Quantifying The Progress of Economic and Social Justice: Charting Changes in Equality of Opportunity in the United States 1960-2000

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Abstract

The notion of Equality of Opportunity (EO) has pervaded much of economic and social justice policy, and research over the last half century. The sense that differences in agent outcomes that are the consequence of their individual choice and effort are acceptable, whereas variation in agent outcomes that are the consequence of circumstances beyond their control are not, has underpinned much gender, race, education, and family law and policy over that period, making it a many dimensioned issue. In this context the empirical analysis of EO has been hampered in the sense that the usual techniques are one-dimensional in nature. Here a new approach to evaluating levels of, and changes in EO which readily accommodates these many dimensions is introduced, and progress in the extent of equality of opportunity for 18 year old's in the United States is examined over the period 1960-2000. The evidence is that gains were made in all categories throughout the period, more so for males than females (though females were better off in an EO sense to start with), more so for children in single parent circumstances, and more so for the poorly endowed.

Key Words: Overlap Measure; Stochastic Dominance; Equality of Opportunity

1 Introduction

Atkinson (2012) in discussing public policy reform in the realm of political economy, argued that the aim is "to remedy injustice rather than characterize perfect justice". In this he was following Sen (2009), who in the introduction to *The Idea of Justice* avows it to be "... an attempt to investigate realization-based comparisons that focus on the advancement or retreat of justice.". The objective for both is to seek progressive reform rather than transcendental optimality. Accordingly, techniques for evaluating such progress should be capable of measuring the degree and significance of such advances or retreats in economic and social outcomes. Here a new method is proposed and implemented, and the effects of the many Economic and Social Justice policy reforms in the United States over the last half century are evaluated in terms of the extent to which various notions of economic and social justice have advanced the imperative.

In recent years there has been considerable interest amongst economists and philosophers in quantifying various notions of Economic and Social Justice in its many dimensions (Arrow et al. 2000; Brighouse and Robeyns 2010; Dworkin 2011; Rawls 2001; Sen 2009). One aspect of social and economic justice that has perhaps resonated most with law and policy makers alike is the notion of Equality of Opportunity (EO), the sense that inequalities that are the result of differences in individual choice and effort are acceptable, whereas inequalities that derive from differences in individual circumstances are not.[1] Roemer (2006) in referring to equalizing opportunities as a "field levelling" exercise suggests that education is an institution central to such an exercise. Thus EO has underpinned much gender, race, and education law and policy in the past 50 years, all of which can be interpreted as attempts to release one generation from the constraints of inherited circumstances.[2] This notion of justice is essentially a statement about the nature of the desired joint density function of a collection of outcome and circumstance variables, and a measure of the degree to which justice exists would be the proximity of the existing joint density to the desired joint density that characterizes a particular notion of EO.

Hitherto most techniques for assessing such concepts of social and economic justice have been of the generational regression variety or of the generational transition (Markov

[1]See Arneson (1989), Cohen (1989), Dworkin (1981a), Dworkin (1981b), Dworkin (2000), Roemer (1998), Roemer et al. (2003) and Roemer (2006) for the philosophical foundations.

[2]As such it can also be construed as part of an agenda associated with advocates of a functionings and capabilities approach to societal wellbeing, and Human Development (Sen 2009; Nussbaum 2011).

chain) matrix variety, each of which in its own way maps parental outcomes into child outcomes, and seeks to evaluate the degree of dependence/independence in the mapping. Recently an approach equivalent to comparing the distributions of the outcomes of agents from different circumstance classes, for the absence of stochastic dominance relationships between the different circumstance group distributions has also been suggested. Each technique has its problems for this purpose, especially when policy is pursued asymmetrically over a variety of child and parent outcomes.

An illuminating discussion in the survey by Ramos and Van de Gaer (2012) points out that in the theoretical literature, agents are identified by an outcome which is the object of the policy, an R dimensioned vector of *efforts*, and a C dimensioned vector of *circumstances*. The joint distribution of agents across both *effort* and *circumstance* states, an $R \times C$ matrix P, is what the policy maker confronts. Many EO measures have been founded on fundamental theoretical principles of compensation and reward. The two compensation principles, *ex-post* compensation (EPC) – equalizing outcomes of individuals with the same effort, and *ex-ante* compensation (EAC) equalizing unambiguous inequalities in circumstance, were shown by Fleurbaey and Peragine (2013) to be mutually incompatible. This suggests that if one wants to evaluate outcomes from the perspective of EO, a choice has to be made between *ex-ante* and *ex-post* compensation. The three major reward principles: *Liberal Reward* – "Tax by Type", individuals from the same type regardless of effort should incur the same tax burden; *Utilitarian Reward* – respecting the income differences due to differences in effort requires zero inequality aversion (proportionate tax) with respect to such differences, and *Inequality Averse Reward* (Roemer 2010) – which takes an increasing concave transformation of incomes as the relevant outcome variable, have all been found to be incompatible with the *ex post* compensation principle. Hence it would seem that if the policy maker's concern is with compensation, the focus should be on EAC. However Ooghe et al. (2007) point out that the *ex-post* inequality of opportunity concern with inequalities across effort (columns of P), and the *ex-ante* inequality of opportunity concern with inequalities across circumstance (columns of P) has an important implication: when effort is distributed independently of circumstance, full equality of *ex-post* opportunities implies full equality of *ex-ante* opportunities. Thus proximity to the state of independence between effort and circumstance states would be a measure of the success of either compensation policies. Further, if inequality aversion underlays the policy maker's objective, then progress toward EO for the poorly endowed should be weighted more heavily than progress toward EO for the circumstantially well-endowed.

A pure equal opportunity policy mandate which values releasing a child from its circumstance regardless of it being good or bad, coheres with the aforementioned *Liberal* and *Utilitarian* reward principles, wherein the outcomes of each agent have equal weight in the sense that the distance from the desired outcome of each agent carries the same weight. However the pursuit of such a pure equal opportunity goal has not been unequivocal; Cavanagh (2002) and Roemer (2010) expressed some philosophical reservations, Jencks and Tach (2006) have questioned whether an equal opportunity imperative should require the elimination of all sources of economic resemblance between parents and children[3], and in a similar vein Dardanoni et al. (2006) question how demanding the pursuit of equal opportunity should be in terms of the feasibility of such a pursuit. Furthermore, most observed law and policy practice has been more qualified in its approach to relieving poorly endowed children of dependence upon their circumstance, whilst leaving the dependence of richly endowed children upon their circumstance more or less intact[4], with an emphasis on facilitating the upward mobility of the poorly endowed. This coheres with the aforementioned *Inequality Averse* reward system wherein the distance from the desired outcome for each agent is weighted more heavily for poorly endowed agents. Methods for evaluating the degree of progress should be able to reflect this asymmetric, conditional and progressive nature of the policy objective. Here as examples, two constructions of the Equality of Opportunity imperative are considered, one (EO) reflecting an unadulterated requirement of independence between outcomes and circumstances, and another more qualified view, hence Qualified Equality of Opportunity (QEO), which reflects an emphasis on upward mobility.

In the following, new techniques are developed and employed to examine the progress toward equality of opportunity over the last four decades of the 20^{th} century in the United States in the context of the educational attainment of 18 year olds and an array of cir-

[3]Indeed in terms of the nature versus nurture debate, it is doubtful that resemblances due to nature can be totally eliminated or compensated for.

[4]Equal opportunity programs observed in "Liberal" societies do seem to be of this flavour. For example, when questioned on the widening gap between the rich and poor, the British Prime Minister responded that "... the issue is not in fact whether the very richest person ends up being richer. The issue is the poorest person is given the chance they don't otherwise have. The most important thing is to level up, not level down." Interview with the Prime Minister on BBC News Newsnight on June 4, 2001. Transcript available from `http://news.bbc.co.uk/2/hi/events/newsnight/1372220.stm`.

cumstances they faced (Note that in a general sense, equality of opportunity in the work place, and the impact of "the glass ceiling" for example, are not being considered here). The novelty is a new multivariate measure of equality of opportunity which calibrates the proximity to statistical independence between an agent's outcome and circumstance sets. Existing approaches to measuring equality of opportunity are discussed, and the new measure is introduced in section 2. Section 3 provides some background to the equal opportunity policies that were enacted in the preceding decades in the U.S., and section 4 reports the empirical results of the examination of these policies. Some conclusions are drawn in section 5.

