_k^s \in \Gamma_{-G_i^s}, G_k^s \succ G_i^s\}$, and denote the subset of $\Gamma^s_{-G_i^s}$ with dominated policies as $\Gamma^s_{-G_i^s,\prec G_i^s} = \{G_k^s : G_k^s \in \Gamma_{-G_i^s}, G_k^s \prec G_i^s\}$.

Corollary 1: *For $G_i^s \in \Gamma^s$, if $\Gamma^s_{-G_i^s,\succ G_i^s} \neq \emptyset$, and $\Gamma^s_{-G_i^s,\prec G_i^s} \neq \emptyset$ at the s^{th} order, it is an irrelevant policy at the s^{th} order.*

Proof. The result follows directly from proposition 5. ∎

This result holds for all higher orders of s by the definition of stochastic dominance as highlighted by the following corollary.

Corollary 2: *If $G_i^s \in \Gamma^s$ is an irrelevant alternative at the s^{th} order of dominance, it is an irrelevant policy at all higher orders of dominance.*

Proof. Without loss of generality, suppose for $G_i^s(x) \in \Gamma^s$, $\Gamma^s_{-G_i^s,\succ G_i^s} \neq \emptyset$, and $\Gamma^s_{-G_i^s,\prec G_i^s} \neq \emptyset$ at the s^{th} order, so that G_i^s is an irrelevant alternative at s. By the definition of stochastic dominance, for order $s' > s$, the policies in $\Gamma^s_{-G_i^s,\succ G_i^s}$ will continue to dominate G_i^s at s'. Similarly, the policies in $\Gamma^s_{-G_i^s,\prec G_i^s}$ will remain dominated at s' by G_i^s. So that $\Gamma^{s'}_{-G_i^{s'},\succ G_i^{s'}} \neq \emptyset$ and $\Gamma^{s'}_{-G_i^{s'},\prec G_i^{s'}} \neq \emptyset$, and the result follows. ∎

As an aside to the above discussion, it is tempting to interpret independence of irrelevant alternatives of distribution G_i^s as being irrelevant if and only if:

$$\begin{aligned} G_i^s &\leq \mathcal{L}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) \\ G_i^s &\geq \mathcal{M}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right) \end{aligned}$$

However, in this case, since the boundaries that define $S_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$ are altered, it is equivalent to changing the evaluation criterion, and there are no guarantees that the ordering would persist here. This is primarily because the ordering is generated against the baselines defined by the boundaries of $S_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$, that is $\mathcal{L}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$ and $\mathcal{M}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$, so that if the boundaries changes, $\mathcal{T}_s\left(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}\right) \neq \mathcal{T}_s\left(G_k^s; G_k^s \in \Gamma^s\right)$, and in consequence $\mathcal{H}_s\left(G_k^s; G_k^s \in \Gamma^s_{-G_i^s}\right) \neq \mathcal{H}_s\left(G_k^s; G_k^s \in \Gamma^s\right)$. Insofar as the desirability of *independence of irrelevant alternatives* is in its ability to focus on a pertinent set of choices, a distribution that strictly dominates $\mathcal{L}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$ is not irrelevant when the objective is in selecting the best policy. Indeed, it would make all other alternatives irrelevant. Reversing this argument, when G_i^s is dominated by $\mathcal{M}_{s,-G_i^s}\left(\Gamma^s_{-G_i^s}\right)$, it is absolutely irrelevant.

It now remains to be demonstrated that $\mathcal{H}_s$ generates a complete ordering. To formalize this, we need to define features necessary for a complete ordering (Grätzer 2003).

Definition 3: *For a, b and c on* $\mathbb{R}$,

1. *Reflexivity:* $a \leq a$.
2. *Antisymmetry:* $a \leq b$ *and* $b \leq a \Rightarrow a = b$.
3. *Transitivity:* $a \leq b$ *and* $b \leq c \Rightarrow a \leq c$.
4. *Linearity:* $a \leq b$ *or* $b \leq a$

Definition 4: *An ordering is a chain, and is consequently a complete ordering if and only if it satisfies the above four properties of definition 3.*

Given propositions 1 to 4, both $\mathcal{T}_s$ and $\mathcal{H}_s$ are real valued functionals that map the distributions in Γ^s onto $[0, 1]$. Then by definitions 3 and 4, the orderings generated by $\langle \mathcal{T}_s; \leq \rangle$ and $\langle \mathcal{H}_s; \leq \rangle$ are *partially ordered sets* (POSETs), and are *chains*, and the theorem follows.

