Totally Disconnected Sierpiński Relatives

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Outline

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Introduction

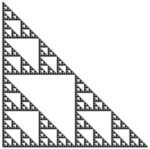

Figure: The Sierpiński gasket

The gasket is the attractor for the IFS $\{g_1, g_2, g_3\}$, with

$$g_1(x,y) = \frac{1}{2}(x,y), \quad (1)$$

$$g_2(x,y) = \frac{1}{2}[(x,y)+(1,0)], \quad (2)$$

$$g_3(x,y) = \frac{1}{2}[(x,y)+(0,1)] \quad (3)$$

Some Details about the Gasket

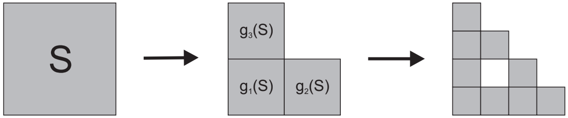

Figure: The first three approximations of Sierpiński gasket

- Each map in the IFS takes the unit square S to a square scaled down by $1/2$.
- The gasket G is the unique set that satisfies

$$G = g_1(G) \cup g_2(G) \cup g_3(G) \tag{4}$$

- The gasket is self-similar, and its fractal dimension is $\ln 3/\ln 2 \approx 1.585$.
- G is multiply-connected.

Symmetries of the Square

The eight symmetries of the square form a group $\Sigma = \{a, b, c, d, e, f, g, h\}$.

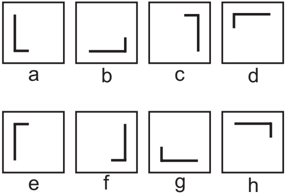

Figure: The Eight Symmetries of the Square

IFS for Sierpiński Relatives

Let $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ and let T_2 and T_3 be translations:

$$T_2(x, y) = (x + 1/2, y), \quad T_3(x, y) = (x, y + 1/2) \qquad (5)$$

Define the IFS $\{f_1, f_2, f_3\}$ with

$$f_1 = \frac{1}{2}\sigma_1, \quad f_2 = T_2 \circ \frac{1}{2}\sigma_2, \quad f_3 = T_3 \circ \frac{1}{2}\sigma_3 \qquad (6)$$

The relative $R_{\sigma_1\sigma_2\sigma_3}$ is the unique attractor of $\{f_1, f_2, f_3\}$.

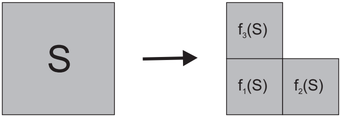

Figure: IFS for Relatives

Examples of Relatives

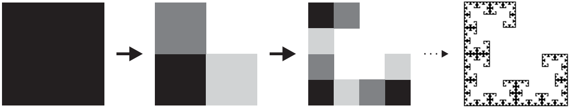

Figure: R_{abd}

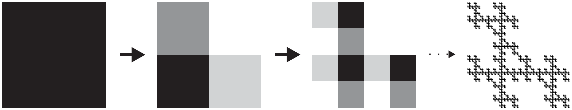

Figure: R_{ccc}

Same Fractal Dimension, Different Topologies

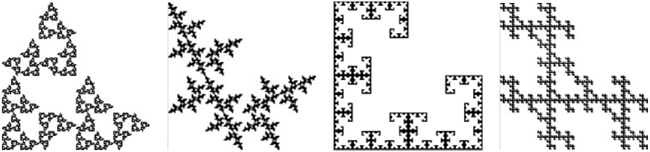

Figure: Four relatives: R_{ggf} (totally disconnected), R_{cda} (disconnected with paths), R_{abd} (simply-connected), and R_{ccc} (multiply-connected)

Totally Disconnected

Let $R_i = f_i(R)$. The **level 1 sub-relatives of** R are R_1, R_2 and R_3. Then

$$R = R_1 \cup R_2 \cup R_3 \tag{7}$$

Theorem

[2] R is totally disconnected if and only if one of the following holds:

1. *R_1, R_2 and R_3 are pair-wise disjoint*
2. *One R_i is disjoint from the other two and there are no straight line segments in R*

Examples of Totally Disconnected Relatives

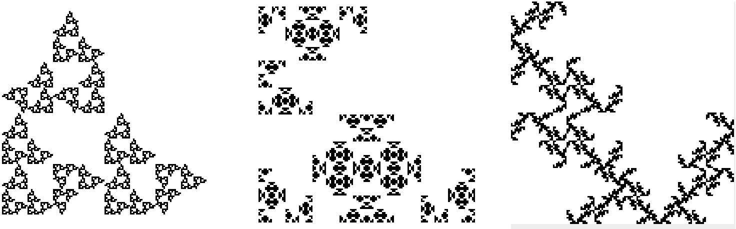

Figure: (a) R_{ggf}, (b) R_{bge}, (c) R_{hfg}

Outline

Why Study Totally Disconnected Relatives?

The totally disconnected relatives are all topologically equivalent to Cantor dust so why bother studying them?

- To develop new ways to characterize and classify classes of fractals with the same fractal dimension and same topology

What can be used to distinguish them?