2 Measuring Equality of Opportunity

2.1 Classical Techniques

Generally EO has been studied and evaluated in the context of an agent's outcomes being measurably independent of its circumstances (usually measured as the agent's corresponding parental outcomes, which has led to a generational mobility interpretation), and the two dominant approaches are Generational Regressions and Transition Matrices approach. Generational Regressions, where child outcomes are regressed on indicators of their circumstances (usually parental characteristics), evaluate EO by the proximity of the circumstance coefficients to zero (note this approach weights equally trends away from dependence of the richly and poorly endowed upon their circumstances). Transition Matrices between "parent outcomes" and corresponding "child when adult" outcomes, evaluate EO by the proximity of the matrix structure to that which would be engendered by independence between outcomes and circumstances (again this approach weights equally trends away from dependence of the richly and poorly endowed upon their circumstances). Here it is contended that both approaches present problems for evaluating the progress of equal opportunity beyond not reflecting the asymmetric nature of policy imperatives, and an alternative technique is proposed which does not suffer these deficiencies.

The main difficulty with Generational Regressions hinges on the notion that zero covariance, the basis for inferences about mobility in regressions, does not imply independence (an interpretational difficulty that is compounded when outcomes and circumstances are represented by a collection of variables, i.e. when the regression model becomes a system of multivariate regressions). This renders changes in β in the equation

$y = \alpha + \mathbf{x}\beta + e$ (where $y \in Y$ is the child outcome, and $\mathbf{x}$ the k vector of parental circumstance, including for instance parental quality, effort in child care, etc., with typical element $x_i \in X_i$, $i = \{1, \ldots, k\}$, and e represents all other influences) a poor foundation for thinking about changes in generational disadvantage, or equal opportunity[5]. Viewed as a technological relationship, it is a strange equation, since the incremental change in the child outcome per incremental change in a particular parental quality is the same regardless of the level of that parental quality. Furthermore the variability of all other influences (other than parental) is the same at all levels of parental ability, in other words the pure randomness of child outcomes is unaffected by parental abilities or effort. So we have a constant returns technology with parental indifference to (or inability to influence) other influences at all levels of the ability spectrum. In the context of this simple generational regression framework, it can be readily shown (Anderson et al. 2014) that a Qualified Equal Opportunity (QEO) policy seeks to "convexify" with respect to $\mathbf{x}$, the deterministic component of the relationship, and engender greater negative (relative to $\mathbf{x}$) heteroskedasticity in the random component e, reflecting less variation in the parent-child connection for high outcome parents. These problems are somewhat ameliorated by using non-parametric quantile regression techniques (Li and Racine 2008), however seldom is attention given to the potential heteroskedasticity in the errors which a QEO regime would demand.

The implications of intergenerational mobility for the transition matrix approach have been examined in Van de Gaer et al. (2001). They axiomatically developed measures which could distinguish between (a.) mobility as agents move within a society, (b.) calibrated equal opportunity and (c.) indicated life chances, all three of which they showed to be incompatible concepts. Perhaps the most serious concern in the context of Markov methods is that transition matrices, and the related summary statistics are very hard to formulate when y and x are each measured in many discrete and continuous dimensions, and not in the same metric. Furthermore, this approach does not yield a measure by which advances and retreats can be measured and examined for their significance. Aside from

[5] Much time is spent in introductory statistics courses stressing that, while independence implies zero covariance, zero covariance (the basis of inference on β) does not imply independence! Think about an exact non-independent relationship $y = 0.5 + 2(x - x^2)$ for $0 < x < 1$ (a fairly plausible technology), with parent quality uniformly distributed in $[0, 1]$. A random sample of agents from this would yield zero covariance between y and x, and hence a zero estimate of β implying independence for what is a completely dependent relationship.

these conceptual difficulties, a practical difficulty with the transition matrix approach is that it relies on partitioning the outcome and circumstance space into common segments in order to define the transition matrix. One minor problem is that proximity to independence in this context is difficult to evaluate, and usually some function of the matrix components (frequently in the form of the trace or the determinant of the transition matrix) is considered. Another minor problem is the potential test inconsistency issue, which arises when the difference between two continuous distributions is compared at discrete points of support (which is effectively what is done in the transition matrix approach). Test inconsistency is about the chance of correctly rejecting the null hypothesis not going to 1 as the sample size grows without bound (see Cox and Hinkley (1979)). In the present context, the independence of x and y (defined by their joint density $h(x,y) = f(x)g(y)$, where $f(.)$ and $g(.)$ are the respective marginal densities, for all possible values of x and y) is evaluated by considering the closeness to zero of objects of the form:

$$\int_a^b \int_c^d \left(\frac{h(x,y)}{g(y)} - f(x) \right) dydx \tag{1}$$

where (a,b) and (c,d) are chosen partition points. Clearly the possibility exists for this object to be equal to 0 while $h(x,y) \neq f(x)g(y)$ for some values of $x \in (a,b) \subset X$ and $y \in (c,d) \subset Y$ (hence the test inconsistency issue). Practically the implication here is that an injudicious choice of partition structure could result in inferring a magnitude of independence which in truth did not prevail, thereby rendering the test inconsistent or at least substantially weakening its power. A more comprehensive way of evaluating the extent and type of dependency between two quite distinct collections of variables is required.

2.2 Alternative Techniques of Quantifying EO

The statistics literature abounds with types of dependence. Lehmann (1966) outlines three types of dependence, all of which deal with monotone relations between X and Y (see also Bartolucci et al. (2001)). The most recent approach to measurement of social and economic justice (Lefranc et al. 2008, 2009) focuses on monotone dependencies by examining the stochastic dominance relationships between the outcome variables associated with different dependency classes. While this approach will undoubtedly capture the effects of QEO policies in the simple two variable case, it becomes much more difficult to employ in multivariate situations, and does not yield a statistic that readily facilitates measurement of progress.[6] A more omnibus notion of dependence is required, since relationships between parent and child characteristics need not be monotone. Thus here a more general concept of "distance from independence" is employed which admits both monotone and non-monotone relationships (as were conjectured in footnote 5), and which will always provide consistent tests.

Letting $\mathbf{x}$ be an k-dimensional vector, and $f_a(\mathbf{x})$ and $f_b(\mathbf{x})$ be two continuous multivariate distributions. The extent to which $f_a(\mathbf{x})$ and $f_b(\mathbf{x})$ overlap can be measured as:

$$\mathbf{OV} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \min\{f_a(\mathbf{x}), f_b(\mathbf{x})\}\, d\mathbf{x} \tag{2}$$

If $f_a(\mathbf{x})$ is the unrestricted joint p.d.f. of $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^k$ and $f_b(\mathbf{x})$ is the joint distribution when the $\mathbf{x}$'s are independent, then $0 \leq \mathbf{OV} \leq 1$ is an index of independence, and $1 - \mathbf{OV}$ is a general index of dependence, be it monotone or not. Although, it is possible that other suitably normalized variations, such as $\int |f_a(\mathbf{x}) - f_b(\mathbf{x})|\, d\mathbf{x}$ may likewise seem appropriate, the fact remains that the measure as defined in (2) is a very convenient index, bounded between 0 and 1, and most importantly has well defined statistical properties that facilitate inference. Indeed, Anderson et al. (2012) has shown that the kernel estimator of $\mathbf{OV} := \theta = \int \min\{f_a(\mathbf{x}), f_b(\mathbf{x})\} d\mathbf{x}$ is distributed as follows:

$$\sqrt{n}(\widehat{\theta} - \theta) - \alpha_n \longrightarrow N(0, v) \tag{3}$$

where

$$\begin{aligned}
v &= p_0\sigma_0^2 + p_a(1-p_a) + p_b(1-p_b) \\
p_0 &= \Pr(\mathcal{X} \in C_{f_a,f_b}); \qquad C_{f_a,f_b} = \{\mathbf{x} \in \mathbb{R}^n : f_a(\mathbf{x}) = f_b(\mathbf{x}) > 0\} \\
p_a &= \Pr(\mathcal{X} \in C_{f_a}); \qquad C_{f_a} = \{\mathbf{x} \in \mathbb{R}^n : f_a(\mathbf{x}) < f_b(\mathbf{x})\} \\
p_b &= \Pr(\mathcal{X} \in C_{f_b}); \qquad C_{f_b} = \{\mathbf{x} \in \mathbb{R}^n : f_a(\mathbf{x}) > f_b(\mathbf{x})\}
\end{aligned}$$

and where α_n and σ_0^2 are bias correction factors. Since this estimator does not depend upon arbitrarily chosen points in the support, it will provide for consistent testing of