Theorem 1: *The ordering generated by* $\mathcal{T}_s$ *and* $\mathcal{H}_s$ *forms a chain, and thus generates a complete ordering.*

It should be noted that when there is resolution in the stochastic dominance tests for the set of policies under consideration, the ordering generated by the indices proposed here are consistent with those of the dominance tests.

Theorem 2: *For a given* Γ^s *set of policies, if the stochastic dominance test yields a chain, then the ordering from* $\mathcal{T}_s$ *and* $\mathcal{H}_s$ *will likewise be consistent across all orders.*

Proof. If stochastic dominance yields a chain $\forall\ G_k, G_i \in \Gamma^s$, then,

$$\begin{aligned} G_k^s &\succ_s G_i^s \\ \Rightarrow G_k^{s'} &\succ_{s'} G_i^{s'} \end{aligned}$$

$\forall\ G_i, G_k \in \Gamma^s$ and $s' > s$.

In turn, the above implies that by proposition 3,

$$\begin{aligned} \mathcal{T}_s(G_i^s) - \mathcal{T}_s(G_k^s) &> 0 \\ \Rightarrow \mathcal{H}_s(G_i^s) - \mathcal{H}_s(G_k^s) &< 0 \end{aligned}$$

and similarly,

$$\begin{aligned} \mathcal{T}_{s'}(G_i^{s'}) - \mathcal{T}_{s'}(G_k^{s'}) &> 0 \\ \Rightarrow \mathcal{H}_{s'}(G_i^{s'}) - \mathcal{H}_{s'}(G_k^{s'}) &< 0 \end{aligned}$$

for all orders of dominance $s' > s$. Thus the ordering from $\mathcal{T}_s$ and $\mathcal{H}_s$ are consistent with the stochastic dominance test. ■

This in turn means that for a given Γ^s set of policies, if stochastic dominance tests yields only a partial ordering (POSET), the ordering produced by $\mathcal{T}_s$ and $\mathcal{H}_s$ may exhibit switching of ordering as the order considered increases. Nonetheless, this ceases at the order s where stochastic dominance yields a resolution. This behavior should be understood from the point of view of the objective these indices were meant to solve; the provision of a solution to the problem of stochastic dominance tests typically not yielding resolution, and the cumbersome nature of pairwise comparisons. Put another way, when stochastic dominance tests yields a chain, $\mathcal{H}_s$ reduces the number of comparisons required to generate the ordering. When stochastic dominance tests do not provide any resolution to the ordering, $\mathcal{H}_s$ provides a method of generating an ordering given the priorities of the policymaker. This thus highlights the importance of choosing the order based on the policymaker's priorities prior to using $\mathcal{H}_s$.

3.3 Statistical Properties

Finally, if a statistical comparison of the indices is required, note that the difference between two non-normalized indices may be written as:

$$\begin{aligned} & [\mathcal{T}_s(G_j^s) - \mathcal{T}_s(G_k^s)] \\ = & \int\limits_{x\in\mathcal{X}} \left[G_j^s(x) - \mathcal{L}_s(x)\right] dx - \int\limits_{x\in\mathcal{X}} \left[G_k^s(x) - \mathcal{L}_s(x)\right] dx \\ = & \int\limits_{x\in\mathcal{X}} \left[G_j^s(x) - G_k^s(x)\right] dx \end{aligned} \tag{16}$$

which can be estimated, and appropriate inference performed following Davidson and Duclos (2000), the details of which are outlined in the appendix A.1.

