COMPUTATIONAL TOPOLOGY AND FRACTAL TREES

by

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Threefold wisdom of the tree:
Leaf-wisdom of change,
ever releasing;
Branch-wisdom of growth,
ever reaching;
Root-wisdom of endurance,
ever deepening.

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Abstract

This thesis presents a study of symmetric binary fractal trees using methods of computational topology. Fractal trees can be used to model various natural systems, such as the cardiovascular system or river drainage networks.

Symmetric binary fractal trees were first introduced by Mandelbrot in [30]. A symmetric binary fractal tree is defined by two parameters: the branching angle θ (between 0 and 180 degrees) and a scaling ratio r (between 0 and 1). A trunk of length 1 splits into two branches, one on the left and one on the right, with lengths equal to the scaling ratio and forming an angle θ with the extension of the trunk. Each of these branches splits into two new branches, and the branching is continued *ad infinitum*. The resulting object is the fractal tree, which can be seen as a representation of the free monoid M_{LR} on two generators L and R.

We study the self-avoiding and self-contacting trees. Motivated by techniques from shape theory and computational topology, we will be considering these trees along with their closed epsilon-neighbourhoods as ϵ ranges over the non-negative real numbers. We investigate various features of the closed ϵ-neighbourhoods, based on the holes in these neighborhoods.

Due to the nice geometric nature of the trees, we can refine our approach by classifying holes according to their shape and location in the tree. The action of M_{LR} on the tree brings a natural grading by level to these holes. We will see that the level 0 holes form a kind of fundamental domain, and we can restrict our attention to the level 0 holes. To describe the location of a hole, we have generalized the notion of contact address (for self-contacting trees) to hole locator address and hole locator pairs.

We determine the hole sequence of these trees together with the persistence intervals of the holes as the 'topological barcodes' (as defined by Carlsson *et al.*) of these trees. We find that the notion of persistence has some interesting and perhaps unexpected properties in this context.

From various notions and properties of holes we derive several classifications of

the symmetric binary fractal trees. These are the complexity, location, type and hole sequence classifications. They lead to the determination of certain critical values for the angle θ with respect to location, the scaling ratio r as a function of θ and with respect to complexity, and ϵ as a function of both r and θ and with respect to the hole sequence.

We illustrate the theory with a presentation of a collection of specific trees and their closed ϵ-neighbourhoods. We discuss four particularly interesting trees which scale according to the golden ratio.

Acknowledgements

This thesis has been an incredibly challenging, surprising and rewarding journey, and I could never have done it on my own.

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Chapter 1

Introduction

This thesis presents a study of some of the topological aspects of symmetric binary fractal trees. This class of fractals was first introduced by B. Mandelbrot in his seminal book "The Fractal Geometry of Nature" [30] and was studied in more detail by Mandelbrot and M. Frame in a more recent paper [31]. A practical motivation for this work is the application to natural systems such as the cardiovascular system and river networks. We develop new methods of computational topology to analyze fractal trees. Various classifications of symmetric binary fractal trees arise, along with new invariants for these trees. The methods and results can be extended to other kinds of fractal trees, more general classes of fractals and spaces that are not fractals.

1.1 Fractals and Fractal Trees

What are fractals? This is not an easy question to answer mathematically, but we often recognize fractals when we see them. Mandelbrot first used the word "fractal" to describe objects that were too irregular to fit into traditional geometric settings [30]. A classic example of a fractal is the *Cantor set*. The Cantor set is obtained by removing the middle third interval from the unit interval, then removing the middle thirds from the remaining subintervals, and continuing this process *ad infinitum*. See Figure 1.1.

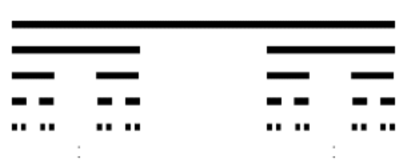

Figure 1.1: The first few iterations of removing middle thirds to obtain the Cantor set

The resulting set is a perfect, disconnected set of measure 0. For basic definitions and results in topology, see [38]. An important feature of the Cantor set is that it is equal to the union of two scaled down versions of itself. This feature of self-similarity is a common characteristic of fractals. In this thesis, we consider fractals with properties such as this. We discuss fractals, including the Cantor set, in greater detail in Appendix A.

Mandelbrot introduced fractal trees in “The Fractal Geometry of Nature” [30]. In general, fractal trees are compact, connected subsets of $\mathbb{R}^n$ that exhibit some kind of branching pattern at arbitrary levels. We study a class of trees that are subsets of $\mathbb{R}^2$. It is important to note that for applications, it would be more realistic to study trees that are subsets of $\mathbb{R}^3$. However, the trees that are subsets of $\mathbb{R}^2$ can be thought of as projections of trees in $\mathbb{R}^3$, so our results about planar trees may still be useful for three-dimensional trees.

A simple type of fractal tree is a binary fractal tree. A binary fractal tree $T(r_1, r_2, \theta_1, \theta_2)$ is specified by four parameters. The first two parameters are the scaling ratios r_1 and r_2, which can take any real values between 0 and 1. The last two parameters are the branching angles θ_1 and θ_2, which can take any real values between $0°$ and $360°$. An intuitive description of a binary fractal tree is as follows. Every tree has a trunk, which is a closed vertical line segment of unit length. This trunk splits into two new branches at the top. One branch has length r_1 and forms an angle of θ_1 with the affine hull of the trunk, and the other branch has length r_2 and forms an angle of θ_2 with the affine hull of the trunk. Each of these two branches forms the trunk of a subtree, *i.e.*, it splits into two more branches, following the same rule. So the branch of length r_1 splits into one branch that has length r_1^2 and forms an angle of θ_1 with the affine hull of the branch of length r_1, and the other has length $r_1 r_2$ and forms an angle of θ_2 with the affine hull of the branch of length r_1. The binary fractal tree is the object obtained by applying the branching process *ad infinitum.*

A surprising result is that we can use binary trees with scaling ratio $r_1 = 1$ to represent some structures that do not look very tree-like. For example, Figures 1.2, 1.3, 1.4 and 1.5 show that the Sierpinski gasket can be obtained as $T(.5, 1, 240°, 240°)$ (see [30] for more details about the Sierpinski gasket). Many open questions remain

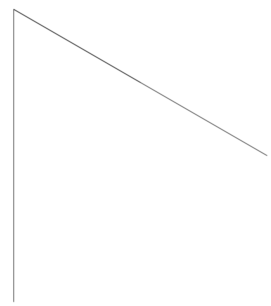

Figure 1.2: First branching iteration of $T(.5, 1, 240°, 240°)$

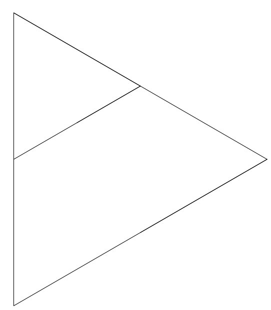

Figure 1.3: Second branching iteration of $T(.5, 1, 240°, 240°)$

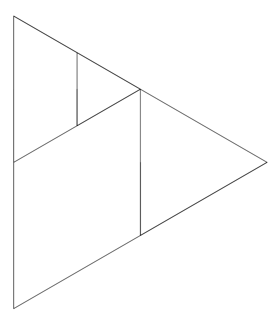

Figure 1.4: Third branching iteration of $T(.5, 1, 240°, 240°)$

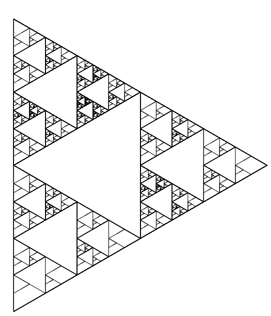

Figure 1.5: The Sierpinski gasket: $T(.5, 1, 240°, 240°)$

regarding binary fractal trees, including a general definition of and criteria for a notion of 'self-contact'.

A particular class of asymmetric binary trees are those with symmetric angles, so with $\theta_2 = 360° - \theta_1$ and $\theta_1 \in (0°, 180°)$. D. Brown *et al.* recently studied the path length and height of equiangular trees (referred to as asymmetric binary trees) in [40]. Another class of binary trees are the isoscalar trees, for which $r_1 = r_2$.

A special class of binary fractal trees is the class of symmetric binary trees. These trees are the trees that we study in this thesis. A symmetric binary fractal tree $T(r, \theta)$ is $T(r, r, \theta, 360° - \theta)$, for some $r \in (0, 1)$ and $\theta \in (0°, 180°)$. In this case, the trunk splits into two branches of equal length, with the same angles to the left and to the right of the trunk. Figures 1.6, 1.7 and 1.8 show three different symmetric binary fractal trees.

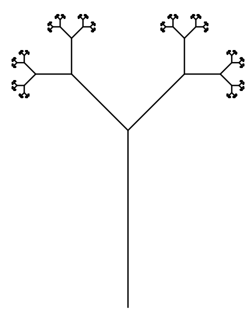

Figure 1.6: A self-avoiding symmetric binary fractal tree: $T(0.45, 45°)$

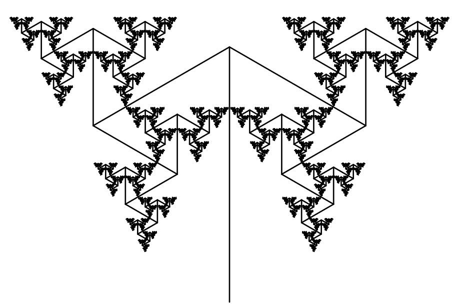

Figure 1.7: A self-contacting symmetric binary fractal tree: $T\left(\dfrac{-1+\sqrt{5}}{2}, 120°\right)$

A binary fractal tree is self-similar in the sense that it is similar to a proper subset of itself. The self-similarity of a symmetric binary fractal tree forms a representation of the free monoid on two generators in the affine maps from $\mathbb{R}^2$ to itself. This concept will be discussed in much greater detail in Chapter 2 of this thesis.

A binary fractal tree may be classified as self-avoiding, self-contacting or self-overlapping. A self-avoiding tree has no self-intersection. In the case of symmetric binary fractal trees, a tree is self-contacting if it has self-intersection but no branch crossings (this is the case when the right subtree contains points with x-coordinate equal to zero, but no points with negative x-coordinates); if it has branch crossings it is self-overlapping (this is the case when the right subtree contains points with negative x-coordinates). For general binary fractal trees, a precise definition of self-contact has not been established.