[6] While parent-child education relationships are clearly monotonic it is not clear that other relationships, e.g. parental income-child education are not necessarily monotonic given notions of diminishing marginal returns to scale in investment in education.

hypotheses. One slight wrinkle, is that $\mathbf{x}$ is often a mixture of discrete and continuous variables. Denoting them by $\mathbf{x}_d \in \mathcal{X}_d$ and $\mathbf{x}_c \in \mathcal{X}_c$ respectively, so that $\mathbf{x}' = [\mathbf{x}_d', \mathbf{x}_c']$, the appropriate overlap measure is:

$$\mathbf{OV}_{mix} = \int_{x_c \in \mathcal{X}_c} \sum_{x_d \in \mathcal{X}_d} \min\{f_a(x), f_b(x)\} dx_c \tag{4}$$

Here, summation is over the discrete components, and integration is over the continuous components. The discrete version of $\mathbf{OV}$ has been developed in Anderson et al. (2010) so the properties of $\mathbf{OV}_{mix}$ can be derived as a mixture of the two cases as in Anderson and Hachem (2012). Moreover, $\mathbf{OV}_{mix}$ lends itself quite naturally to a measure of the degree of independence[7], as well as other notions of social justice by letting $f_a(\mathbf{x})$ be the empirical distribution, and $f_b(\mathbf{x})$ the desired distribution under a given particular definition of social justice. To see how, with some abuse of notation, let $\mathbf{y}$ be a vector of agent outcomes with joint distribution $g(\mathbf{y})$, and $\mathbf{x}$ be a vector of their circumstances with joint distribution $f(\mathbf{x})$, and the joint distribution of outcomes and circumstances is denoted by $h(\mathbf{y}, \mathbf{x})$. Under independence, $h(\mathbf{y}, \mathbf{x}) = f(\mathbf{x})g(\mathbf{y})$, and the following measure of their independence will be:

$$\mathbf{OV} = \int \sum \min\{h(\mathbf{x}, \mathbf{y}), (f(\mathbf{x})g(\mathbf{y}))\} d\mathbf{z} \qquad \in (0,1) \tag{5}$$

or alternatively in its conditional form,

$$\mathbf{OV} = \int \sum \min\left\{\frac{h(\mathbf{x}, \mathbf{y})}{f(\mathbf{x})}, g(\mathbf{y})\right\} d\mathbf{z} \qquad \in (0,1) \tag{6}$$

where integration is over $\mathbf{z}$, the continuous components of $\mathbf{y}$ and $\mathbf{x}$, and summation is over the discrete components of $\mathbf{y}$ and $\mathbf{x}$. A greater degree of dependence between $\mathbf{y}$ and $\mathbf{x}$ implies less overlap between $h(\mathbf{x}, \mathbf{y})$ and $f(\mathbf{x})g(\mathbf{y})$, leading to lower values of $\mathbf{OV}$. Furthermore, the statistic can be calculated conditionally on particular aspects of circumstances to check for example whether equality of opportunity improvements are symmetric with respect to poorly or richly endowed children, or on the marital status of

[7]The Overlap Measure proposed in this paper can be adapted to the three conceptions of intergenerational mobility suggested by Van de Gaer et al. (2001), since each transition matrix implies a particular structure for the joint density matrix, which the empirical joint density can be measured against. Further, the third mobility measure for Markov chains proposed by Van de Gaer et al. (2001) is related to the Overlap Measure in the sense that it measures the complement to the overlapping region of the conditional probabilities.

the parents to check whether equality of opportunity policies have affected those groups differentially.[8]

The simple "pure" version of EO, which accords with the basic Liberal and Utilitarian reward principles mentioned earlier, as a justice imperative has met with its critics[9], who refer to it as "Luck Egalitarianism". Basically their concern is that because good outcomes are strongly correlated with good circumstances, if there is insufficient capacity in the system (Anderson et al. 2014) to upgrade the poorly endowed to the status of the richly endowed, high type inheritors have to be disinherited to achieve the just equal opportunity outcome. So for example, high achieving children, who are so because they have genetically inherited benefits from their high achieving parents, have to be penalized or disinherited, in essence destroying inherited social capital. A resulting compromise policy is to follow a "Qualified" Equality of Opportunity imperative which seeks equality of opportunity for the poorly endowed whilst preserving the outcomes of those who have been more fortunate in their inheritance. This accords more closely with the inequality associated with an Inequality Averse reward principle referred to earlier, and may be interpreted as following a second, Pareto-like imperative, wherein no child should be made worse off by an EO policy, so that the focus is on elevationg the outcomes of the poorly endowed. A simple way of characterizing the just outcome in this case can be achieved by modifying the previous measures of equation (5) and (6) by considering the target joint density of the "just" society as:

$$h^{**}(\mathbf{x},\mathbf{y}) \quad = \quad (1-G(\mathbf{y}^*))f(\mathbf{x})g(\mathbf{y}) + G(\mathbf{y}^*)h(\mathbf{x},\mathbf{y}) \tag{7}$$

Here $G(.)$ is the cumulative density of $\mathbf{y}^*$ which is a monotonic non-decreasing function of a subset or the entire set of parental qualities (perhaps a mixture of income and educational status), so that high type circumstance outcomes tend to preserve the status quo, whilst low type outcomes engender independence between outcome and circumstance. The same measures of distance (that is the overlap) between $h(\mathbf{x},\mathbf{y})$ and $h^{**}(\mathbf{x},\mathbf{y})$ provide indicators of the extent of this sense of justice in this society.

[8]Appendix A.1 provides a brief description of how the measure is estimated. These indices are confined to the unit interval with proximity to one representing the ultimate in social justice however defined.

[9]Anderson (1999), Cavanagh (2002), Hurley (1993), Piketty (2000), and Swift (2005) are some opponents.

3 Background of Policies with Effect on EO in the U.S.

Legislation promoting EO in the United States has fallen into two broad categories, policies promoting outcomes of the poorly endowed, and more general anti-discrimination policies. The former group is comprised of family law policies (usually a state level issue) which promote the outcomes of children in disadvantaged home circumstances and schooling legislation, which has largely been a federal matter manifesting in policies such "No Child Left Behind". The latter group have generally appeared under a civil rights banner.

Divorce law changes, associated with facilitating Unilateral Divorce (either party in the marriage having the ability to leave the marriage without consent from the other party), and No-Fault Divorce (the party leaving not needing to prove he/she is leaving because the other party has transgressed during the marriage) occurred first in California in 1969, and by 1980s, almost all states had them.[10] They can be interpreted as having reduced the incidence of children living in the context of disadvantageous circumstances. Child custody law changes which also began in California in 1980, (but at least 3 states had acknowledge the possibility of joint custody in the 1970s) can be construed to have a similar intent in securing more and better resources for investment in child development. By 2000, only 7 states had not implemented such laws, namely Nebraska, New York, North Dakota, Rhode Island, Vermont, West Virginia, and Wyoming. The impact of the law's adoption is exemplified in California, where joint custody decisions rose from 2.2% of all final decrees in 1979, to 13% in 1981 (Maccoby and Mnookin 1994). Further, its breadth of influence was evident among states which permitted divorced parents to reevaluate custodial arrangements made prior to the regime shift, obtaining fresh judgments based on current application of the new law (Mason 1999). Basically, the change in statute allows for both parents to share in the custody whereas previously, the law acknowledged maternal preference in their rulings.[11]

[10]Studies into the effect of divorce law changes include the following, (a.) Divorce rates (Peters 1986; Allen 1992; Peters 1992; Friedberg 1998; Wolfers 2006), (b.) Marriage rates (Rasul 2006), (c.) Child outcomes (Gruber 2004; Johnson and Mazingo 2000), (d.) Marriage specific investments (Stevenson 2007), and (e.) Domestic violence rates (Stevenson and Wolfers 2006).

[11]Studies of the effect of custodial law changes include, (a.) Implications for a non-custodial parent's willingness to make child custody payment (Weiss and Willis 1985; Del Boca and Ribero 1998), and (b.)

With respect to schooling policies, Title 1 was the centerpiece of the Elementary and Secondary Education Act signed into Law by Lyndon Johnson in 1965 to provide financial assistance to local education agencies in districts with high incidences of poverty. Its implementation and impact have been critically discussed in Kosters and Mast (2003), and Cohen and Moffitt (2009). Over one billion dollars were assigned over and above the regular school budget (i.e. this was not a reallocation of the school budget) in the first year. In the ensuing 35 years almost $200 billion was allocated. The asymmetric nature of these family and schooling laws may well reflect the presence of a second policy imperative which modifies or qualifies a pure equal opportunity policy so that poor in circumstance children are relieved of their circumstance connection, whereas the connection between richly endowed children and their circumstance is maintained[12].