4 Two Illustrative Examples

Two examples are provided here to illustrate the usefulness of the indices in different, but equally pertinent situations. The first, a standard public policy choice problem where the policymaker is confronted with three revenue neutral policies, illustrates how differences between policy alternatives can be quantified under different priorities confronting the planner. The second example augments the equality of opportunity literature (Lefranc et al. 2008, 2009) by providing a comprehensive index of multi-dimensional wellbeing including health concerns. Using a U.S. panel data consisting of youths tracked through to adulthood, the impact of childhood circumstance and behaviors are examined in the context of a triple of wellbeing outcomes when adult.

4.1 Example 1: Comparing Redistribution Policies

Contemplate three alternative mutually exclusive policies, A, B, and C that yield the same per capita return in terms of expected post-tax income to society. The different policies have different redistributional effects, which will be characterized through different revenue neutral tax policies on the initial distribution, with the initial distribution denoted as policy A. Using the results of Moyes and Shorrocks (1994), it is assumed that the effect of policy B is that of a proportionate tax $t_p(x) = t$, where $0 < t < 1$, the aggregate proceeds of which are distributed equally across the population at a level of M per person. The effect of policy C is equivalent to a progressive tax $t_{pr}(x) = t_1 + t_2F(x)$ (where $F(x)$ is the cumulative distribution of $f(x)$, the income size distribution of pre-tax income $x \in \mathcal{X} \subset [0,\infty))$, where $0 < t_1 + t_2F(x) < 1$, so that $0 < t_1 < 1$ and $0 < t_2 < 1 - t_1$, and again the aggregate per capita proceeds M is distributed equally across the population. For policy B, given post-tax income is $(1 - t)x + M$, revenue neutrality implies:

$$\begin{aligned} \int_{\mathcal{X}} (tx - M)dF(x) &= 0 \\ \Rightarrow M &= t\,\mathbf{E}(x) \end{aligned}$$

For policy C, post-tax income will be $(1 - t_1 - t_2 F(x))x + M$, with revenue neutrality implying:

$$\begin{aligned} \int_{\mathcal{X}} \left[(t_1 + t_2 F(x))\, x - M \right] dF(x) &= 0 \\ \Rightarrow t_2 &= \frac{M - t_1\, \mathbf{E}(x)}{\int_{\mathcal{X}} x F(x) dF(x)} \end{aligned}$$

The empirical analogues of the policies applied to a random sample of n pre-tax weekly incomes x_i, $i = 1, \ldots, n$ (where incomes x are ranked highest 1 to lowest n) drawn from the Canadian Labour Force Survey for January 2012 (wage rate multiplied by usual hours of work per week) would yield post-policy incomes y_i of,

- A: $y_i = x_i$
- B: $y_i = (1 - t)x_i + M$
- C: $y_i = \left[1 - t_1 - t_2 \left(1 - \frac{\text{rank}(x_i)}{n}\right)\right] x_i + M$

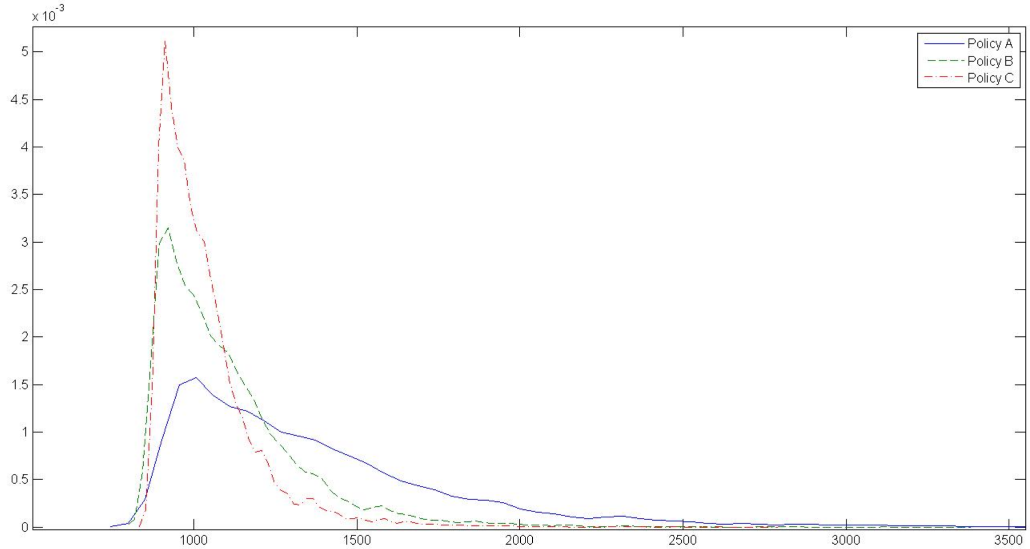
Income distributions that are the result of the three policy alternatives are illustrated in figure 1. The sample size was $52,173$, the parameters were chosen as $t = 0.5$, $t_1 = 0.3$, and as a consequence $t_2 = 0.2976$, and summary statistics for the three policies are presented in Table 2. All three distributions have the same average income with the dispersion ranking $A > B > C$, all are right skewed with policy C being the least skewed.