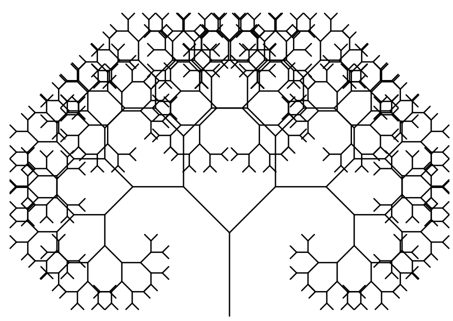

Figure 1.8: A self-overlapping symmetric binary fractal tree: $T(0.78, 45°)$

Further generalizations of binary trees include n-ary trees, trees with branches of non-zero width, trees with more complex branching rules, and trees that are subsets of $\mathbb{R}^3$.

The intuitive concept of fractal tree has been formalized in other ways. Fractals in nature are often a result of a growth process, but mathematical fractals are often seen as static. The binary trees are also the result of an infinite sequence of approximations, but the growth happens in a different fashion, in a more globally defined way. In 1968, the biologist A. Lindenmayer introduced a formalization of the description of plant growth that is also suitable in computer implementations. This formalization is now known as parallel rewriting systems, or L-systems. See [53], [54], [43] and [44] for more information about L-systems and other models. L-systems consist of axioms and production rules.

To demonstrate how L-systems work, we give a basic example. The axiom F

is a horizontal line segment. The production rules are $F \to FfF$ and $f \to fff$. $F \to FfF$ means replace a line segment by three line segments of equal length, of which the middle is not drawn, so that the total length is the same as the original line segment. $f \to fff$ means replace a blank line segment by three blank line segments of equal length, whose total length is the length of the original blank segment. Continue this process *ad infinitum*, and the result is the Cantor set. Any of the trees in this thesis could be realized with an L-system. L-systems can be used to create far more complex fractal trees, see [44]. Figure 1.9 displays a 'bush' created using an L-system.

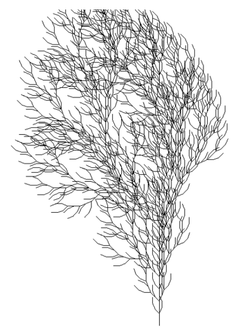

Figure 1.9: L-system bush

1.1.1 Applications and Fractal Dimension

Applications of fractal tree geometries to natural systems provide a great inspiration for this thesis. Although the thesis does not explore any new applications, it does present theory that could yield a mathematical foundation for new applications. We now offer a brief overview of applications of fractal geometries, with the emphasis on fractal tree geometries.

Since the publication of Mandelbrot's "Fractal Geometry of Nature", there have been widespread attempts to use fractal geometry to explain natural phenomena. Indeed, Mandelbrot himself was first inspired by such naturally occurring objects as coastlines and snowflakes. Many objects in nature cannot be completely described in terms of traditional geometric language, they are too complex to be thought of in terms of straight lines and perfect circles. Fractal geometry has provided one way to model objects that have higher levels of complexity, in the sense that there is detail at arbitrary scales, or at least within a certain range. One can find fractal geometry in applications ranging from the very small, such as DNA sequences, to the very large, such as galaxy clustering. The new feature that the fractal geometry provides is a more precise description of the notion of scaling, in the form of the "fractal dimension".

The fractal dimension has proved to be a very useful tool. For example, according to A. Goldberger *et al.*, a healthy heartbeat displays a fractal pattern [19], [23]. In contrast, sick hearts display more predictable, less complex patterns (with lower fractal dimensions). One can use the fractal dimension to measure the variability in heartbeat patterns. Their theory is that fractal variability helps the heart deal with variable situations. So one way to detect and prevent disease is to monitor the variability of the heartbeat. Current research is testing the hypothesis that disease can be treated by restoring variability to a system [19], [23].

Another example of the use of fractal dimension in the medical field is in tumor analysis. The fractal dimension can indicate whether a tumor is cancerous or not, see [9], [18], [55], [11] and [49]. One can measure the fractal dimension of the blood vessels within the tissue of a tumor. Tumor vessels have greater length and diameter than healthy ones, and also contain more loops in the network [11]. There are many publications regarding general fractal geometry applications in nature. See [22] for a more detailed introduction.

A fractal tree is a specific kind of fractal, and fractal trees are an interesting and worthwhile class of fractal to study. There are many examples of branching networks in natural systems which can be modelled using fractal trees. These networks include the cardiovascular and bronchial systems of animals, insect tracheal tubes, and the structure of plants and trees. See [63] and [64] regarding fractal properties of arterial trees and [28] regarding the modelling of blood vessel development. Brain vessel data have been segmented and topologically classified via cubical homology in [39]. See [24] regarding the fractality of general biological tree-like structures. Fractal trees can also be used as a model for drainage systems [16].

It is difficult to study the blood vessels of an organism, because the vasculature is

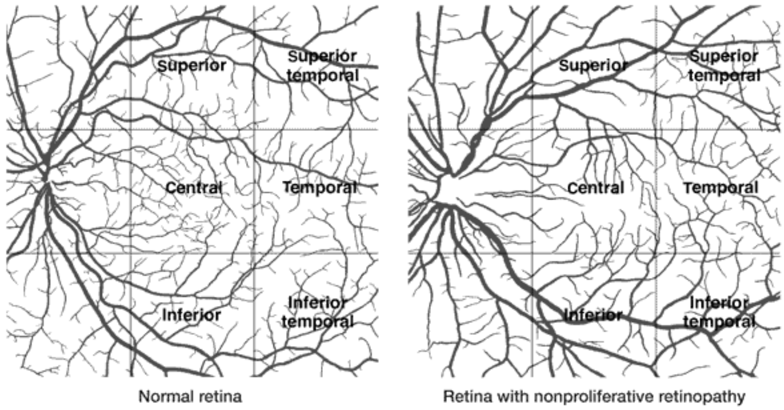

Figure 1.10: Comparison of normal retina and retina with non-proliferative retinopathy (from [2])

a complex, tree-like structure that is embedded in three-dimensional opaque tissue. Researchers use fractal geometry to measure and model the morphological stimulation and inhibition of blood vessel growth [41], [2]. For example, fractal dimension analysis has been used to measure the decrease of blood vessels in early-stage diabetic retinopathy (right side of Figure 1.10) compared with healthy blood vessels in the normal retina (left side of Figure 1.10).

Recent research involving fractal trees offers an explanation for the quarter-power scaling laws that manifest throughout biology [59]. As mentioned by H. Kurz and K. Sandau [29], "The design of living beings is not only a matter of molecular biology but also of geometry and physics". Variables such as life-span, age at first reproduction, and duration of embryonic development all share the property of being proportional to the mass of the organism raised to the one quarter power. This relationship holds in almost all organisms, from microbes to higher plants and animals, and has been a long-standing mystery to biologists. Metabolic rate varies in proportion to the 3/4 power of an organism's mass. The larger the animal, the slower its metabolism. If metabolic rate reflected only geometric constraints, then 1/3 powers would be more logical, due to the 3-dimensionality of living organisms.

In [59], G.B. West, J.H. Brown and B.J. Enquist first put forth an explanation for these power-laws based on fractal geometry. Their argument is that living things are sustained by the transport of materials through space-filling fractal networks of branching tubes. The three assumptions of their model [59] are:

- A space-filling, fractal-like branching pattern is required of the network to be able to supply the entire volume of the organism
- The final branch of the network is a size-invariant unit
- The energy required to distribute resources is minimized

Based on these three assumptions, the researchers developed a model for the design of distribution networks that incorporates both fractal geometry and hydrodynamics. The model predicted values for the scaling of structural and functional variables that were more in agreement with measured values than any other model before. The researchers claim that fractal geometry adds the fourth dimension. The authors of [59] continue to test and expand their controversial theory. See [60], [29], [61], [5] for more information.

The fractal dimension is a main characteristic of fractal-like objects. However, it does not fully characterize a fractal. We can obtain fractals with the same dimension that are quite different topologically. Consider the Sierpinski gasket, and its relatives, as described in [42] and [32]. The three similarities that define the Sierpinski gasket each map a square to a smaller version of itself, with side lengths equal to one half the side length of the original square. The similarities of a Sierpinski relative each map a square to three smaller versions scaled by one half, and may also involve the reflection and rotation symmetry properties of the square. All Sierpinski relatives have the same dimension, which is $\ln 3/\ln 2$. Topologically, these fractals can be dusts (totally disconnected), dendrites (simply-connected, one curve with no loops), multiply-connected (connected with loops) or hybrids (infinitely many components each containing a curve). See Figure 1.11, which displays the Sierpinski gasket along with three relatives. The Sierpinski gasket and the fourth relative are multiply-connected, the second relative is a dendrite, and the third is a hybrid. There are 456 distinct Sierpinski relatives [42].

Just as there are many advocates for the use of fractal geometry to model natural objects, there are many skeptics. One source for skepticism is the heavy reliance on

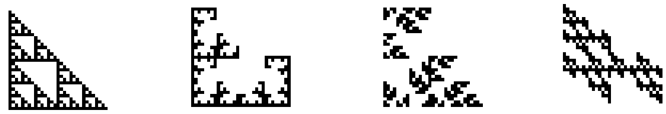

Figure 1.11: The Sierpinski Gasket and Three Relatives

the fractal dimension in the literature to date. However, there are other ways to characterize a fractal. One example is lacunarity, which attempts to distinguish between fractals with the same dimension and distinct coarse-scale structure. Lacunarity is a measure for the distribution of the holes of various sizes in a fractal, it is a way to describe the texture. Unfortunately, there is not a precise definition of lacunarity that is widely-accepted and applicable. See [1], [33] and [17] for more information about lacunarity.

There is a strong need for new ways to characterize and classify fractals. Consider the following quote from H. Kurz [27]:

"*My personal encounter with fractal geometry always was extremely stimulating and rewarding. But, alas, after having gained quite an experience in applying fractal geometry, I found that not much could be learned about the mechanistic links between physical forces and the emergence of biological form. So I sometimes feel inclined to follow Wittgenstein's advice who recommended to throw away a ladder once it had been used to climb up on it upon a new level.* **Unless new concepts for defining and measuring fractals are developed and carefully applied** *[our emphasis] I would not expect much progress for developmental or evolutionary biology from current main-stream fractal thinking. I eagerly await disproval.*"

Our goal is to provide new ways to characterize and classify fractals. While this is important from a purely mathematical point of view, it will also be exciting to see how the theory will be applied to biological theories, and more general theories of natural systems.

1.2 Topology and Fractal Trees

This thesis uses topology to develop new ways to characterize fractal trees. What is topology? Generally it involves a type of geometry that ignores concrete spatial notions such as straightness, convexity, and distance. Topology considers properties such as connectivity (how many components, are there any holes, etc) and continuity. See [38], [20], [52] and [26] for more details about topology.