Title IV of the Civil Rights Act of 1964 desegregated public education in the U.S., and was largely a response to the Supreme Court's ruling on Brown versus the Board of Education of Topeka 1954 (Brown). Brown was the best known of a sequence of cases initiated by the National Association for the Advancement of Colored People (NAACP) Legal Defense and Educational Fund to break down racial segregation in the field of education and beyond. In the ruling, Justice Warren declared that "in the field of public education, the doctrine of 'separate but equal' has no place. Separate educational facilities are inherently unequal.". Previously, Court decisions held that educational segregation was acceptable as long as conditions and curriculum in the separate schools were equal[13]. The 1964 Act enforced the assignment of students to schools without regard to their race, colour, religion or national origin, and was explicit in averring that desegregation did not mean assigning students to schools to overcome racial imbalance. It offered technical assistance, training assistance, and grants to school boards to facilitate desegregation.

Title IX of the Education Amendment Act of 1972 addressed discrimination with respect to gender in education. Modeled on Title IV of the 1964 act, the preamble to Title

Implications for divorce and marriage rates, and consequent impact on child investments (Rasul 2006; Brinig and Buckley 1998; Halla 2008; Halla and Holzl 2007; Leo 2008; Nunley and Seals 2009).

[12]Anderson et al. (2014) showed that in a constrained world with no growth in average child outcomes, movement toward an equal opportunity outcome for one group of children must necessarily make another group of children worse off.

[13]Interestingly some school boards met the equality mandate by penalizing white schools. King George County, Virginia, for example, chose to equalize its curriculum by dropping several advanced courses from its white high school rather than add them at the black school, an example of a symmetric equal opportunity policy which we argue is not generally observed in the 1960-2000 period.

IX declared that: "No person in the United States shall, on the basis of sex, be excluded from participation in, be denied the benefits of, or be subject to discrimination under any educational programs or activity receiving federal financial assistance ...". With respect to employment opportunities, Section 703 (a) of Title VII of the 1964 Civil Rights Act, made it unlawful for an employer to "fail or refuse to hire or to discharge any individual, or otherwise to discriminate against any individual with respect to his compensation, terms, conditions or privileges or employment, because of such individual's race, color, religion, sex, or national origin.". The final bill also allowed gender to be a consideration when it is a *bona fide* occupational qualification for the job. Title VII of the act also created the Equal Employment Opportunity Commission (EEOC) to implement the law.

Generally all of these legislative changes took place in the early part of our data period, in the 1960s and 1970s. However the policies often took some time to implement, and clearly very often they took some time to have an effect. Hutchinson (2011) notes that "... it was only in the 1980s that ... school district were obligated by federal courts to implement mandatory busing plans ... that high schools long formally desegregated still had different bells for black and white students ... separate basketball teams ... at Mississippi's Charleston High School ... only in 2009 ... the first integrated prom dance occurred.". Here the long term effects of these policy changes will be quantified in terms of the extent to which a young person's academic achievement was influenced by the circumstance they confronted.

Since so much of an individual's circumstance can be associated with their parents, much "Equal Opportunity" research and policy, under the banner of Generational Mobility, has had a one dimensional focus on the degree to which an individual's outcome can be considered independent of the corresponding parental outcome. However the "level playing field" motif suggests that, given similar effort and choices, all should have the same chance of success (or failure) regardless of color, gender and socioeconomic background, namely a multitude of circumstances that are not purely parental. Thus if progress in EO is to be evaluated, a technique is required which will relate an outcome measure (or a collection of outcome measures) to a variety of circumstance measures simultaneously, so that the distance of the existing joint density from that of one reflecting independence from circumstances can be evaluated.

4 Results

4.1 Descriptive Statistics

Census data from 1960-2000 drawn from the Integrated Public Use Microdata Series (IPUMS) was employed, utilizing the educational status of the child as the effort variable, and the parental marital status, educational attainment and income variables, and the child's gender and race as circumstance variables. To accommodate the gradual nature of policy adoption, measures of EO for 1960 (prior to enactment of the policies), and 2000 (post adoption of policies) will be compared, and where they are of interest, some intermediate stage results will be reported. Using the parental marital status[14], the observations were separated into three family structures; intact, divorced or separated, and widowed parent families. The grade attainment indicator is: 1 if preschool or had no education, 2 if grade 1-4, 3 if grade 5-8, 4 if grade 9, 5 if grade 10, 6 if grade 11, 7 if grade 12, 8 if 1-3 years of college and 9 if more than 4 years of college. For intact families the maximum of the parental educational attainments was employed, while family income is in constant dollars and is family size deflated according to the square root rule (Brady and Barber 1948). The analysis below focuses on children of age 18 because for most states, compulsory education ceased to be binding then, consequently it should be stressed that the notions of EO reported here, apply to the achievements of 18 year old's, and not the career achievements of adults in general[15].

As background, consider the summary statistics of the data for the years 1960 and 2000 for 18 year old children by gender presented in table 1, panels A to C, while panels D and E presents similar information by race. For intact families the boy−girl differences in 1960 were significant with girls significantly outperforming boys (a "t" statistic of -15.8201, $\Pr(T < t) = 0$). The gap was still significant in 2000 ($t = -6.4057$, $\Pr(T < t) = 0$), albeit had been substantially reduced (the "t" for the difference-in-difference is -8.6458, $\Pr(T < t) = 0$). None of the boy−girl parental differences are particularly significant as is to be expected.

[14]The respective coded parental marital status responses are as follows: Married, spouse present is 1; Married, spouse absent is 2; Separated is 3; Divorced is 4; Widowed is 5; Never married/single is 6. This paper does not examine children born outside of wedlock, nor marriages where one parent is "missing"(responses 2 and 6).

[15]Nonetheless, the results for children of age 16 and 17 are similar, and are available from the authors upon request.

Table 1: Summary Statistics by Gender & Family Structure

Year		Boys			Girls		
		Panel A					
		Intact Parents			Intact Parents		
		Child's Education	Parent's Education	Parent's Income	Child's Education	Parent's Education	Parent's Income
1960	Mean	5.7447	5.5721	12.7190	6.1364	5.6210	13.0830
	s.d.	(1.3724)	(2.0841)	(9.3442)	(1.1734)	(2.0530)	(9.4965)
	N	5819	5819	5819	4766	4766	4766
2000	Mean	6.3644	8.0600	27.7640	6.4862	8.0280	27.4920
	s.d.	(0.9263)	(0.9496)	(20.6680)	(0.9784)	(1.0249)	(20.6220)
	N	7312	7312	7312	6052	6052	6052
		Panel B					
		Divorced & Separated Parent			Divorced & Separated Parent		
		Child's Education	Parent's Education	Parent's Income	Child's Education	Parent's Education	Parent's Income
1960	Mean	5.3201	4.7385	5.8884	5.7585	4.9488	6.6655
	s.d.	(1.5333)	(1.9607)	(5.9571)	(1.3100)	(2.0348)	(5.2244)
	N	334	334	334	261	261	261
2000	Mean	6.3029	7.5558	14.4500	6.4158	7.5350	14.1170
	s.d.	(1.0232)	(1.2095)	(14.0700)	(0.9888)	(1.2053)	(13.9870)
	N	1780	1780	1780	1448	1448	1448
		Panel C					
		Widowed Parent			Widowed Parent		
		Child's Education	Parent's Education	Parent's Income	Child's Education	Parent's Education	Parent's Income
1960	Mean	5.2145	4.3028	5.7357	5.9312	4.7319	6.0581
	s.d.	(1.5934)	(2.1370)	(6.8933)	(1.3892)	(2.1304)	(6.1696)
	N	423	423	423	375	375	375
2000	Mean	6.2419	7.4312	12.7840	6.4548	7.2876	12.5830
	s.d.	(1.1013)	(1.5429)	(12.6070)	(1.1309)	(1.4812)	(9.9644)
	N	266	266	266	197	197	197

Note: The means and standard deviations are all weighted statistics.