Table 2: Income Distribution Summary Statistics

	Policy A	Policy B	Policy C
Mean Income	836.89	836.89	836.89
Median Income	750.00	793.44	831.73
Standard Deviations	534.41	267.20	205.10
Maximum Income	5769.60	3303.24	2740.27
Minimum Income	4.80	420.84	421.80

Table 3 reports the dominance relationships between the policies in terms of the maximum and minimum differences between the 1st, 2nd, and 3rd orders of integration of the respective distributions (positive maximums together with negative minimums imply no dominance relationship at that order of integration, see Linton et al. 2005). As is evident,

Figure 1: Density Functions of Policy Outcomes



there are no dominance relationships between the policy outcomes at the 1st order, at the 2nd order A is dominated by both B and C, though there is no dominance relationship between B and C[12], and at the 3rd order comparison outcome C universally dominates, and will be the hull of the three distributions at that level of dominance comparison. Note incidentally that if a *Rawlsian*, infinite order dominance, priority confronted the policymaker, policy C would be the choice since it presents the best outcome for the poorest person. Nonetheless, the primary point here is stochastic dominance's inability to provide a resolution to the policy choice problem should the policymaker's priority be *Utilitarian*, or *Daltonian* in nature.

To highlight the merit of the comparison technique proposed, consider the Non-Standardized Policy Evaluation Indices reported in Table 4. Under a *Utilitarian* priority, policy A would be chosen (although the magnitudes of each respective policy index suggests that there is very little to choose between the policies at this order of dominance comparison). Under a second order inequality averse imperative, policy C would be chosen, and under a third order inequality averse imperative, where poorer agents are of greater concern, policy C would still be chosen (note here the Index is zero because

[12] However the extremely small negative value could be due to rounding error, and is insignificantly different from zero statistically speaking, which would permit a C dominates B conclusion.

Table 3: Between Policy Dominance Comparisons ($A \succ_k B$ implies k^{th} order dominance of A over B)

	$A - B$	$A - C$	$B - C$
1st Order minimum difference	-0.1272	-0.1844	-0.0734
1st Order maximum difference	0.2359	0.2527	0.1019
	No Dominance	No Dominance	No Dominance
2nd Order minimum difference	0.0000	0.0000	$-1.9895e^{-12}$
2nd Order maximum difference	0.1225	0.1531	0.0318
	$B \succ_2 A$	$C \succ_2 A$	No Dominance
3rd Order minimum difference	0.0000	0.0000	0.0000
3rd Order maximum difference	0.1529	0.1738	0.0209
	$B \succ_3 A$	$C \succ_3 A$	$C \succ_3 B$

policy C's distribution, in being uniformly dominant at the third order over the other distributions, will constitute the lower envelope at that order).

Table 4: Policy Evaluation Indices

	Policy A	Policy B	Policy C
$\mathcal{T}_1$	22.2199	22.2359	22.2310
$\mathcal{H}_1$	0.5018	0.5012	0.5012
$\mathcal{T}_2$	25.2379	3.0398	$1.4205e^{-10}$
$\mathcal{H}_2$	0.0000	0.9003	1.0000
$\mathcal{T}_3$	140.9837	17.1928	0.0000
$\mathcal{H}_3$	0.0000	0.8822	1.0000

4.2 Example 2: Adult Quality of Life and Child Circumstance

The second example is drawn from the quality of life, social justice, and human development literatures which examine the extent of dependency of adult outcomes on their childhood circumstances. Information on whether circumstances influence outcomes and, if they do, which circumstance classes promote the best adult outcomes, is of interest to social policymakers (Bound et al. 1999).