Mandelbrot and Frame classified all symmetric binary self-contacting fractal trees according to the topological type of the canopy of the tree. They identify two branching angles as topologically critical; these angles are $90°$ and $135°$ [31]. Mandelbrot and Frame restrict their attention to the self-contacting trees, while our work studies both the self-avoiding and self-contacting trees. The self-avoiding symmetric binary fractal trees are all simply-connected and they are all topologically equivalent. On the other hand, the self-contacting trees are infinitely complicated in the sense that they have infinitely many holes (with the two exceptions of the two space-filling contractible self-contacting trees with branching angles $90°$ and $135°$). However, they are topologically equivalent in the ways described by Mandelbrot and Frame. We discuss the general homeomorphism classes of non-overlapping binary trees in Section 3.7 of this thesis. The homology type of the self-contacting trees is too complicated though. So it seems that from a topological point of view, symmetric binary fractal trees are either too trivial or too complex for these invariants to describe them. For this reason, we choose to study the fractal trees together with the way they are embedded in $\mathbb{R}^2$, and to look at certain well-behaved subsets of $\mathbb{R}^2$ that represent the metric structure of these trees. This idea comes from shape theory, where one can study a complicated space, such as a fractal, by studying an inverse system of well-behaved spaces which in some sense approximate the original space. For our research, we use ideas from computational topology to create an inverse system of well-behaved spaces.

Computational topology is the study of topological properties that can be computed requiring data and computations with only finite accuracy. There are many applications of computational topology, such as digital image processing, cartography, computer graphics, solid modelling, mesh generation and molecular modelling. See [12] for a general survey of computational topology. One of the common methods employed in computational topology is to embed an object or space in a larger space, and to study the connections between the space and the embedding.

Algebraic topology is the study of algebraic objects attached to topological spaces.

The algebraic invariants reflect some of the topological structure of the space. The algebraic tools include various homology and cohomology theories, homotopy groups, and groups of maps. See [38], [35], [37], and [48] for more information about algebraic topology.

A particular branch of algebraic topology that is important for this thesis is homology theory. In general, homology theory attempts to distinguish between objects, which will be called spaces in this context, by constructing algebraic and numerical invariants that are related to the connectivity of the space. The origins of homology are in the work of Poincaré. He thought of homology as a relation between manifolds mapped into a manifold. One manifold forms a homology when it forms the boundary of a higher-dimensional manifold inside the other manifold. Poincaré simplified the spaces by using triangulations. He looked at the subcomplexes instead of general objects.

The most basic homology theory is simplicial homology, which is based on the triangulations of spaces, see [37]. Given a triangulation of a space, the homology groups can be calculated using an algorithm based on linear algebra, which in general has rather poor numerical behaviour. However, in many applications it is the rank of the homology group that is needed, not the entire group structure. This rank is represented by the Betti number and is more easily computable.

The problem with fractals is that they require infinitely many simplices in their triangulations, and thus at least one of their non-trivial homology groups would be infinite. This is not possible with simplicial homology, so something else needs to be done for spaces like fractals which have infinitely detailed structure.

Instead of applying homology theory to the original object X, one can apply it to derived spaces (arising from some kind of embedding of the original space) that have a simpler geometric structure. This idea has recently been used by Gunnar Carlsson *et al.* [6], [66], [7]. Their research involves a "study of shape description using a marriage of geometric and topological techniques" [7]. Their derived spaces are constructed using tangential information about the underlying space X as a subset of $\mathbb{R}^n$. First, they have defined the **tangent complex** as the closure of the space of all tangents to all points of X. Homology of the tangent complex can be used to detect sharp features such as edges and corners. To distinguish the soft features, one needs the so-called **filtered tangent complex**. An invariant called a **persistence module** is obtained by applying homology to this filtered tangent complex. The filtered tangent complex can be used to distinguish between a circle and an ellipse, something that wasn't possible with the ordinary tangent complex. Persistent homology is used to define a simple shape descriptor, called the **barcode**. The barcode is a combinatorial invariant that possesses information about the shape of an object.

Another example of a derived space is the closed ϵ-neighbourhood of the space. Suppose X is a subset of a complete metric space, such as $\mathbb{R}^n$ for some $n \geq 2$. Then for any $\epsilon > 0$, the closed ϵ-neighbourhood of X, denoted by $\overline{X_\epsilon}$, is the closed subset of $\mathbb{R}^n$ that consists of all points that are within a distance of ϵ to X. We shall discuss the closed ϵ-neighbourhoods of fractal trees in the next subsection.

1.2.1 Inverse Systems, Čech Homology and Shape Theory

There are two ways to generalize the concept of limit to general index sets and spaces: direct and inverse limit systems. See [51] for more details.

Definition 1.2.1.1 *An* **inverse system** *of topological spaces consists of a directed set* $(\Lambda, \gtrsim)$*, a family* $(X_\lambda)_{\lambda \in \Lambda}$ *of topological spaces, and continuous mappings*

$$p_{\lambda\mu} : X_\mu \to X_\lambda,$$

for each pair $\mu \gtrsim \lambda$*. The maps are called* **bonding morphisms** *and must satisfy the following two conditions:*

$$p_{\lambda\lambda} = 1_{\lambda\lambda} \text{ (Identity map on } X_{\lambda\lambda}) \tag{1.2.1}$$

$$p_{\lambda\mu}p_{\mu\nu} = p_{\lambda\nu} \text{ (for any choice of } \nu \gtrsim \mu \gtrsim \lambda) \tag{1.2.2}$$

The system is called an 'inverse' system because the bonding morphisms act against the order relation.

Notation. Let $\mathbf{X} = (X_\lambda, p_\lambda, \Lambda)$ denote the inverse system.

Definition 1.2.1.2 *The* **inverse limit space***, denoted by* $\lim\limits_{\leftarrow}\mathbf{X}$*, is the subspace of* $\Pi_\Lambda X_\lambda$ *defined by*

$$\lim_{\leftarrow}\mathbf{X} := \{(x_\lambda)|x_\lambda \in X_\lambda \text{ and } p_{\lambda\mu}(x_\mu) = x_\lambda \text{ for } \mu \succsim \lambda\}. \tag{1.2.3}$$

Definition 1.2.1.3 *The* **projections** $p_\mu : \lim_{\leftarrow}\mathbf{X} \to X_\mu$ *are the continuous maps* $p_\mu((x_\lambda)) = x_\mu$.

One can simplify an inverse limit calculation by using a cofinal subset Λ' of Λ (where cofinal means that for any $\lambda \in \Lambda$, there is $\mu \in \Lambda'$ with $\mu \succsim \lambda$). This leads to the following theorem:

Theorem 1.2.1.4 ([51]) *If* Λ' *is a cofinal subset of* Λ*, then the inverse systems* $(X_\lambda, p_{\lambda\mu}, \Lambda)$ *and* $(X_\lambda, p_{\lambda\mu}, \Lambda')$ *have isomorphic limits. That is, their inverse limit spaces are homeomorphic.*

One can similarly define an inverse limit system for the category of groups. Then the bonding morphisms are group homomorphisms and the inverse limit is a subgroup of the direct sum $\bigoplus_\Lambda G_\Lambda$ of the groups of the inverse system. This will be useful for Čech homology.

Čech homology is a more general homology theory than simplicial homology, and for us it is relevant because it can reflect the infinitely detailed structure of a fractal. For finite simplicial complexes, Čech homology agrees with simplicial homology. For further information see [20] or [34]. The foundation of Čech homology is the nerve of a cover to generate simplicial complexes.

Definition 1.2.1.5 *Let* X *be a compact Hausdorff space, and let* $\Sigma(X)$ *denote the family of all finite open coverings of* X*. For a covering* $\mathcal{U} \in \Sigma(X)$*, the* **nerve of the cover** *is constructed by associating each open set* $U \in \mathcal{U}$ *with a vertex, labelled* U*, in the complex. An edge exists between two vertices* U *and* V *if and only if* U *and* V *have non-empty intersection. Higher dimensional simplices are similarly defined,* $(U_{ij})_{j=1}^n$ *is an* n*-simplex when* $\bigcap\limits_{j=1}^{n} U_{ij} \neq \emptyset$.

Definition 1.2.1.6 *A cover* $\mathcal{V}$ *is a* **refinement** *of* $\mathcal{U}$*, denoted* $\mathcal{V} \succsim \mathcal{U}$*, if for any set* $V \in \mathcal{V}$*, there is a set* $U \in \mathcal{U}$ *such that* $V \subseteq U$*. We say that* $\mathcal{V}$ *refines* $\mathcal{U}$*. This partial order relation* $\succsim$ *makes* $\Sigma(X)$ *a directed set.*

The family of all covers with the refinement ordering is the index set for the Čech inverse system. Next we need bonding morphisms.

One can define a projection map $p_{\mathcal{U}\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$ by taking the image of a set $V \in \mathcal{V}$ to be any fixed element $U \in \mathcal{U}$ such that $V \subset U$. The projection map is not necessarily unique. Thus the bonding morphisms are the homotopy equivalence classes of projections $p_{\mathcal{U}\mathcal{V}}$ (two maps are homotopic and of the same homotopy equivalence class if there exists a continuous transformation from one to the other). The **Čech system** is the inverse system $(\mathcal{U}, p_{\mathcal{U}\mathcal{V}}, \Sigma(X))$.

A finite cover $\mathcal{U}$ is a simplicial complex, so we can compute its simplicial homology. Now, however, the coefficient group must be more general than the integers $\mathbb{Z}$.

If one considered inverse limits of approximating spaces instead of nerves of covers, the result would be a generalization of Čech homology. This is part of a general theory called shape theory. See [34] for more information. An important result from shape theory is that every compact metric space is homeomorphic to the limit of an inverse system in the category of finite polyhedra and homotopy equivalence classes of maps. An example of such a system is the Čech system. The generalization in shape theory comes from allowing the approximating spaces to be homotopy equivalent to a finite polyhedron. In fact, for any other inverse system of polyhedra, the corresponding inverse system of homology groups has a limit that is isomorphic to Čech homology.

1.2.2 Inverse Limits and Closed Epsilon-Neighbourhoods

The particular inverse systems that we investigate arise from considering closed ϵ-neighbourhoods.