Table 1 (Continued): Summary Statistics by Race & Family Structure

Year		White Child			Black Child		
		Panel D					
		Intact Parents			Intact Parents		
		Child's Education	Parent's Education	Parent's Income	Child's Education	Parent's Education	Parent's Income
1960	Mean	5.9830	5.7061	13.4960	5.1926	4.2787	5.6810
	s.d.	(1.2617)	(2.0436)	(9.4663)	(1.5207)	(1.9232)	(4.5663)
	N	9755	9755	9755	830	830	830
2000	Mean	6.4163	8.0745	28.3200	6.4535	7.7203	20.0030
	s.d.	(0.9472)	(0.9770)	(20.8910)	(1.0054)	(1.0089)	(15.7750)
	N	12402	12402	12402	962	962	962
		Panel E					
		Single Parent			Single Parent		
		Child's Education	Parent's Education	Parent's Income	Child's Education	Parent's Education	Parent's Income
1960	Mean	5.7082	4.9042	6.8455	4.9432	3.7566	3.2660
	s.d.	(1.4271)	(2.0941)	(6.6097)	(1.6121)	(1.7998)	(3.1938)
	N	1077	1077	1077	316	316	316
2000	Mean	6.3514	7.5968	15.5590	6.3494	7.2934	9.4821
	s.d.	(0.9979)	(1.2246)	(14.6400)	(1.1021)	(1.3110)	(8.9945)
	N	2895	2895	2895	796	796	796

Note: Single parent families include divorced & separated, and widowed parent families. The means and standard deviations are all weighted statistics.

With respect to boy−girl differences, the results for children of divorced or separated, and widowed parents were qualitatively the same as those for children of intact families, with girls significantly out-performing boys in both 1960 and 2000, though the gap had substantially narrowed over the period (indeed for boys and girls of widowed parents, there was no significant difference in 2000). Again generally there are no substantive differences in the parental characteristics of parents of boys and girls. The increase in (a multiple of over 5) the numbers of children in divorced or separated households between 1960 and 2000 is noteworthy (possibly a result of the increased ease with which divorce was obtained over the period). Similarly the numbers of children in widowed parental circumstances almost halved over the period, undoubtedly the result of improved health circumstances of parents over the 40 year period.

On the other hand from panels D and E of table 1, for both intact and single parent family structures, white children's dominance in educational attainment has dwindled, and is no longer statistically significant by 2000. This is despite significant parental educational attainment and income differences, providing some initial evidence of the fruitfulness of the educational and civil rights policies over the five decades that transpired. One interesting feature that is ubiquitous across family types is that in 1960, children of age 18 were on average more educated than their parents, whereas in 2000 they were on average less educated than their parents.

As for differences between family types, table 2 indicates that children from intact families clearly do better than children from single parent families, whether the head of household is divorced/separated or bereaved, and this is the case for both genders in 1960. However, this gap had narrowed by 2000, particularly so amongst children of bereaved families. Using these children of widowed parents as a comparison group, it is clear that the education outcomes of children of divorced/separated parents had improved by 2000, with no significant differences between them[16].

4.2 Progress Towards Equality of Opportunity

Table 3 reports the overall Social Justice Indices for two definitions of Social Justice, accommodating all of the race, gender, family type, parental type circumstances that confront an 18 year old. The first definition, Pure Equality of Opportunity (EO), characterizes social justice as independence of effort from circumstance, where all agents are weighted the same. The second definition, Qualified Equality of Opportunity (QEO), weighs more heavily independence of circumstance for those poorly endowed in circumstance relative to those richly endowed in circumstance. In both cases, the indices represent proximity to the ideal state in terms of the circumstances of race, gender, household type, household income and parental education. Since the measures reported here involve

[16] When considering pure EO measures, and mobility by family structure, a first concern is whether differing familial household structures have different transition structures. By comparing the overlap of the joint densities of intact versus single, and widowed versus divorced/separated parent families, the possibility of common transit structures was examined. Here, an overlap of one implies a common transit structure, and a value of less than one implies otherwise. The hypothesis of common transitional structures were rejected in every case, the results are available from the authors on request.

Table 2: Difference in Means Tests. (Standard Normal Tests & Lower Tail Probabilities)

	Boys	Girls	Boys	Girls	Boys	Girls
	Intact − Divorced	Intact − Divorced	Intact − Widowed	Intact − Widowed	Widowed − Divorced	Widowed − Divorced
1960	4.9482	4.5606	6.6639	2.7847	-0.9246	1.5954
	1.0000	1.0000	1.0000	0.9973	0.1776	0.9447
2000	2.0231	2.1653	1.5475	0.3490	-0.7350	0.4174
	0.9785	0.9848	0.9391	0.6364	0.2312	0.6618
Diff−Diff	3.9886	3.4546	3.6325	1.4944	-0.3159	0.9349
	1.0000	0.9997	0.9999	0.9325	0.3760	0.8251

Note: Divorced refers to both Divorced & Separated Parent

parental educational attainment and income, the densities estimated are a mixture of both continuous and discrete variables. Note that each of the discrete dimension of the densities estimated utilizes the cross-validated kernel smoothing method prescribed by Li and Racine (2003) and Ouyang et al. (2006).

As may be seen from Table 3, there has been a statistically significant progression over the 1960-2000 period in both EO and QEO. Note that most of the improvements in Social and Economic Justice took place over the first 20 years. In the following, the changes in Social and Economic Justice with respect to the specific circumstances mentioned above will be examined more closely. It is of interest to evaluate these progressions in EO by circumstance type (Lefranc et al. (2008) do so by seeking an absence of dominance relations of outcomes across types, but obviously this does not yield a measure of the degree of change). Here changes in the levels of EO are compared across family income types.

The raw EO indices by Parental Income and Gender circumstance for the years 1960, 1980 and 2000 are reported in Table 4. As may be seen, EO is ubiquitously and significantly lower for those in the lowest income circumstance quartile than for those in the highest income quartile for all three observation years. However the gaps are much narrower in 2000 than they were in the 1960's, reflecting the much bigger gains that were made in EO outcomes for the poor in income circumstance relative to the rich in income circumstance[17]. In the two lower quartiles, girls typically enjoyed significantly greater EO than boys, with the exception of the poorest group in 2000.

The raw EO indices by Parental Income and Race Circumstance for the years 1960, 1980 and 2000 are reported in Table 5. As may be seen EO is likewise ubiquitously and significantly lower for those in the lowest income circumstance quartile than for those in the highest income quartile for all three observation years. Again the gaps are much narrower in 2000 as compared to 1960. Generally differences between blacks and whites had been greatly reduced by the year 2000.

Table 6 presents the results for the measure of the QEO imperative. Since in this case the just outcome objective is a variable weighted sum of the EO outcome and the status quo outcome, with weight on the status quo increasing with family income, it will naturally overlap with the status quo better than would the pure EO model, so higher measures of Justice are to be expected than for the pure EO case. Specifically, the hypothesized QEO is that represented in equation (7) using the three variables of parental income and education, and the educational attainment of the child, with the qualification at three differing cutoffs. Indeed for the highest quartile, the overlaps are very close so they were not reported, however the lower quartiles are of interest, and record improvements in social justice for all quartiles over all categories during the period. Worthy of note is the fact that gains for males were made steadily throughout the period, whereas the gains for females were primarily made by the 1980s.

[17] These results are similar to those for Canada (Anderson et al. 2014).

Table 3: Overall Measures of Different Notions of Social Justice

		Six Variable OV*	
		Pure EO	QEO
1960	OV Unbiased	0.8229	0.8416
	S.E.	(0.0007)	(0.0007)
	# Obs.	11978	11978
1980	OV Unbiased	0.8874	0.9015
	S.E.	(0.0005)	(0.0005)
	# Obs.	22386	22386
2000	OV Unbiased	0.8802	0.9009
	S.E.	(0.0006)	(0.0006)
	# Obs.	17055	17055
		"t" Statistic of Improvement in EO	
1980−1960		72.60	67.48
		[1.00]	[1.00]
2000−1980		-8.92	-0.76
		[0.00]	[0.22]
2000−1960		60.94	63.06
		[1.00]	[1.00]

* The six variables are child education (effort variable), and five circumstance variables, gender of child, race of child, family structure, parental income and parental education.

Note: 1. Standard Errors in Parenthesis, and $\Pr(T < t)$ are in brackets. 2. Ouyang et al. (2006) smoothing of discrete variables was not employed in these calculations since over-smoothing led to a lack of discrimination with large numbers of observations(see appendix A.1).