Usually such analyses are performed in a uni-dimensional context (see for example Lefranc et al. (2008, 2009)). In providing a complete ordering of circumstance classes for an outcome triple, this example extends that analysis to a multi-dimensional framework highlighting the usefulness of the technique in nonparametric multi-dimensional scenarios where standard dominance techniques suffer from "curse of dimensionality" problems (Yalonetzky 2014). Although similar in nature to that problem, its genesis here is due to the dilution of the contribution of each observation to the estimated distribution as dimensionality increases, so that in the limit, distributions tend to flatten out, and become more similar. This in turn increases the chance that standard stochastic dominance tests yield inconclusive comparisons, and only partially ordering. On the other hand, the use of the hull index, by focusing on a common synthetic baseline "distribution", yields definitive differences, and a complete ordering.

Following United Nations Development Programme, UNDP (2016) practice, measurement of individual human development is based upon their health, education and income status. Panel data from waves I and IV of the Inter-University Consortium for Political and Social Research (ICPSR), National Longitudinal Study of Adolescent Health 1994-2008 are used to relate an adult's health, education and income outcomes to their parental circumstance, and childhood behaviors when they were young. Here, the health gradient (Allison and Foster 2004; Case et al. 2002, 2005; Currie and Stabile 2003; Currie 2009; Cutler et al. 2011) in terms of an adult's self-reported health (SRH) variate is augmented with their educational and income status variates (Anand and Sen 1997; Atkinson 2003; Grusky and Kanbur 2006; Sen 1995; Stiglitz et al. 2011), so that individual developmental wellbeing is presumed to be represented by a regular (i.e. monotonic quasi-concave) multivariate felicity function. Note that the education and health outcome variables are ordinal in nature, while the proposed techniques pertain to cardinal variables, so it is assumed that these variables have cardinal content, in order to demonstrate the viability of the index in a multi-dimensional context. This facilitates multivariate dominance comparison (Atkinson and Bourguignon 1982) of the distributions of such triples over groups of adults where grouping is governed by circumstance and behavior.

Four principle measures of individual circumstance when young were the focus, namely their parental income and educational status, and their exercise and sports engagement frequency. Parental income and educational status circumstance were then divided into quartiles, while the frequency classifications for exercise and sports engagement were used to delineate the observations into their respective circumstance classes. The rationale for the last 2 circumstance classes is to examine if there are possible long run effects from good childhood health outcomes, or positive lifestyle behaviors (Contoyannis and Jones 2004; Balia and Jones 2008). Although superficially, the two variables seem similar, there are significant differences in their responses across the sample, making examination of the dominance of one conditional outcome distribution over the other worthwhile. Based upon the survey questions, the latter refers to organized school or community based sport (within the purview of the policymaker), while the former pertains more to private engagements requiring indirect incentives. The comparisons of outcomes are then focused on the highest and second highest quartiles/frequency classifications of each circumstance, thus providing eight circumstance classes that individuals could be "selected" into[13].

The self-reported health (SRH) variable is a 5 point scale response ranging from poor to excellent[14]. A sample of the survey questions, and the range of their responses are in appendix A.2. Table 5 presents the summary of the key variables used.

Table 6 reports the indices for the three dimensional outcome of joint log income, educational attainment, and SRH when adult. Each index was calculated based on $(100, 20, 20)$ grid points respectively. Nonetheless, the ordering reported here is robust to both coarser and finer grids. Note that for all orders of comparison, the rankings are stable. It is also clear that the highest joint outcomes are from higher parental educational attainment as opposed to income, suggesting the stronger effects that education can have. It is interesting to note that a high level of sport engagement can lead to better outcomes when adult than having upper middle income parents. Taken together, this

[13] The frequency classifications were used for the exercise and sports engagement variables due to the low response spectrum. Nonetheless, the number of observations in each classification is reasonably balanced.

[14] There is no doubt the concern of endogeneity in utilizing non-experimental data here, but the premise of this example is meant to demonstrate the index's simplicity and usefulness even in a multi-dimensional setting.