Given a compact space X that is a subset of a complete metric space (M, d) and $\epsilon \geq 0$, we have the closed ϵ-neighbourhood:

$$\overline{X_\epsilon} = \{x \in M | d(x, X) \leq \epsilon\}$$

Now we have an inverse system of closed ϵ-neighbourhoods, indexed by $0 \leq \epsilon \leq \epsilon_0$, for some suitable value of ϵ_0. The set X can be considered a metric inverse limit in the sense of M. Moszyńska [36]. Let $\{\epsilon_n\}$ be any monotonically decreasing sequence of positive real numbers whose limit is 0 ($\{1/n\}$ is an example). Let $X_n = \overline{X_{\epsilon_n}}$. Then $\{X_n\}$ is a decreasing sequence of non-empty compact subsets of $\mathbb{R}^2$. Let $p_n^{n+1} : X_{n+1} \to X_n$ denote the inclusion maps. Then $\mathbf{X} = (X_n, p_n^{n+1})$ is a geometric sequence and

$$\lim_{\twoheadleftarrow} \mathbf{X} = \bigcap_{n=1}^{\infty} X_n = \lim_{H} X_n$$

where $\lim\limits_{\twoheadleftarrow}$ denotes the metric inverse limit and $\lim_H$ denotes the Hausdorff limit [36]. For this thesis, the spaces we consider are fractal trees.

Since we can think of a space as the limit of its closed ϵ-neighbourhoods as $\epsilon \to 0$, we are concerned with what happens as ϵ gets closer to 0. So the order relation will be inverted: we say $\lambda \succsim \epsilon$ when $\lambda \leq \epsilon$. It is easy to see that $X_\lambda \subset X_\epsilon$ whenever $\lambda \leq \epsilon$, so for the bonding morphisms we can take the inclusion maps $p_{\epsilon\lambda} : X_\lambda \hookrightarrow X_\epsilon$. Note that if $0 < r < 1$ and $\epsilon_0 > 0$, the decreasing sequence $\{r^n \epsilon_0\}_{n \geq 0}$ yields a cofinal sequence of closed ϵ-neighbourhoods.

The use of closed ϵ-neighbourhoods allows us to give a finer, more interesting classification of symmetric binary fractal trees than the straight topological one given by Mandelbrot and Frame [31]. We can also use the closed ϵ-neighbourhoods to measure how the neighbourhoods of subsystems such as subtrees interact with one another. The self-avoiding fractal trees, which by themselves are all topologically equivalent, can now be associated with systems of closed ϵ-neighbourhoods that possess vastly different topological properties, depending on the branching angle and scaling ratio of the original tree. It is possible for a closed ϵ-neighbourhood of a self-avoiding tree with a specific scaling ratio and branching angle to have non-trivial homology. Since the closed ϵ-neighbourhoods are connected, compact subsets of $\mathbb{R}^2$, determining the homology reduces to counting holes. The number of holes is equal to the rank of the first homology group. To see how it is possible for a simply-connected space to have a multiply-connected closed ϵ-neighbourhood, consider the image in Figure 1.12.

The main idea of the thesis is studying holes in closed ϵ-neighbourhoods. If ϵ scales according to the branch length, then around a tip point (a point only reached after infinitely many branching iterations) the closed ϵ-neighbourhood is just the point

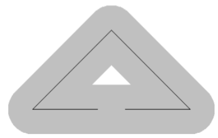

Figure 1.12: Example of a Multiply-connected Closed ϵ-neighbourhood of s Simply-connected Subset of $\mathbb{R}^2$

itself. To determine the minimum ϵ to get contact in the neighbourhood just depends on branches, not tip points at infinity. This approach ignores a great deal of the structure that the fractal trees possess, as we shall see in the different types of holes that arise due to tip points. Perhaps this approach would be more suitable for finite trees. This is something we are currently investigating.

The closed ϵ-neighbourhoods have recently been used by other researchers to study fractals. The main example that we reference is the research of V. Robins *et al.* [45], [46], [47]. Their research presents a study of the extrapolation of topological information about the structure of a space from a finite set of data points. They assume that the underlying set, X, is a compact metric space, and the data, S, are a finite set of points that approximate X. A finite set of points has trivial topological structure. The basic approach is to determine the topological properties of the closed ϵ-neighbourhoods of S as $\epsilon \to 0$, and to extrapolate this information to investigate the connectivity and homology of the underlying set X.

Previous work by Robins *et al.* has focused on holes in the closed ϵ-neighbourhoods that correspond to a hole in the underlying space. The problem is to identify which holes in the closed ϵ-neighbourhoods do correspond to such holes. Persistent Betti numbers count the number of holes that persist in the epsilon-neighbourhood for a certain range of epsilon values [45], [46]. When X has fractal structure, it is possible to see unbounded growth in the persistent Betti numbers as $\epsilon \to 0$, so one can characterize this growth by assuming an asymptotic power law.

In an attempt to only count holes in a closed ϵ-neighbourhood that are generated by a hole in the underlying space, the notion of a persistent Betti number was formulated [45], [46]. For $\lambda \leq \epsilon$, the persistent Betti number β_k^λ is an integer-valued function of two real numbers: λ and ϵ. An equivalence class of cycles $[z_\epsilon] \in H_k(\overline{X_\epsilon})$ **persists**

in $H_k(\overline{X_\lambda})$ if it is in the image of the bonding homomorphism $[z_\epsilon] \in p_{\epsilon\lambda^*}(H_k(\overline{X_\lambda}))$. Then

$$\beta_k^\lambda(\overline{X_\epsilon}) = \text{rank}(p_{\epsilon\lambda^*}(H_k(\overline{X_\lambda}))) \tag{1.2.4}$$

The persistent Betti numbers are necessary for a proper classification of the underlying topology. A classic example to demonstrate this point is Antoine's necklace, see Figure 1.13. The idea of the necklace is that each link (a hollow torus) is broken into a necklace consisting of smaller links, *ad infinitum*. The resulting set is homeomorphic to Cantor dust, and is totally disconnected. Thus the necklace has a trivial first homology group. However, for any $\epsilon > 0$, the closed ϵ-neighbourhood of the necklace does not have a trivial first homology group, because there are holes. As ϵ decreases towards 0, the number of holes in the corresponding closed ϵ-neighbourhood has unbounded growth. Thus the limit of the regular Betti numbers as ϵ goes to 0 does not equal the Betti number of the actual necklace (which is 0). In fact, none of the holes in a given approximation persists even to the next approximation of the necklace.

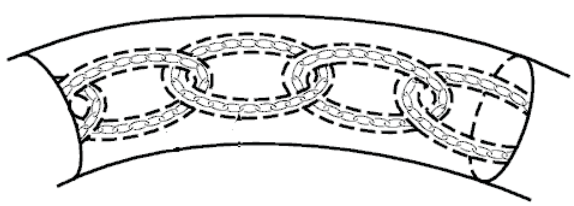

Figure 1.13: Antoine's Necklace [56]

Not surprisingly, if the underlying set is a self-similar fractal, there is a connection between the similarity dimension and the number of holes as $\epsilon \to 0$. If $\beta_k^0(\overline{X_\epsilon}) \to \infty$ as $\epsilon \to 0$, then it is natural to quantify the divergence by assuming an asymptotic power law:

$$\beta_k^0(\overline{X_\epsilon}) \sim \epsilon^{\gamma_k}.$$

The exponent γ_k can be found by the following limit (if it exists):

$$\gamma_k = \lim_{\epsilon \to 0} \frac{\log \beta_k^0(\overline{X_\epsilon})}{\log(1/\epsilon)}. \tag{1.2.5}$$

Robins gives the following conjecture:

Conjecture 1.2.2.1 *If X is a self-similar fractal and $\gamma_i \neq 0$, then*

$$\gamma_i = s$$

where s is the similarity dimension of the fractal. (See A.2.3 for the definition of the similarity dimension.)

Our own work supports this conjecture. In Chapter 6 of this thesis, we discuss examples of trees for which the growth rate of holes in the closed ϵ-neighbourhoods equals the similarity dimension of the tree. In Theorem 7.1.4.3 of Chapter 7 we prove the conjecture for holes in the closed ϵ-neighbourhoods of any tree. Specifically, we prove that for a non-simple tree with scaling ratio r (for which there exist closed ϵ-neighbourhoods that are multiply-connected) and a sequence $\{\epsilon_n\}$ defined by $\epsilon_n = r^n\epsilon_0$ such that $\epsilon_0 > 0$ and there are a finite number of holes in the closed ϵ-neighbourhoods for all ϵ_n, the growth rate of holes

Theorem. Let $T(r, \theta)$ be a non-simple tree. Let $\epsilon_0 > 0$ be such that $E(r, \theta, r^n\epsilon_0)$ has a finite number of hole classes for all $n \geq 0$. For the sequence $\{\epsilon_n\}$ defined by $\epsilon_n = r^n\epsilon_0$, the growth rate of holes is equal to $\dfrac{\log 2}{\log 1/r}$.

Our basic assumptions are different from the work discussed above [45], [46]. Though the closed ϵ-neighbourhoods do provide the basis for our study of fractal trees, we consider the closed ϵ-neighbourhood of the actual fractal tree, not of a finite approximation to the tree. Future work will include a thorough study of the finite approximations of fractal trees, along with a comparison between the actual trees and their finite approximations (as mentioned above). We do study the topological properties of the closed ϵ-neighbourhood of a given tree as a function of ϵ, and consider how the topological properties depend on the two parameters r and θ of a symmetric binary fractal tree. This enables us to obtain different invariants, classifying certain classes of trees. So although the utilization of closed ϵ-neighbourhoods is not new, our goals and methods are quite different from those used in the literature to date.

The fractal trees that we study are either simply-connected (if self-avoiding or space-filling and self-contacting), or have an infinite-dimensional homology (if self-contacting and not space-filling). Contrary to the the earlier work [45], [46], we are interested in all holes that arise in a closed ϵ-neighbourhood, not just ones that are due to holes in the underlying tree. Our goal is not just to classify the topology of the underlying fractal tree, but to use the closed ϵ-neighbourhoods to gain finer invariants to compare trees with different parameters. We are also interested in persistence as another characteristic of the holes. In this case, persistence says something about the size of the holes, and it can be considered a topological way to describe 'lacunarity'. Persistence of holes also reflects how 'space-filling' the holes are, and this characteristic may be particularly useful for applications. For example, consider the two retina images given in Figure 1.10. The persistence of holes in the healthy retina would be smaller than the persistence of holes in the non-healthy retina.

1.3 Overview of Thesis

We now provide a brief overview of the thesis.

This thesis presents a study of non-overlapping symmetric binary fractal trees using methods of computational topology. Motivated by techniques from shape theory and computational topology we will be considering these trees along with their closed epsilon-neighbourhoods as ϵ ranges over the non-negative real numbers. We investigate various features of the closed ϵ-neighbourhoods, based on the holes in these neighborhoods. A fractal tree can be seen as a representation of the free monoid M_{LR} on two generators L and R, and we use this fact to describe many of the scaling features of the trees and their closed ϵ-neighbourhoods.