- Geometry (consider isometries between relatives)
- Double Points (consider location and number of)
- Epsilon-Hulls (to characterize how close to being connected a relative is)
- Morphisms between relatives

Congruency of Relatives

- The gasket G is symmetric about the line $y = x$, and there are 8 different IFS that yield the same attractor.
- If a relative is not symmetric, then it has a congruent match. This occurs when $R \neq g(R)$.
- If $R \neq g(R)$, then the congruent match of $R = R_{\sigma_1\sigma_2\sigma_3}$ is $R' = R_{\sigma_1'\sigma_2'\sigma_3'}$ where $\sigma_1' = g \circ \sigma_1 \circ g$, $\sigma_2' = g \circ \sigma_3 \circ g$, and $\sigma_3' = g \circ \sigma_2 \circ g$ [2].
- Each totally disconnected relative is congruent to one other totally disconnected relative.

Example of Congruent Pair of Totally Disconnected Relatives

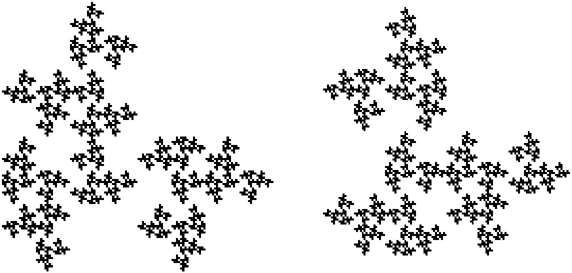

Figure: R_{edb} and its congruent match R_{fdb}

Double Points

- A **double point** is a point that corresponds to two different addresses, *i.e.*, is in the intersection of two distinct sub-relatives.
- The **level 0** double points are the double points that are in the pair-wise intersections of the sub-relatives R_1, R_2 and R_3, so the elements of

$$R_1 \cap R_2, \quad R_1 \cap R_3, \quad \text{and} \quad R_2 \cap R_3$$

- For the totally disconnected relatives, the number of level 0 double points is 0, 1, 2, 3, 4 or infinite [1].

Examples of Totally Disconnected Relatives

Figure: R_{ggf} does not have double points

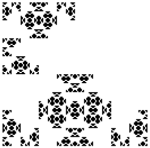

Figure: R_{bge} does have double points

Epsilon-Hulls of Relatives

- Although the totally disconnected relatives are all topologically equivalent, some "look" more connected than others.

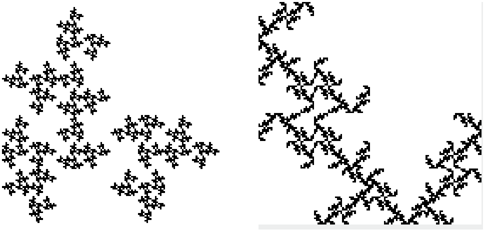

Figure: R_{edb} and R_{hfg}

Epsilon-Hulls of Relatives (continued)

- One way to illustrate this concept is through the use of ϵ-hulls.
- For $\epsilon \in [0, \infty)$, define the ϵ-hull of the relative R as

$$R(\epsilon) = \{(x, y) | d((x, y), R) \leq \epsilon\}$$

- We similarly define the ϵ-hulls of the sub-relatives R_1, R_2 and R_3, and denote them by $R_1(\epsilon), R_2(\epsilon), R_3(\epsilon)$.
- Thus $R(\epsilon) = R_1(\epsilon) \cup R_2(\epsilon) \cup R_3(\epsilon)$.

Contact Epsilon of a Relative

- For a given relative R, define the **contact epsilon**, denoted by $\epsilon_{con}(R)$, to be

$$\min\{\epsilon \in [0,\infty) | R(\epsilon) \text{ is path-connected}\}$$

Theorem

[1] The contact ϵ for totally disconnected relatives is

$$\min\{\epsilon \in (0,\infty) | R_1(\epsilon) \cap R_2(\epsilon) \neq \emptyset \textit{ and } R_1(\epsilon) \cap R_3(\epsilon) \neq \emptyset\}$$

Idea: Consider distances between pairs of points where a pair consists of a point in R_1 and a point from R_2 or R_3. One can use the scaling nature of the relatives to show that if the condition of the theorem is met, then the $R(\epsilon)$ must be path-connected.

Morphisms Between Relatives

- The totally disconnected relatives are all topologically equivalent.
- Given any two distinct totally disconnected relatives, there are infinitely many homeomorphisms between them.
- One class of morphisms to consider are the address-preserving morphisms. These are straightforward homeomorphisms for relatives with no double points, but they are more interesting where at least one of the relatives does have double points.

Outline

Conclusions and Future Work

- The class of totally disconnected relatives has great potential to show how to use different ways to characterize and classify fractals with the same fractal dimension and the same topology.
- This work in progress will lead to ideas for dealing with other classes of relatives, as well as other classes of fractals.

Acknowledgements

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Thank you!

T. D. Taylor.
Totally disconnected Sierpiński relatives.
in progress.

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