Table 4: Gender Equality of Opportunity Indices by Parental Income Quartile & Year

		Panel A: Equality of Opportunity by Gender & Year					
Income		Males			Females		
Quartile		1960	1980	2000	1960	1980	2000
1^{st} Quartile	OV	0.8388	0.8684	0.9285	0.8717	0.9014	0.8996
	Standard Error	(0.0005)	(0.0004)	(0.0006)	(0.0006)	(0.0005)	(0.0006)
	# of Observations	1714	3126	2439	1298	2477	2050
2^{nd} Quartile	OV	0.8915	0.9316	0.9486	0.9094	0.9637	0.9531
	Standard Error	(0.0005)	(0.0006)	(0.0006)	(0.0006)	(0.0006)	(0.0007)
	# of Observations	1668	2998	2473	1308	2592	2012
3^{rd} Quartile	OV	0.9176	0.9512	0.9474	0.9423	0.9582	0.9482
	Standard Error	(0.0006)	(0.0006)	(0.0007)	(0.0007)	(0.0007)	(0.0007)
	# of Observations	1586	2989	2267	1409	2607	1891
4^{th} Quartile	OV	0.9462	0.9471	0.9355	0.9272	0.9611	0.9157
	Standard Error	(0.0006)	(0.0006)	(0.0008)	(0.0008)	(0.0007)	(0.0009)
	# of Observations	1608	3022	2179	1387	2575	1744
		Panel B: Difference Between Quartiles (t Statistics & [$\Pr(T < t)$])					
	1st - 2nd Quartile	-73.55	-91.56	-23.20	-41.87	-77.69	-57.58
		[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
	1st - 3rd Quartile	-99.45	-116.67	-21.04	-75.76	-69.99	-51.44
		[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
	1st - 4th Quartile	-131.10	-110.14	-6.95	-56.61	-72.80	-14.87
		[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]

Note: Standard errors are in parenthesis.

Table 4 (Continued): Gender Equality of Opportunity Indices by Parental Income Quartile & Year

	Panel C: Difference Between Years (t Statistics & [$\Pr(T < t)$])						Panel D: Difference Between Genders (t Statistics & [$\Pr(T < t)$])		
	Males			Females			1960	1980	2000
	1960-1980	1960-2000	1980-2000	1960-1980	1960-2000	1980-2000	Males-Females	Males-Females	Males-Females
1^{st} Quartile	-45.63	-114.20	-83.93	-37.66	-31.57	2.33	-40.79	-52.77	33.52
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.99]	[0.00]	[0.00]	[1.00]
2^{nd} Quartile	-53.18	-71.26	-20.13	-59.61	-46.21	11.32	-21.84	-37.66	-4.90
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[1.00]	[0.00]	[0.00]	[0.00]
3^{rd} Quartile	-39.83	-32.99	4.26	-16.73	-5.93	10.41	-26.88	-8.03	-0.84
	[0.00]	[0.00]	[1.00]	[0.00]	[0.00]	[1.00]	[0.00]	[0.00]	[0.20]
4^{th} Quartile	-1.03	10.39	11.58	-33.75	9.86	41.05	19.22	-15.71	16.52
	[0.15]	[1.00]	[1.00]	[0.00]	[1.00]	[1.00]	[1.00]	[0.00]	[1.00]

Table 5: Racial Equality of Opportunity Indices by Parental Income Quartile & Year

		Panel A: Equality of Opportunity by Race & Year					
Income		Whites			Blacks		
Quartile		1960	1980	2000	1960	1980	2000
1st Quartile	OV	0.8376	0.8764	0.9147	0.8867	0.9014	0.9270
	Standard Error	(0.0004)	(0.0004)	(0.0005)	(0.0008)	(0.0005)	(0.0009)
	# of Observations	2240	4006	3580	772	1597	909
2nd Quartile	OV	0.8945	0.9433	0.9461	0.8827	0.8958	0.9386
	Standard Error	(0.0004)	(0.0005)	(0.0005)	(0.0012)	(0.0010)	(0.0015)
	# of Observations	2722	4990	4077	254	600	408
3rd Quartile	OV	0.9341	0.9591	0.9507	0.9537	0.8979	0.9349
	Standard Error	(0.0005)	(0.0005)	(0.0005)	(0.0016)	(0.0015)	(0.0018)
	# of Observations	2908	5249	3890	87	347	268
4th Quartile	OV	0.9408	0.9564	0.9208	0.9102	0.9029	0.9557
	Standard Error	(0.0005)	(0.0004)	(0.0006)	(0.0030)	(0.0017)	(0.0022)
	# of Observations	2962	5375	3750	33	222	173
		Panel B: Difference Between Quartiles (t Statistics & $[\Pr(T < t)]$)					
	1st - 2nd Quartile	-91.79	-111.34	-44.24	2.72	4.99	-6.83
		[0.00]	[0.00]	[0.00]	[1.00]	[1.00]	[0.00]
	1st - 3rd Quartile	-147.88	-137.31	-50.15	-38.12	2.22	-4.03
		[0.00]	[0.00]	[0.00]	[0.00]	[0.99]	[0.00]
	1st - 4th Quartile	-154.65	-135.81	-7.83	-7.54	-0.81	-12.17
		[0.00]	[0.00]	[0.00]	[0.00]	[0.21]	[0.00]

Note: Standard errors are in parenthesis.

Table 5 (Continued): Racial Equality of Opportunity Indices by Parental Income Quartile & Year

	Panel C: Difference Between Years (t Statistics & [$\Pr(T < t)$])						Panel D: Difference Between Races (t Statistics & [$\Pr(T < t)$])		
	Whites			Blacks			1960	1980	2000
	1960-1980	1960-2000	1980-2000	1960-1980	1960-2000	1980-2000	Whites-Blacks	Whites-Blacks	Whites-Blacks
1st Quartile	-65.46	-115.21	-60.94	-15.62	-34.14	-25.39	-53.65	-39.33	-12.23
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
2nd Quartile	-77.55	-77.65	-4.14	-8.34	-29.44	-24.10	9.23	43.04	4.83
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[1.00]	[1.00]	[1.00]
3rd Quartile	-37.74	-23.54	12.20	25.85	7.93	-16.02	-11.94	39.42	8.53
	[0.00]	[0.00]	[1.00]	[1.00]	[1.00]	[0.00]	[0.00]	[1.00]	[1.00]
4th Quartile	-23.50	25.61	47.44	2.12	-12.17	-18.88	9.97	29.90	-15.32
	[0.00]	[1.00]	[1.00]	[0.98]	[0.00]	[0.00]	[1.00]	[1.00]	[0.00]

Table 6: Qualified Equality of Opportunity Hypothesis by Year, Gender & Race

		Overall			Males			Females		
		1960	1980	2000	1960	1980	2000	1960	1980	2000
25^{th} Percentile	OV	0.9625	0.9704	0.9797	0.9587	0.9655	0.9818	0.9672	0.9756	0.9742
& Below	s.e.	(0.0002)	(0.0002)	(0.0002)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0004)
Independent	# of Obs.	11978	22386	17055	6576	12135	9358	5402	10251	7697
50^{th} Percentile	OV	0.9305	0.9496	0.9657	0.9216	0.9402	0.9670	0.9403	0.9608	0.9604
& Below	s.e.	(0.0002)	(0.0002)	(0.0002)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0004)
Independent	# of Obs.	11978	22386	17055	6576	12135	9358	5402	10251	7697
75^{th} Percentile	OV	0.9010	0.9346	0.9797	0.8890	0.9225	0.9492	0.9155	0.9494	0.9474
& Below	s.e.	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)
Independent	# of Obs.	11978	22386	17055	6576	12135	9358	5402	10251	7697
					Whites			Blacks		
					1960	1980	2000	1960	1980	2000
25^{th} Percentile	OV				0.9603	0.9716	0.9799	0.9653	0.9666	0.9722
& Below	s.e.				(0.0002)	(0.0002)	(0.0003)	(0.0006)	(0.0004)	(0.0007)
Independent	# of Obs.				10832	19620	15297	1146	2766	1758
50^{th} Percentile	OV				0.9331	0.9561	0.9648	0.9535	0.9437	0.9555
& Below	s.e.				(0.0002)	(0.0002)	(0.0003)	(0.0006)	(0.0004)	(0.0007)
Independent	# of Obs.				10832	19620	15297	1146	2766	1758
75^{th} Percentile	OV				0.9076	0.9464	0.9496	0.9311	0.9255	0.9530
& Below	s.e.				(0.0002)	(0.0002)	(0.0003)	(0.0006)	(0.0004)	(0.0007)
Independent	# of Obs.				10832	19620	15297	1146	2766	1758

Note: Standard errors are in parenthesis, and $\Pr(T < t)$ are in brackets.