We determine the hole sequence (the sequence of the number of holes as ϵ varies from infinity to zero) of these trees together with the persistence intervals of the holes (the values of ϵ for which a particular hole is part of the closed ϵ-neighborhood) as the 'topological barcodes' (as defined by Carlsson *et al.* [7]) of these trees. We find that the notion of persistence has some interesting and perhaps unexpected properties in this context.

Due to the nice geometric nature of the trees, we can refine our approach by classifying holes according to their shape and location in the tree. The action of M_{LR} on the tree brings a natural grading by level to these holes. We will see that the level 0 holes form a kind of fundamental domain, and we can restrict our attention to the level 0 holes.

To describe the location of a hole, we have generalized the notion of contact address (for self-contacting trees) to hole locator address and hole locator pairs. From these notions and properties we derive several classifications of the symmetric binary fractal trees. These are the complexity, location, type and hole sequence classifications. They lead to the determination of certain critical values for the angle θ, the scaling ratio r (as a function of θ) and ϵ (as a function of r and θ).

The methods developed in this thesis could be extended to other classes of fractal trees and more general fractals. Other future work could include a study of the connections between our theory and applications of fractal tree geometries to natural systems.

Chapter Summaries:

In Chapter 2, we give precise definitions for the generator maps that we use to define a tree as a representation of the free monoid on two generators, and present basic notations and results about the trees. The important notion of 'level' is introduced.

We continue the discussion on symmetric binary fractal trees in Chapter 3. This chapter discusses various properties of the trees, beginning with height, width and relative size. The main part of this chapter deals the notion of self-contact. This includes definitions of self-avoidance, self-contact, and self-overlap; criteria for self-contact; and methods of finding the unique scaling ratio for a given angle. Different types of points of a tree are discussed: contact, secondary contact, corner, top vertex and canopy. We give descriptions of the boundaries of holes for self-contacting trees. The homeomorphism classes of non-overlapping trees are discussed. All self-avoiding trees are homeomorphic, and non-space-filling self-contacting trees are homeomorphic if and only if they have the same self-contact addresses.

In Chapter 4, we finally discuss closed ϵ-neighbourhoods of trees. This chapter presents basic notations, definitions and results regarding closed ϵ-neighbourhoods of trees and other sets, and holes in a given closed ϵ-neighbourhood. We define a hole class, along with the persistence interval and complexity of a hole class. Important theorems regarding symmetry and level of holes are presented, along with criteria and methods for finding holes. Other characteristics related to holes of closed ϵ-neighbourhoods are given, such as number of holes and the location of a hole.

Chapter 5 expands the theory on closed ϵ-neighbourhoods of trees. First we continue the discussion on hole locations, by studying hole class location as a function of branching angle and using hole location sets to compare trees or angles. The location of a hole class can be used to identify the type of the hole class. Based on the persistence intervals of the hole classes of a give tree, we define the critical set of ϵ-values for the tree, along with the hole partition of ϵ values and the corresponding hole sequence. We present four different ways to classify the set of self-avoiding and self-contacting trees based on location, type, hole sequence, and complexity. We also discuss topological critical values. For a given angle, there are critical scaling ratios. For the entire set of self-avoiding and self-contacting trees, there are critical angles.

Chapter 6 presents a collection of different examples of trees and their closed ϵ-neighbourhoods, to clarify the theory and illustrate the rich, varied structure that the trees possess. The four 'golden trees' are discussed (these special trees have self-contacting scaling ratio equal to $1/\phi$, where ϕ is the golden ratio). The golden trees are particularly interesting, because of their additional symmetry properties, and various properties of their closed ϵ-neighbourhoods.

The thesis concludes with Chapter 7. Based on the examples, we discuss the theory on a deeper level. In particular, we discuss critical values of the parameters. We summarize the main results and state some of the immediate questions that are unresolved. We also give a broad picture of future work.

Chapter 2

Symmetric Binary Fractal Trees

In this chapter, we introduce the fractal trees that are the main object of study for this thesis. These are the self-contacting and self-avoiding symmetric binary trees as introduced by Mandelbrot and Frame in [31]. We gave an intuitive description of these trees in the introduction of this thesis. We also introduced the idea of a 'self-contacting' tree, which is a tree that exhibits self-intersection but not branch overlap. Precise definitions and conditions for self-avoidance, self-contact and self-overlap are presented in the next chapter. For now we only mention that for every branching angle θ, there exists a unique scaling ratio r that yields a self-contacting tree [31]. The two angles $90°$ and $135°$ are identified in [31] as being topological critical points for the class of self-contacting symmetric binary fractal trees. The corresponding self-contacting trees for the angles $90°$ and $135°$ are the only self-contacting trees that are space-filling. Self-contacting trees with angles in the same angle range, $(0°, 90°)$, $(90°, 135°)$, or $(135°, 180°)$, are homeomorphic, *i.e.*, have the same topological type [31]. Mandelbrot and Frame restrict their attention to the self-contacting trees, though they do claim that "the structure of self-avoiding and self-contacting trees is instructive and entertaining" [31]. One of the main results of our research is that we identify other critical branching angles, and furthermore, for each branching angle we identify critical values of the scaling ratios apart from the self-contacting scaling ratios.

The symmetric binary fractal trees possess a high level of symmetry and scaling, and this proves to be very useful in the geometric analysis of the closed ϵ-neighbourhoods that we use to define additional topological properties of the trees. To highlight the symmetric properties of the trees, we have developed a new way to present the symmetric binary fractal trees. This is in terms of the free monoid on two generators. To a given tree, *i.e.*, to a given pair (r, θ), we associate two generator maps. These maps act on compact subsets of $\mathbb{R}^2$ such that the image of the trunk under one map is the first branch on the left side, while the image of the trunk under the other map is the first branch on the right side. A tree is then the union of the images of the trunk under all possible maps that are formed from composition on the two generators maps.

These generator maps map parts of the tree to subsets of those parts. In particular, the image of the whole tree under one of the maps is the left subtree, while the image under the other map is the right subtree. In this chapter we will further extend this to give addresses for various subtrees. The generator maps also help us to identify certain paths in the tree by considering the images of the trunk under a certain sequence of compositions of generator maps.

In this chapter, assume that

$$r \in (0,1) \quad \text{and} \quad \theta \in (0^\circ, 180^\circ). \tag{2.0.1}$$

2.1 Trees as a Representation of the Free Monoid on Two Generators

Recall that a **monoid** $(M, *)$ is a set with an associative binary operation and a unique unit element $e \in M$, *i.e.*, $(m_1 * m_2) * m_3 = m_1 * (m_2 * m_3)$ and $e * m = m = m * e$.

Examples

1. For any space X with a collection $\mathrm{Hom}\,(X, X)$ of transformations from X to itself which is closed under composition and contains the identity mapping, we can give the set $\mathrm{Hom}\,(X, X)$ the structure of a monoid where the operation is composition $\varphi * \psi = \psi \circ \varphi$ in the opposite order and the unit element is the identity mapping. For example, the collection $\mathrm{Sim}\,(\mathbb{R}^n, \mathbb{R}^n)$ of similarities in $\mathbb{R}^n$ forms a monoid, and so does the set $\mathrm{Con}\,(\mathbb{R}^n, \mathbb{R}^n)$ of all non-expanding similarities.

2. The **free monoid with n generators** $\{R_1, R_2, \cdots, R_n\}$ is the set of all 'words' in this alphabet, *i.e.*, strings of finite length with elements in $\{R_1, R_2, \cdots, R_n\}$. The monoid operation is defined as concatenation of strings and the unit element is the empty string.

A **representation** of a monoid M is a monoid map $\varphi\colon M \to \operatorname{Hom}(X, X)$ for a space X with a collection of transformations which forms a monoid. The fact that φ is a monoid map means that $\varphi(m_1 * m_2) = \varphi(m_2) \circ \varphi(m_1)\colon X \to X$ and $\varphi(e) = \mathrm{id}_X$, the identity map on X.

For the purpose of this thesis, we are interested in representations of the free monoid M_{LR} on two generators $\{L, R\}$ into $\operatorname{Con}(\mathbb{R}^2, \mathbb{R}^2)$. Note that any representation $\varphi\colon M_{LR} \to \operatorname{Con}(\mathbb{R}^2, \mathbb{R}^2)$ is completely determined by $\varphi(L)$ and $\varphi(R)$, since φ of any other word is simply the corresponding composition of $\varphi(R)$s and $\varphi(L)$s.

2.1.1 The Generator Maps m_R and m_L

Definition 2.1.1.1 *For any pair (r, θ) with $0 < r < 1$ and $0° < \theta < 180°$, we define the representation*

$$m(r, \theta)\colon M_{LR} \to Con(\mathbb{R}^2, \mathbb{R}^2)$$

in the following way. The similarity map $m(r, \theta)(L)\colon \mathbb{R}^2 \to \mathbb{R}^2$, also denoted by $m_L(r, \theta)$ or just m_L when r and θ are obvious from the context, is defined by

$$m_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.1.1}$$

Similarly, $m(r, \theta)(R)$, or $m_R(r, \theta)$ or m_R for short, is defined by

$$m_R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.1.2}$$

What do the generator maps actually do? Given any subset U of $\mathbb{R}^2$, a generator map rotates U around the origin over an angle of θ clockwise (m_R) or counterclockwise (m_L), contracts the resulting set by a factor of r and then translates the set up one unit. So the generator maps are compositions of a rotation, contraction, and a translation. The following lemma is a natural consequence of the definition of the generator maps.

Lemma 2.1.1.2 *Let $A \in \{R, L\}$, and let C be any compact subset of $\mathbb{R}^2$. Then the set $U = m_A(C)$ is a compact subset of $\mathbb{R}^2$ that is similar to C with contraction factor r.*

Proof. This lemma follows directly from the definition of the generator maps.

Observation. For any set U that is symmetric about the y-axis, the images $m_R(U)$ and $m_{\mathrm{L}}(U)$ will be symmetric to each other in the sense that $m_L(U)$ is equal to the reflection of $m_R(U)$ across the y-axis. This property of the generator maps is the source of the left-right symmetry of the symmetric binary fractal trees.

2.1.2 Addresses

We are going to call the elements of the monoid $< L, R >$ 'addresses' and then define a grading of the monoid by 'level' (*i.e.*, number of symbols in an address). We then add the infinite addresses by defining them in terms of finite level approximations.