Table 5: Summary Statistics

	Sports Engagement	Exercise Frequency	Max. Parental Education	log Hh. Income
Mean	1.46	1.60	6.09	3.55
Std. D.	1.29	1.09	4.96	0.98
Max.	3	3	9	6.91
Min.	0	0	1	-2.30
# Obs.	3319	3319	3319	3319
	SRH when Young	SRH when Adult	Income when Adult	Educ. Attnmt. when Adult
Mean	3.91	3.69	9.40	5.75
Std. D.	0.79	0.80	10.22	4.31
Max.	5	5	13.12	11
Min.	1	1	-2.30	1
# Obs.	3319	3319	3319	3319

result suggests the significant direct impact that public policy on education, and sports engagement within the educational system could have in raising overall lifetime wellbeing of the populace both in the short and long run. In other results not presented here, it was found with the same data that the most signicant impact associated with sports and exercise engagement is from the highest level of engagement only, and that moderate levels of engagement had little signicant impact, relative to no engagement in either.

Table 6: $\mathcal{T}_s$ & $\mathcal{H}_s$ Ranking of Welfare (Joint CDF of SRH, Income, & Educational Attainment) in Adulthood

	1st Order Comparison							
Ordering	Top Educ. Quartile	Top Income Quartile	2nd Top Educ. Quartile	Top Sports Engmnt.	2nd Top Income Quartile	Top Exercise Freq.	2nd Top Sports Engmnt.	2nd Top Exercise Freq.
$\mathcal{T}_1$	20.1011	130.6952	240.7610	299.2590	373.2837	435.6056	522.0092	602.6007
$\mathcal{H}_1$	0.9682	0.7932	0.6191	0.5265	0.4094	0.3108	0.1741	0.0466
	2nd Order Comparison							
Ordering	Top Educ. Quartile	Top Income Quartile	2nd Top Educ. Quartile	Top Sports Engmnt.	2nd Top Income Quartile	Top Exercise Freq.	2nd Top Sports Engmnt.	2nd Top Exercise Freq.
$\mathcal{T}_2$	62.8447	2392.0994	4244.1474	5782.7589	8608.3555	9690.3365	11258.2134	14278.0629
$\mathcal{H}_2$	0.9956	0.8336	0.7048	0.5977	0.4012	0.3259	0.2168	0.0068
	3rd Order Comparison							
Ordering	Top Education Quartile	Top Income Quartile	2nd Top Educ. Quartile	Top Sports Engmnt.	2nd Top Income Quartile	Top Exercise Freq.	2nd Top Sports Engmnt.	2nd Top Exercise Freq.
$\mathcal{T}_3$	949.8380	103622.9557	153348.0303	252452.4899	394756.0493	429859.2653	497806.5872	619170.1736
$\mathcal{H}_3$	0.9985	0.8338	0.7540	0.5951	0.3668	0.3105	0.2016	0.0069

5 Conclusions

A practitioner's difficulties in applying stochastic dominance techniques are twofold. Generally the technique only offers a partial ordering, and furthermore it never yields the policymaker a number by which she can assess "by how much" one policy is better than another. This presents a severe problem when alternative policies are not combinable so that the first best solution is not available. Here indices are proposed which are founded on stochastic dominance principles, and which provide the policymaker with an index of how much better one policy is than another in the context of the particular priorities she confronts. The index provides a complete ordering at any level of comparison deemed appropriate by the policymaker, and it is shown to possess an Independence of Irrelevant Alternatives property. Two examples on redistributional policy choice, and multi-dimensional wellbeing gradient comparisons illustrate the use of the statistic where the lack of a completeness property presents a problem.