Table 7 reports the degree of overlap between the empirical density against the hypothetical EO density conditional on different family structures. It is clear from panel A of Table 7 that in the 1960s, neither boys or girls in any family structure exhibit a great deal of EO, however with the exception of girls from divorced/separated families, there is significant evidence of increased EO between 1960 and 2000. Panel B reports the significance tests of within year differences between boys and girls, while panel C reports the tests of cross year differences for each gender, derived from the indices of panel A. With regard to the gender comparisons, in 1960s girls enjoyed significantly more EO than boys in all family structures. However, by 2000 this pattern is no longer true with the exception of children from widowed parent families. In fact, for children in divorced/separated parent families, boys enjoy significantly more EO than girls. The changes within each gender across the four decades as reflected in panel B are that although all children experienced significant increases in EO, the improvements are larger for boys than they are for girls. The extension of the examination to cross family structure comparisons by gender and year in panel C, finds that with the exception of the divorced/separated versus intact comparisons for boys, in 1960 and 2000, children of single parent families exhibited significantly less EO than their intact counterparts. On the other hand, with the exception of boys of widowed parents, the gaps had significantly diminished by 2000 reflecting the impact of family law legislation that took place over the preceding period.

The growing similarity in EO says little explicitly about differences in the distributions of academic attainments across race or gender. Table 8 reports the corresponding stochastic dominance tests[18], utilizing similar ideas in the mixture method of the density comparisons above, estimating the continuous variable using kernel density estimation methods (Linton et al. 2005). Notice that despite the improvements among boys in terms of EO, the educational outcomes of 18 year old girls continue to first order stochastically dominate boys regardless of their familial background. In other words, all things equal girls have a higher probability of performing better in school relative to boys. The comparisons pertaining to race on the other hand affirms the improvements that the civil rights policies have afforded African Americans, since whereas in 1960, whites first order stochastically dominated blacks, there is no longer any dominance relationship in 2000 for all family structures.

[18]This test corresponds to those presented by Lefranc et al. (2008, 2009), though in this case the respective circumstance classes are race and gender.

Table 7: Equal Opportunity Measures by Family Structure & Gender

		Panel A: Overlap Measure & Difference Between Family Structures by Year & Gender								
		1960			1980			2000		
		Intact	Divorced + Separated	Widowed	Intact	Divorced + Separated	Widowed	Intact	Divorced + Separated	Widowed
Male	OV Unbiased	0.8505	0.8870	0.7145	0.9152	0.8787	0.8631	0.9156	0.9273	0.8176
	Standard Error	(0.0010)	(0.0043)	(0.0038)	(0.0008)	(0.0020)	(0.0032)	(0.0009)	(0.0018)	(0.0048)
	# Observations	5819	334	423	9943	1581	611	7312	1780	266
Female	OV Unbiased	0.8816	0.8525	0.8435	0.9385	0.9043	0.9161	0.9180	0.9028	0.9149
	Standard Error	(0.0011)	(0.0049)	(0.0041)	(0.0009)	(0.0022)	(0.0033)	(0.0010)	(0.0021)	(0.0054)
	# Observations	4766	261	375	8392	1309	550	6052	1448	197
		Panel B: Difference Between Genders by Year & Family Structure (t Statistics & [$\Pr(T < t)$])								
	Males − Females	-20.25	5.31	-23.18	-20.05	-8.74	-11.51	-1.77	8.84	-13.43
		[0.00]	[1.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.04]	[1.00]	[0.00]

Note: Standard errors are in parenthesis.

Table 7 (Continued): Equal Opportunity Measures by Family Structure & Gender

	Panel C: Difference Between Family Structures by Gender & Year (t Statistics & [$\Pr(T < t)$])								
	Intact − (Divorced + Separated)	Intact − Widowed	(Divorced + Separated) − Widowed	Intact − (Divorced + Separated)	Intact − Widowed	(Divorced + Separated) − Widowed	Intact − (Divorced + Separated)	Intact − Widowed	(Divorced + Separated) − Widowed
Male	-8.26	34.43	30.03	17.20	15.95	4.18	-5.66	20.16	21.44
	[0.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[0.00]	[1.00]	[1.00]
Female	5.82	9.05	1.43	14.68	6.49	-2.97	6.60	0.57	-2.07
	[1.00]	[1.00]	[0.92]	[1.00]	[1.00]	[0.00]	[1.00]	[0.72]	[0.02]
	Panel D: Difference Between Years by Gender & Family Structure (t Statistics & [$\Pr(T < t)$])								
	1960−1980			1960−2000			1980−2000		
Male	-49.97	1.74	-29.96	-46.85	-8.62	-16.90	-0.30	-17.97	7.94
	[0.00]	[0.96]	[0.00]	[0.00]	[0.00]	[0.00]	[0.38]	[0.00]	[1.00]
Female	-40.02	-9.73	-13.83	-23.78	-9.54	-10.52	15.30	0.49	0.20
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[1.00]	[0.69]	[0.58]

Table 8: Stochastic Dominance Test by Family Type & Race

	1960		2000	
	Panel A: Stochastic Dominance Test by Gender			
Family Structure\Hypothesis	Boys $\succ^i$ Girls	Girls $\succ^i$ Boys	Boys $\succ^i$ Girls	Girls $\succ^i$ Boys
Intact	6.3076	-0.0676	3.3651	0.0896
	[1.0000]	[0.0000]	[1.0000]	[0.0159]
Result	Girls $\succ^1$ Boys		Girls $\succ^1$ Boys	
Single	3.4246	-0.0106	1.7016	0.0318
Parent	[1.0000]	[0.0000]	[0.9969]	[0.0020]
Result	Girls $\succ^1$ Boys		Girls $\succ^1$ Boys	
	Panel B: Stochastic Dominance Test by Race			
Family Structure\Hypothesis	White $\succ^i$ Black	Black $\succ^i$ White	White $\succ^i$ Black	Black $\succ^i$ White
Intact	0.0000	21.8600	5.8470	1.6321
	0.0000	1.0000		
Result	White $\succ^1$ Black		No Dominance	
Single	0.0000	11.9580	0.4736	3.0115
Parent	0.0000	1.0000		
Result	White $\succ^1$ Black		No Dominance	

Note: $\succ^i$ denotes ith order stochastic dominance. The tests performed tests up to third order dominance, and if no dominance relationship is revealed, it is concluded that there is "No Dominance". The upper tail probabilities of the test statistic are in brackets

5 Conclusions

A general readily applicable method for quantifying the progress of Social Justice has been presented, and applied to the Equality of Opportunity notion of Social Justice. The method relies upon measuring a sense of the distance of the joint probability distribution of agent characteristics from that which would be desired under a particular notion of Social Justice. The method does not run into the problems that bedevil regression and transition matrix techniques which are commonly employed, and it is sufficiently flexible to admit a variety of agent characteristics that may be either discretely or continuously measured. The technique has been used to measure the progress of Equality of Opportunity, and a similar notion of Qualified Equal Opportunity for 18 year old children in the United States over the last 4 decades of the last century.

Those decades saw considerable efforts through various family, education, and civil rights law and policy to equalize opportunity in the U.S., especially with regard to elevating outcomes of those who were disadvantaged in their circumstance whether it be gender, race or family background based. While these efforts have been much lauded, their success has been contested by some. Here a new measure that provides a metric for the level of equality of opportunity has been provided, which has well defined statistical properties facilitating inference and which can handle collections of circumstance, and outcome variables that can be discrete or continuous. Using the measure to relate 18 year old school attainments to their circumstances in the form of their gender, race, family background (intact, or divorced/separated, or widowed parents) and the educational status and income of the family, it is possible to conclude that the efforts have met with some qualified success. With the exception of one group, daughters in the lowest parental income quartile (especially those daughters of a widowed parent), all groups in all family types, and of both genders have experienced significant improvements in Equality of Opportunity over the period. Some have advanced more than others, though it should be said that the genders started from different positions, with girls generally experiencing greater equality of opportunity than boys in the 1960s era.