Definition 2.1.2.1 *A* **finite address** *is a finite string of symbols* $\mathbf{A} = A_1A_2\cdots A_k$*, for some positive integer* k*, where* $A_i \in \{R, L\}$ *for* $1 \leq i \leq k$*. We use bold-faced, upper case letters to denote addresses.*

Definition 2.1.2.2 *A finite address has* **level** k *if it contains* k *elements in its string. The level is denoted* $l(\mathbf{A})$*.*

Definition 2.1.2.3 *For a positive integer* k*,* $\mathcal{A}_k$ *denotes the set of all addresses of level* k*.*

$$\mathcal{A}_k = \{A_1A_2\cdots A_k \mid A_i \in \{R, L\},\ 1 \leq i \leq k\} \tag{2.1.3}$$

The **empty address** *is denoted by* $\mathbf{A}_0$*, and* $\mathcal{A}_0 = \{\mathbf{A}_0\}$*.*

Definition 2.1.2.4 *An* **infinite address** *is an infinite string of symbols* $\mathbf{A} = A_1A_2\cdots$*, where* $A_i \in \{R, L\}$*, for all positive integers* i*. Let* $\mathcal{A}_\infty$ *denote the set of all infinite addresses.*

$$\mathcal{A}_\infty = \{A_1A_2\cdots \mid A_i \in \{R, L\},\quad 1 \leq i\ \} \tag{2.1.4}$$

Notation. Let $\mathbf{A}$ be an infinite address. Let $[\mathbf{A}]_n$ be the substring consisting of the first n characters of $\mathbf{A}$. $[\mathbf{A}]_n$ is called the **level n approximation** of $\mathbf{A}$.

Notation. The collection of all finite addresses, denoted $\mathcal{A}$, is defined by

$$\mathcal{A} = \bigcup_{k\geq 0} \mathcal{A}_k \tag{2.1.5}$$

The collection of all finite and infinite addresses, denoted $\bar{\mathcal{A}}$, is

$$\bar{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}_\infty \tag{2.1.6}$$

Definition 2.1.2.5 *Let* $\mathbf{A} = A_1 \cdots A_k \in \mathcal{A}_k$ *and* $\mathbf{A}' = A'_1 \cdots A'_{k'} \in \mathcal{A}_{k'}$. *The* **concatenation** *of* $\mathbf{A}$ *and* $\mathbf{A}'$*, denoted* $\mathbf{A}\mathbf{A}'$*, is the address in* $\mathcal{A}_{k+k'}$ *formed from the concatenation of the two strings:*

$$\mathbf{A}\mathbf{A}' = A_1 \cdots A_k A'_1 \cdots A'_{k'} \tag{2.1.7}$$

2.1.3 Address Maps and r-Similarity Relations

The representation of the monoid gives rise to a family of 'address maps'.

Definition 2.1.3.1 *For any string* $\mathbf{A} = A_1 A_2 \cdots A_k$ *in* M_{LR} *we define the* **address map** $m_{\mathbf{A}}$ *to be* $m(r,\theta)(\mathbf{A})$*, i.e.,*

$$m_{\mathbf{A}} = m_{A_k} \circ \cdots \circ m_{A_2} \circ m_{A_1}. \tag{2.1.8}$$

If $\mathbf{A} = \mathbf{A}_0$ *(the empty address), then* $m_{\mathbf{A}}$ *is the identity map on* $\mathbb{R}^2$. *Given* $l(\mathbf{A}) = k$ *for some* $k \geq 0$*, the address map* $m_{\mathbf{A}}$ *is a* **level** k *address map. The level of an address map* $m_{\mathbf{A}}$ *is denoted* $l(m_{\mathbf{A}})$*, i.e.,* $l(m_{\mathbf{A}}) = l(\mathbf{A})$.

As an immediate consequence of Lemma 2.1.1.2, we get:

Lemma 2.1.3.2 (**Address Map Lemma**) *Let* r *and* θ *be given. Let* C *be any compact subset of* $\mathbb{R}^2$. *Let* $k \geq 0$ *and let* $m_{\mathbf{A}}$ *be any level* k *address map. Then the set* $U = m_{\mathbf{A}}(C)$ *is a compact subset of* $\mathbb{R}^2$ *that is similar to* C *with contraction factor* r^k.

The Address Map Lemma may seem obvious, but it will prove to be extremely useful throughout the remainder of this thesis.

Because we will see many instances of a pair of sets where one is similar to the other with contraction factor r^k, we will define a relation on compact subsets of $\mathbb{R}^2$ to express this idea.

Definition 2.1.3.3 *Let $r \in (0,1)$. Let $k \geq 0$. We define the* r**-similarity relation** $\sim_r$ *on the compact subsets of $\mathbb{R}^2$ as follows. Let U, V be any compact subsets of $\mathbb{R}^2$. $U \sim_r V$ if and only if there exists an integer k such that U is similar to V with factor r^k. When the specific integer k is known, we write $U \sim_r^k V$.*

Note. The r-similarity relation is a finer relation than the usual idea of similarity, because two sets are related only if they are similar with a factor of an integer power of r.

2.2 Symmetric Binary Fractal Trees

We are now in the position to define branches, address points, mirror images and finite approximation trees.

2.2.1 Branches, Address Points and Mirror Images

We now give a precise definition of the parts of a fractal tree such as trunk and branches, and this allows us to introduce some notation.

Definition 2.2.1.1 *The trunk T_0 is the closed vertical line segment between the points $(0,0)$ and $(0,1)$:*

$$T_0 = \{(0,y) \in \mathbb{R}^2 \mid y \in [0,1]\} \tag{2.2.1}$$

Note: The trunk is independent of r and θ.

Observation. For any r and θ, m_R maps T_0 to a closed line segment of length r, with one endpoint at $(0,1)$, the other endpoint at $(r\sin\theta, r\cos\theta+1)$; i.e. m_R maps the trunk to the first branch on the right side of the trunk. m_L maps T_0 to a closed line segment of length r, with one endpoint at $(0,1)$, the other endpoint at $(-r\sin\theta, r\cos\theta+1)$.

Definition 2.2.1.2 *Given $k \geq 0$, a* **level k branch b** *is the image of the trunk T_0 under a level k address map. Let $\mathcal{B}_k$ denote the set of all level k branches.*

$$\mathbf{b} \in \mathcal{B}_k \Leftrightarrow \exists \mathbf{A} \in \mathcal{A}_k \text{ such that } \mathbf{b} = m_{\mathbf{A}}(T_0) \tag{2.2.2}$$

Note that the trunk T_0 is the only level 0 branch. Moreover, there are 2^k level k branches.

Note: The 'level' of various objects will be defined, and in general $l(\ \ )$ denotes the level of the specific object.

Notation. We often use the notation $\mathbf{b} = b(\mathbf{A})$ to denote the branch $\mathbf{b} = m_{\mathbf{A}}(T_0)$.

Each branch is a closed line segment, and has a starting point and an endpoint. We consider $(0, 0)$ to be the starting point of the trunk and $(0, 1)$ to be the endpoint.

Definition 2.2.1.3 *Let $\mathbf{b} = b(\mathbf{A})$, where $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 0$. The* **starting point of the branch b** *is $m_{\mathbf{A}}((0, 0))$ and the* **endpoint of the branch b** *is $m_{\mathbf{A}}((0, 1))$.*

Note that

$$m_R((0, 0)) = m_L((0, 0)) = (0, 1) \tag{2.2.3}$$

Definition 2.2.1.4 *A* **vertex of a tree** *is a point in $\mathbb{R}^2$ that is the endpoint of a branch on the tree. That is, a vertex corresponds to a finite address.*

Observation. For $k \geq 0$, each level k branch is such that its endpoint is the starting point of 2 branches of level $k + 1$, because of Equation 2.2.3.

Definition 2.2.1.5 *Let $\mathbf{b} \in \mathcal{B}_k$ for some $k \geq 0$. The* **linear extension of the branch b** *is the unique line that the branch* **b** *is contained in, and is denoted by $lin(\mathbf{A})$.*

For example, the linear extension of the trunk is just the y-axis. Note that "affine hull" might be the more usual term than "linear extension", but we have chosen "linear extension" because that is the term used in [31].

For a finite address $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 0$, the **point with address A** is $m_{\mathbf{A}}((0,1))$, i.e., the endpoint of the branch $b(\mathbf{A})$. This point is denoted by $P_{\mathbf{A}}$. For an infinite address $\mathbf{A} = A_1 \cdots \in \mathcal{A}_\infty$, let $\mathbf{A}_i = [\mathbf{A}]_i$ (the level i approximation of $\mathbf{A}$). Then the sequence $\{P_{\mathbf{A}_i}\}$ converges, because of the absolute convergence of the geometric series with $0 < r < 1$. Moreover, the sequence of branches $\{m_{\mathbf{A}_i}(T_0)\}$ has a limit as i goes to infinity in the space of compact subsets of $\mathbb{R}^2$, and the limit is a singleton set. A **tip point** is specified by an infinite address $\mathbf{A} \in \mathcal{A}_\infty$, and we write

$$P_{\mathbf{A}} = \lim_{i \to \infty} P_{\mathbf{A}_i} \tag{2.2.4}$$

The **tipset** for a given r and θ, denoted $\mathbf{Tip}(r,\theta)$ (or $\mathbf{Tip}$), is the collection of points with infinite addresses.

$$\mathbf{Tip}(r,\theta) = \{P_{\mathbf{A}} | \mathbf{A} \in \mathcal{A}_\infty\}. \tag{2.2.5}$$

Because of the left-right symmetry of the generator maps, each address has a corresponding 'mirror' address. Let $\mathbf{A} = A_1 A_2 \cdots$ be a finite or infinite address. Then the **mirror image address of A**, denoted by $\mathbf{A}^*$, is the address obtained by switching each choice of direction. Thus $A^* = A_1^* A_2^* \cdots$, where $A_i^* = R$ if $A_i = L$ or $A_i^* = L$ if $A_i = R$. For example, if $\mathbf{A} = RLLRR$, then $\mathbf{A}^* = LRRLL$. The address map $m_{\mathbf{A}^*}$ is equal to the composition of the address map $m_{\mathbf{A}}$ with a reflection across the y-axis (hence the name 'mirror image'). Let $P_{\mathbf{A}} = (x_{\mathbf{A}}, y_{\mathbf{A}})$ be the point with address $\mathbf{A}$. The **mirror image of the point** $P_{\mathbf{A}}$ has coordinates $(-x_{\mathbf{A}}, y_{\mathbf{A}})$, is denoted by $P_{\mathbf{A}}^*$, and is the point with address $\mathbf{A}^*$.

Notation. Throughout the remainder of this thesis, a superscript of '$*$' denotes mirror images of various objects (i.e. the given image reflected across the y-axis).