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A Appendix

A.1 Estimating difference in $\mathcal{T}$ of Equation 16

Following Davidson and Duclos (2000), the s^{th} order stochastic dominance criteria are based upon (3) which may be estimated from a random sample of y's by:

$$\widehat{F}_s(x) \quad = \quad \frac{1}{N(s-1)!}\sum_{j=1}^{N}(x-y_j)^{s-1}\mathbb{I}(y_j < x)$$

Here $\mathbb{I}(z)$ is the indicator function which equals one if z is true, and zero otherwise. For a sequence of values $x_1, x_2, \ldots, x_K$, the estimates of the vector $[F_s(x_1), F_s(x_2), \ldots, F_s(x_K)]'$ can be shown to be asymptotically normally distributed i.e.

$$\begin{pmatrix} \widehat{F}_s(x_1) \\ \widehat{F}_s(x_2) \\ \vdots \\ \widehat{F}_s(x_K) \end{pmatrix} \sim N\left(\begin{pmatrix} F_s(x_1) \\ F_s(x_2) \\ \vdots \\ F_s(x_K) \end{pmatrix}, \begin{pmatrix} C_s(x_1,x_1) & C_s(x_1,x_2) & \ldots & C_s(x_1,x_K) \\ C_s(x_2,x_1) & C_s(x_2,x_2) & \ldots & C_s(x_1,x_K) \\ \vdots & \vdots & \ddots & \vdots \\ C_s(x_K,x_1) & C_s(x_K,x_2) & \ldots & C_s(x_K,x_K) \end{pmatrix}\right)$$

where the covariance terms:

$$C_s(x_j, x_k) \quad = \quad \mathbf{E}\left(\left(\widehat{F}_s(x_j) - F_s(x_j)\right)\left(\widehat{F}_s(x_k) - F_s(x_k)\right)\right)$$

for $j, k = \{1, 2, \ldots, K\}$, may be estimated as:

$$\widehat{C}_s(x_j, x_k) \quad = \quad \frac{1}{N[(s-1)!]^2}\sum_{n=1}^{N}\left[\begin{array}{cc} & (x_j - y_n)^{s-1}\mathbb{I}(y_n \leq x_j) \\ \times & (x_k - y_n)^{s-1}\mathbb{I}(y_n \leq x_k) \end{array}\right] - N^{-1}\widehat{F}_s(x_j)\widehat{F}_s(x_k)$$

When distributions $f(.)$ and $g(.)$ are independently sampled, interest centers on the vector of differences $\mathbf{F} - \mathbf{G} = [F_s(x_1) - G_s(x_1), F_s(x_2) - G_s(x_2), \ldots, F_s(x_k) - G_s(x_k)]'$, which under the null of equality is jointly distributed as $N(0, C_{s,f} + C_{s,g})$, where $C_{s,f}$ and $C_{s,g}$ are respectively the covariance matrices under f and g[15].

To examine (16), letting x be greater than the maximal value in the pooled sample, the last component of the vector $F_{s+1}(x) - G_{s+1}(x)$, and its corresponding variance estimate would be used for inference purposes.

[15] If f and g were sampled in a panel, then the between distribution covariances also need to be included in the calculus.

A.2 Variables Used in Example 2

1. During the past week, how many times did you play an active sport, such as baseball, softball, basketball, soccer, swimming, or football?

    (a) 0 $\longrightarrow$ Not at all.

    (b) 1 $\longrightarrow$ 1 or 2 times.

    (c) 2 $\longrightarrow$ 3 or 4 times.

    (d) 3 $\longrightarrow$ 5 or more times.

2. During the past week, how many times did you do exercise, such as jogging, walking, karate, jumping rope, gymnastics or dancing?

    (a) 0 $\longrightarrow$ Not at all.

    (b) 1 $\longrightarrow$ 1 or 2 times.

    (c) 2 $\longrightarrow$ 3 or 4 times.

    (d) 3 $\longrightarrow$ 5 or more times.

3. In general, how is your health? Would you say

    (a) 1 $\longrightarrow$ Excellent

    (b) 2 $\longrightarrow$ Very good

    (c) 3 $\longrightarrow$ Good

    (d) 4 $\longrightarrow$ Fair

    (e) 5 $\longrightarrow$ Poor

4. You are physically fit.

    (a) 1 $\longrightarrow$ Strongly agree

    (b) 2 $\longrightarrow$ Agree

    (c) 3 $\longrightarrow$ Neither agree nor disagree

    (d) 4 $\longrightarrow$ Disagree

    (e) 5 $\longrightarrow$ Strongly disagree