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A Appendix

A.1 Estimation of Mixed Overlap Measure

In instances such as the current one where at least one of the variables considered is continuous, the measure proposed by Anderson et al. (2010) is subject to biases. However Anderson et al. (2012) proposed a similar measure using Kernel estimation techniques. This means we can mix the two techniques without any serious implications for the bias and asymptotic distributions of the results. In addition, in the current application, we have utlized the cross-validation kernel smoothing method for the discrete variable as well as suggested by Ouyang et al. (2006). To illustrate the technique used consider the simple bivariate case, leting $x \in X$ be the discete variable, and $y \in Y$ be the continuous variable. Then the overlap measure that compares of two densities $f(.)$ and $g(.)$ is,

$$\mathbf{OV} = \sum_{x \in X} \left\{ \int_{y \in Y} \min\{f(x,y), g(x,y)\} dy \right\} \tag{A-1}$$

Notice that to estimate this mixed overlap index, we need to sum over the estimated density at each discrete realization, besides integrating over the support of the continuous variable. Keep in mind that $g(.)$ may refer to either the hypothesized density in question or the density of another population under comparison. We will describe the comparison of two mixed distributions here, and the examinations of more general Equal oportunity hypotheses can be easily adapted from the procedure given below.

1. First calculate the overlap index,

$$\widehat{\mathbf{OV}} = \sum_{j=1}^{J} \int_{y \in Y} \min\{f_n(x_j, y), g_n(x_j, y)\} dy \tag{A-2}$$

   where j indexes the J unique discrete realizations. The estimators for the densities are,

$$f_n(x_j, y) = \frac{1}{nb^d} \sum_{i=1}^{n} L_\lambda(X_i^f, x_j) K\left(\frac{y - Y_i^f}{b}\right) \tag{A-3}$$

$$g_n(x_j, y) = \frac{1}{nb^d} \sum_{i=1}^{n} L_\lambda(X_i^g, x_j) K\left(\frac{y - Y_i^g}{b}\right) \tag{A-4}$$

   where d denotes the number of continuous variable dimensions ($d = 1$ one in the current case) in the observed variables $\{X_i^f\}_{i=1}^n$, $\{X_i^g\}_{i=1}^n$, $\{Y_i^f\}_{i=1}^n$ and $\{Y_i^g\}_{i=1}^n$.

Further, for $f_n(.)$ and $g_n(.)$,

$$L_\lambda(X_i, x_j) = \prod_{j=1}^{J} \left\{ \frac{\lambda}{J-1} \right\}^{I(X_i \neq x_j)} (1-\lambda)^{I(X_i = x_j)} \quad \text{(A-5)}$$

where λ is estimated via cross-validation method as prescribed in Ouyang et al. (2006). While the kernel function used is the Normal kernel as suggested in Anderson et al. (2012), and the bandwidth used in estimating the overlap index is the Silverman's rule of thumb ($b_s = 1.84sn^{-1/5}$, where s is the sample standard deviation).

2. However, the above measure is biased which needs to be adjusted. To do so, we have to first find the estimated contact set and its complements,

$$\begin{aligned}
\widehat{C}_{f,g} &= \{y \in \mathbb{R}^d : |f_n(x_j, y) - g_n(x_j, y)| \leq c_n, f_n(x_j, y) > 0, g_n(x_j, y) > 0\} \\
\widehat{C}_f &= \{y \in \mathbb{R}^d : f_n(x_j, y) - g_n(x_j, y) < -c_n, f_n(x_j, y) > 0, g_n(x_j, y) > 0\} \\
\widehat{C}_g &= \{y \in \mathbb{R}^d : f_n(x_j, y) - g_n(x_j, y) > c_n, f_n(x_j, y) > 0, g_n(x_j, y) > 0\}
\end{aligned}$$

where the first equation above describes the contact set, while the others are its complement, and c_n is the tuning parameter which was set to $b_s^{3/2}$ if $b_s < 1$, and $b_s^{2/3}$ otherwise. Further, note that this is to be performed for each $j = \{1, \ldots, J\}$.

3. The bias corrected overlap measure and its variance are as follows,

$$\widehat{\mathbf{OV}}^{bc} = \widehat{\mathbf{OV}} - \widehat{a}_n n^{-0.5} \quad \text{(A-6)}$$

$$\widehat{v} = \widehat{p}_0 \sigma_0^2 + \widehat{\sigma}_1^2 \quad \text{(A-7)}$$

where the calculations required to obtain these values, in sequence, are

(a) $||\mathcal{K}||_2^2$ and $||\mathcal{K}||_2$ are

$$\begin{aligned}
||\mathcal{K}||_2^2 &= \int_{\mathbb{R}^d} \mathcal{K}^2(u)du \\
\Rightarrow ||\mathcal{K}||_2 &= \sqrt{\int_{\mathbb{R}^d} \mathcal{K}^2(u)du}
\end{aligned}$$

Note that for the univariate uniform kernel function,

$$\begin{aligned}
||\mathcal{K}||_2^2 &= \int_{-\infty}^{\infty} \mathbf{1}(|u| \leq 0.5)du \\
&= \int_{-0.5}^{0.5} du = 1 \\
\Rightarrow ||\mathcal{K}||_2 &= 1
\end{aligned}$$

(b) $\widehat{a}_n$ is the bias correction factor,

$$\widehat{a}_n = \mathbf{E}\min\{Z_1, Z_2\}\frac{||\mathcal{K}||_2}{2b^{\frac{d}{2}}}\sum_{j=1}^{J}\left(\int_{C_{f,g}} f_n^{\frac{1}{2}}(x_j,y)dy + \int_{C_{f,g}} g_n^{\frac{1}{2}}(x_j,y)dy\right)$$

where $\mathbf{E}\min\{Z_1, Z_2\} = -0.56$, and Z_1 and Z_2 are independent standard normal random variables if the sample sizes are the same for both densities under consideration. But when they are different, letting the sample for $f_n(.)$ be n and that for $g_n(.)$ be m, such that the ratio of the sample sizes are m/n $\rightarrow \tau \in (0,\infty)$, $\mathbf{E}\min\{Z_1, Z_2\} = -\frac{1}{\sqrt{\pi}} = -0.5642$ needs to be augmented with $\mathbf{E}\min\{Z_1, Z_2/\tau\} = -\frac{\sqrt{1+1/\tau}}{2}\sqrt{\frac{2}{\pi}}$.

(c) The kernel constant, σ_0^2, is defined as follows,

$$\sigma_0^2 = ||\mathcal{K}||_2^2\int_{T_0}\text{cov}\left(\min\{Z_1, Z_2\}, \min\left\{\begin{array}{c}\rho(t)Z_1 + \sqrt{1-\rho(t)^2}Z_3 \quad , \\ \rho(t)Z_2 + \sqrt{1-\rho(t)^2}Z_4\end{array}\right\}\right)dt = 0.6135$$

where $T_0 = \{t \in \mathbb{R}^d : ||t|| \leq 1\}$ and

$$\rho(t) = \frac{\int_{\mathbb{R}^d}\mathcal{K}(u)\mathcal{K}(u+t)du}{||\mathcal{K}||_2^2}$$

Note that the value for the kernel constant above is for case when the sample sizes considered are the same. When this is not the case, the kernel constant for that case can be calculated as,

$$\sigma_0^2(\tau) = \frac{1+\tau^{-1}}{2}\sigma_0^2$$

(d) $\widehat{p}_0$ in the calculation of the variance can is estimated by

$$\widehat{p}_0 = \frac{1}{2}\sum_{j=1}^{J}\left(\int_{\widehat{C}_{f,g}} f_n(x_j,y)dy + \int_{\widehat{C}_{f,g}} g_n(x_j,y)dy\right)$$

(e) Finally, $\widehat{\sigma}_1^2$ is estimated by

$$\widehat{\sigma}_1^2 = \widehat{p}_f(1-\widehat{p}_f) + \widehat{p}_g(1-\widehat{p}_g)$$

where

$$\widehat{p}_f = \sum_{j=1}^{J}\int_{\widehat{C}_f} f_n(x_j,y)dy$$

$$\widehat{p}_g = \sum_{j=1}^{J}\int_{\widehat{C}_g} g_n(x_j,y)dy$$

And if the samples are not equal,

$$\widehat{\sigma}_1^2(\tau) \quad = \quad \widehat{p}_f(1-\widehat{p}_f) + [\widehat{p}_g(1-\widehat{p}_g)/\tau]$$

For a complete discussion of the asymptotic results, and intuition regarding the measure and the proofs, see Anderson et al. (2012).

When testing for mobility, the overlap index generated by this mixture of discrete and continuous variables is,

$$\widehat{\mathbf{OV}} \quad = \quad \sum_{j=1}^{J} \int_{y \in Y} \min\{f_n(x_j, y), g_n^1(x_j) g_n^2(y)\} dy \qquad \text{(A-8)}$$

where j indexes the J unique discrete realization as before.