2.2.2 Finite Approximation Trees

Definition 2.2.2.1 *Let $k \geq 0$. The* **level k (finite) approximation tree** *$T_k(r, \theta)$, or just T_k, is the union of all branches of level i, where $0 \leq i \leq k$.*

$$T_k = \bigcup_{l(\mathbf{A}) \leq k} m_{\mathbf{A}}(T_0) \tag{2.2.6}$$

For example, the **level 1 tree** $T_1(r, \theta)$, or just T_1 when the values of r and θ are understood, is the tree with one level of branching. It is the trunk together with two branches. Each branch has length r and starting point $(0, 1)$. The right branch has endpoint $(r \sin\theta, r\cos\theta + 1)$, and the left branch has endpoint $(-r\sin\theta, r\cos\theta + 1)$. See Figures 2.1 and 2.2 for two examples of level 1 trees. More examples of finite approximation trees are shown in Figures 2.3, 2.4, 2.5 and 2.6.

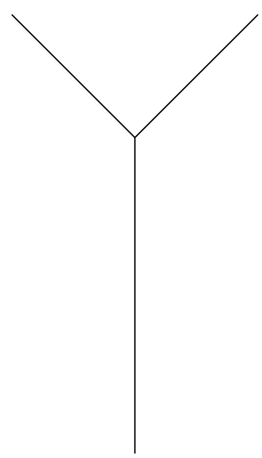

Figure 2.1: $T_1(0.55, 45°)$: Level 1 tree with $r = 0.55$, $\theta = 45°$

Observations. Let k be a non-negative integer. Then the tree T_k has branches of levels 0 through k. For $0 \leq j \leq k$, the length of a level j branch is r^j, and there are 2^j level j branches (which are not necessarily distinct as line segments in $\mathbb{R}^2$). T_k is a compact subset of $\mathbb{R}^2$ which is path-connected.

Proposition 2.2.2.2 *For any r and θ, and for any $k \geq 0$, the finite approximation tree $T_k(r, \theta)$ is symmetric about the y-axis, i.e., the image of $T_k(r, \theta)$ under reflection across the y-axis is just $T_k(r, \theta)$.*

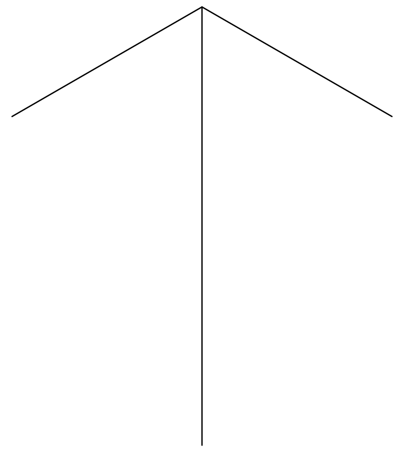

Figure 2.2: $T_1(0.5, 120°)$: Level 1 tree with $r = .5$, $\theta = 120°$

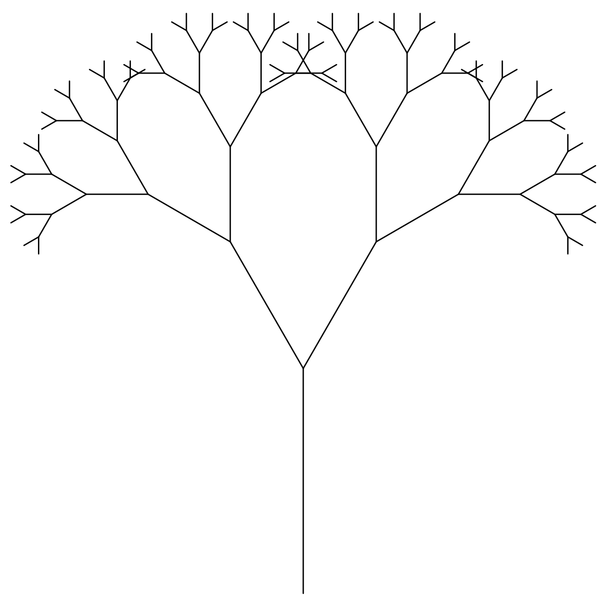

Figure 2.3: $T_6(0.65, 30°)$

Proof. For $1 \leq j \leq k$, each level j branch $\mathbf{b}$ on $T_k(r, \theta)$ has a corresponding level j branch reflected across the y-axis, namely the branch $\mathbf{b}^*$.

Lemma 2.2.2.3 *For a given r and θ, and for non-negative integers j, k such that $j \leq k$, we have $T_j(r, \theta) \subseteq T_k(r, \theta)$.*

Proof. This follows directly from the definition of the finite approximation trees.

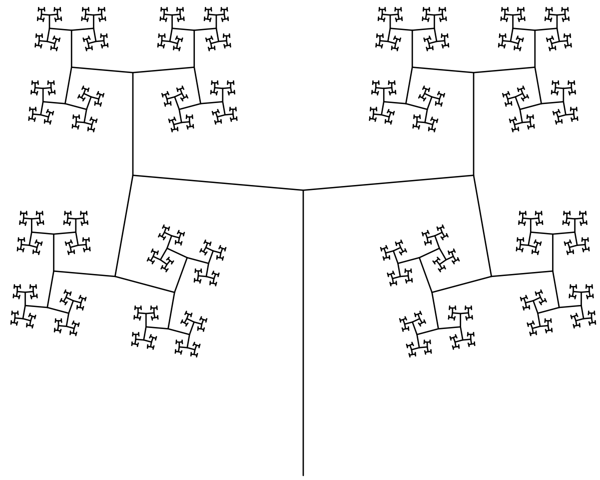

Figure 2.4: $T_{10}(0.6, 85°)$

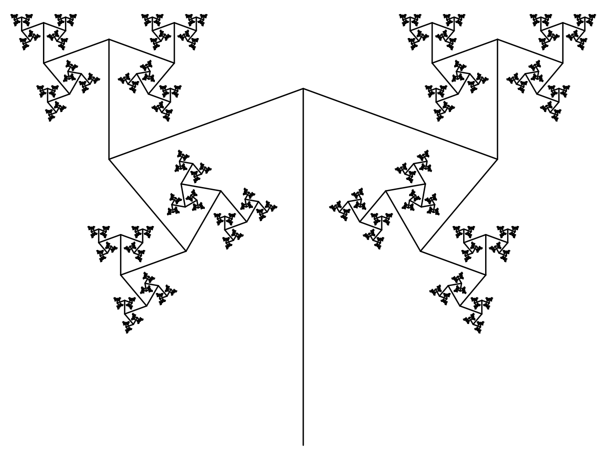

Figure 2.5: $T_{11}(0.58, 110°)$

2.2.3 Definition of Symmetric Binary Fractal Tree

For a given r and θ, $T_k(r, \theta)$ is a compact set in $\mathbb{R}^2$ for any integer $k \geq 0$ since it is the finite union of branches (which are all compact). With the Hausdorff distance as metric on the space of compact subsets of $\mathbb{R}^2$, the sequence of compact sets $\{T_k(r, \theta)\}$ has a limit as $k \to \infty$.

Theorem 2.2.3.1 *For a given r and θ, the sequence $\{T_k\}$ converges with respect to the Hausdorff metric.*

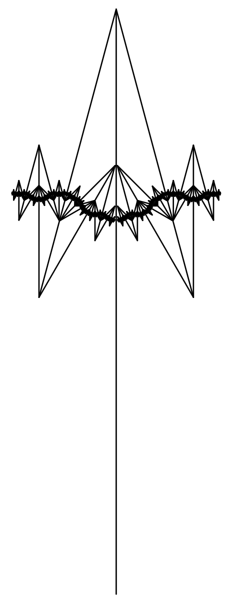

Figure 2.6: $T_{10}(0.51, 165°)$

Proof. Let r and θ be given. The sequence $\{T_k\}$ is a monotonically increasing sequence of sets (since each is a subset of the next). These sets are bounded. Let T_k be the finite approximation tree for some $k \geq 0$. Then T_k consists of all branches of levels 0 through k. By the triangle inequality, T_k is a subset of the trunk and the closed ball of radius $r_k = r + r^2 + \cdots r^k$ centered at the top of the trunk. For any $k \geq 0$, this ball is a subset of the closed ball with the same centre and with radius $r_\infty = r + r^2 + \cdots$, which is finite because i$r + r^2 + \cdots$ is a geometric series with $0 < r < 1$. Thus the sequence $\{T_k\}$ is a monotonically increasing sequence that is bounded, and so it must converge. $\square$

Definition 2.2.3.2 *The* **symmetric binary fractal tree** $T(r, \theta)$*, or just* T*, is the limit of the level* k *trees as* k *goes to infinity.*

$$T(r, \theta) = \lim_{k \to \infty} T_k(r, \theta) \tag{2.2.7}$$

Notation. $\mathcal{T}$ denotes the collection of all symmetric binary fractal trees with scaling ratios between 0 and 1 and branching angles between 0° and 180°.

$$\mathcal{T} = \{T(r, \theta) \mid r \in (0, 1) \text{ and } \theta \in (0°, 180°)\} \tag{2.2.8}$$

Proposition 2.2.3.3 *Let $T(r,\theta) \in \mathcal{T}$. Let B be any union of branches that are on the tree. That is, there exists a collection of addresses $\mathcal{A}' \subset \mathcal{A}$ such that*

$$B = \bigcup_{\mathbf{A}' \in \mathcal{A}'} m_{\mathbf{A}'}(T_0) \tag{2.2.9}$$

Let $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 0$. Then $m_{\mathbf{A}}(B) \subset T$, $m_{\mathbf{A}}(B)$ is a union of branches that are on the tree, and $m_{\mathbf{A}}(B) \sim_r^k B$.

Proof. Let $T(r,\theta) \in \mathcal{T}$. Let B be any union of branches that are on the tree. Then

$$m_{\mathbf{A}}(B) = m_{\mathbf{A}}\left(\bigcup_{\mathbf{A}' \in \mathcal{A}'} m_{\mathbf{A}'}(T_0)\right)$$

for some collection of addresses $\mathcal{A}'$. Thus

$$m_{\mathbf{A}}(B) = \bigcup_{\mathbf{A}' \in \mathcal{A}'} m_{\mathbf{A}'\mathbf{A}}(T_0) \subset T$$

and each branch $\mathbf{b}$ in B is mapped to another branch $m_{\mathbf{A}}(\mathbf{b})$ on the tree. Moreover, B is compact, so the image of B under the address map $m_{\mathbf{A}}$ is similar to B with contraction factor r^k, due to the Address Map Lemma 2.1.3.2. □

Theorem 2.2.3.4 *The tree $T(r,\theta)$ is the smallest connected, compact subset T of $\mathbb{R}^2$ containing the points $(0,0)$ and $(0,1)$ such that $m_{\mathbf{A}}(T) \subset T$ for any $m_{\mathbf{A}} \in M_{\mathcal{A}}$.*

Proof. Let T be the smallest connected, compact subset of $\mathbb{R}^2$ containing the points $(0,0)$ and $(0,1)$ such that $m_{\mathbf{A}}(T) \subset T$ for any $m_{\mathbf{A}} \in M_{\mathcal{A}}$. Then T contains the points $(0,0)$, $(0,1)$ and all images of $(0,1)$ under any address map $m_{\mathbf{A}}$ for any $\mathbf{A} \in \mathcal{A}$. The smallest connected and compact set that contains $(0,0)$ and $(0,1)$ is the line segment between the two points, which is T_0 (the trunk). Once the trunk is part of T, then we necessarily have all branches and all tip points, since we must have $m_{\mathbf{A}}(T_0)$ for all $\mathbf{A} \in \mathcal{A}$ and $m_{\mathbf{A}}((0,1))$ for all $\mathbf{A} \in \mathcal{A}_\infty$. Moreover, $m_{\mathbf{A}}(T) \subseteq T$ for any $m_{\mathbf{A}} \in \mathcal{A}$ by Proposition 2.2.3.3. Thus T is indeed $T(r,\theta)$. □

With the preceding definitions, it makes sense to think of a tree $T(r,\theta) \in \mathcal{T}$ as a "symmetric binary fractal" tree. For any r and θ, there is left-right symmetry in the tree because the reflection of the tree across the y-axis is just the tree itself. The y-axis is the **axis of symmetry**. The trees are binary in the sense that when branching occurs, it is one line segment that branches into two new branches. The trees are fractal in the sense that given any tree $T(r,\theta)$, we can find arbitrarily small similar versions of $T(r,\theta)$ that are contained within the tree itself.

2.3 Ancestry, Paths and Subtrees

2.3.1 Ancestors, Descendants and Paths

Given two different branches, it may or may not be possible to obtain one as the image of the other under some address map. When it is possible, one branch is an ancestor, and the other is a descendant.

Definition 2.3.1.1 *Let r and θ be given. Let*

$$\mathbf{A} = A_1 \cdots A_k \in \mathcal{A}_k, \quad \mathbf{A}' = A'_1 \cdots A'_{k'} \in \mathcal{A}_{k'}$$

for some $k, k' \geq 0$. The branch $\mathbf{b} = b(\mathbf{A})$ is an **ancestor** *of the branch $\mathbf{b}' = b(\mathbf{AA}')$, and the branch $\mathbf{b}'$ is a level k'* **descendant** *of $\mathbf{b}$. That is, $\mathbf{b}' = m_{\mathbf{A}'}(\mathbf{b})$, i.e., the image of the branch $\mathbf{b}$ under a level k' address map.*

Note that all branches are descendants of the trunk. In general, given a set U that is a subset of a tree $T(r,\theta)$, and given an address $\mathbf{A}$, we refer to $m_{\mathbf{A}}(U)$ as a **descendant** of U.

Given a vertex that is the starting point of a specific level k branch and given an address $A_1A_2\cdots$, there exists a 'natural' path on a tree . Starting with the level k branch, we would turn A_1 (left if $A_1 = L$, right if $A_1 = R$) at the endpoint of the branch to choose which level $k+1$ branch to take. Then we would turn A_2 at the end of the level $k+1$ branch to choose a level $k+2$ branch, and so on. At any given level along this path, there is a unique branch which is a descendant of the branches of lower levels and is an ancestor of the branches at higher levels. We can express this idea of 'path' in terms of the address maps.

Definition 2.3.1.2 *Let* $\mathbf{A} \in \mathcal{A}_k$ *and* $\mathbf{A}' \in \mathcal{A}_{k'}$ *for some* $k, k' \geq 0$. *We define the* **finite path** $p_{\mathbf{A}}(\mathbf{A}')$ *as follows. If* $k = 0$, *then the path is just the branch* $b(\mathbf{A}')$. *Otherwise, given* $\mathbf{A} = A_1 \cdots A_k$ *for some* $k \geq 1$, *let* $\mathbf{A}_i = A_1 \cdots A_i$, *for* $1 \leq i \leq k$. *Then*

$$p_{\mathbf{A}}(\mathbf{A}') = \bigcup_{i=0}^{k} m_{\mathbf{A}_i}(b(\mathbf{A}')) = \bigcup_{i=0}^{k} b(\mathbf{A}'\mathbf{A}_i) \tag{2.3.1}$$

We say that the path $p_{\mathbf{A}}(\mathbf{A}')$ starts at the branch $b(\mathbf{A}')$.

Note that this definition of path yields a set that is path-connected, since the endpoint of one branch is either the starting point of another branch, or it is the highest level branch on the path. For a path $p_{\mathbf{A}}(\mathbf{A}')$, we consider the branch $b(\mathbf{A}')$ to be the start of the path, and we consider the **level of the path** to be the level of the branch $b(\mathbf{A}')$.

Notation. When a path starts with the trunk, so is of the form $p_{\mathbf{A}}(\mathbf{A}_0)$, we often denote the path $p_{\mathbf{A}}(T_0)$.

Definition 2.3.1.3 *The* **path-length** *(or just length) of a finite path is the total number of branches on the path. Thus the path-length is* $k + 1$ *if a path is given by an address of level* k.

Proposition 2.3.1.4 *Let* $\mathbf{A} = A_1 \cdots \in \mathcal{A}_\infty$. *Let* $\mathbf{A}_i = [\mathbf{A}]_i = A_1 \cdots A_i$. *The paths* $p_{\mathbf{A}_i}(T_0)$ *have a limit as* i *goes to infinity, and this limit is equal to*

$$\lim_{i \to \infty} p_{\mathbf{A}_i}(T_0) = \bigcup_{i \geq 0} m_{\mathbf{A}_i}(T_0) \cup P_{\mathbf{A}}, \tag{2.3.2}$$

where $P_{\mathbf{A}}$ *denotes the tip point with address* $\mathbf{A}$.

Proof. Let $\mathbf{A} = A_1 \cdots \in \mathcal{A}_\infty$. Let $\mathbf{A}_i = [\mathbf{A}]_i = A_1 \cdots A_i$. For each $i \geq 0$, we have

$$p_{\mathbf{A}_i}(T_0) = \bigcup_{l=0}^{i} m_{\mathbf{A}_l}(T_0)$$

by definition. If $i \leq j$, then we clearly have $p_{\mathbf{A}_i} \subseteq p_{\mathbf{A}_j}$. To prove that the limit exists, we could use the same ideas as in the proof of Proposition 2.2.3.1, so we will not repeat the argument here. We know that the branches $m_{\mathbf{A}_i}(T_0)$ have a limit as i goes to infinity, and this limit is a singleton set, which is precisely the tip point $P_{\mathbf{A}}$.

Definition 2.3.1.5 *Let* $\mathbf{A} = A_1A_2\cdots \in \mathcal{A}_\infty$. *Let* $\mathbf{A}_i = [\mathbf{A}]_i = A_1\cdots A_i$, *for* $i \geq 1$. *Let* $\mathbf{A}' \in \mathcal{A}_k$ *for some* $k \geq 0$. *The* **infinite path** $p_{\mathbf{A}}(\mathbf{A}')$ *is the limit of the paths* $p_{\mathbf{A}_i}(\mathbf{A}')$ *as* i *goes to infinity in the space of compact subsets of* $\mathbb{R}^2$.

Note that this notion of infinite path is well-defined by the previous proposition.

Lemma 2.3.1.6 *A tree* $T(r,\theta) \in \mathcal{T}$ *is the union of all infinite paths that start with the trunk.*

$$T(r,\theta) = \bigcup_{\mathbf{A}\in\mathcal{A}_\infty} p_{\mathbf{A}}(T_0) \tag{2.3.3}$$

Proof. This result follows directly from the definition of $T(r,\theta)$ and the definition of infinite paths. □

Proposition 2.3.1.7 *A symmetric binary fractal tree* $T(r,\theta)$ *is equal to the union*

$$T(r,\theta) = \bigcup_{k\geq 0} T_k(r,\theta) \cup \mathbf{Tip}(r,\theta) \tag{2.3.4}$$

Proof. This result follows from Proposition 2.3.1.4 and Lemma 2.3.1.6. □

Level 0 paths in the tree for which every other branch is vertical turn out to play an important role. This warrants special symbols for the sets of addresses for such paths:

Definition 2.3.1.8 *For any* $k \geq 0$, *let*

$$\mathcal{AL}_{2k} = \{A_1A_2\cdots A_{2k} \mid A_{2i-1}A_{2i} \in \{RL, LR\},\quad 1 \leq i \leq k\} \tag{2.3.5}$$

Let

$$\mathcal{AL}_\infty = \{A_1A_2\cdots \mid A_{2i-1}A_{2i} \in \{RL, LR\},\quad \forall i\} \tag{2.3.6}$$

For example, the address $RLRLLLRRL$ is in $\mathcal{AL}_8$. The address $(RLLR)^\infty$ is in $\mathcal{AL}_\infty$. The '$\mathcal{AL}$' refers to the fact that a level 0 path given by any such address is such that all even level branches are vertical.

Given any address $\mathbf{A}$, there is a similarity between two paths that start at different branches but are both specified by $\mathbf{A}$.

Proposition 2.3.1.9 *Let $\mathbf{A} \in \mathcal{A}$. Let $\mathbf{A}_1 \in \mathcal{A}_{k_1}$ and $\mathbf{A}_2 \in \mathcal{A}_{k_2}$ for some $k_1, k_2 \geq 0$ such that $k_2 \geq k_1$. Then the path $p_{\mathbf{A}}(\mathbf{A}_2)$ is similar to the path $p_{\mathbf{A}}(\mathbf{A}_1)$ with contraction factor $r^{k_2-k_1}$.*

Proof. Let $\mathbf{A} \in \mathcal{A}$. Let $\mathbf{A}_1 \in \mathcal{A}_{k_1}$ and $\mathbf{A}_2 \in \mathcal{A}_{k_2}$ for some $k_1, k_2 \geq 0$ such that $k_2 \geq k_1$. Consider the three paths $p_{\mathbf{A}}(T_0)$, $p_{\mathbf{A}}(\mathbf{A}_1)$, and $p_{\mathbf{A}}(\mathbf{A}_2)$. $p_{\mathbf{A}}(\mathbf{A}_1)$ is similar to the path $p_{\mathbf{A}}(T_0)$ with contraction factor r^{k_1} via the address map $m_{\mathbf{A}_1}$, and $p_{\mathbf{A}}(\mathbf{A}_2)$ is similar to the path $p_{\mathbf{A}}(T_0)$ with contraction factor r^{k_2} via the address map $m_{\mathbf{A}_2}$. Thus $p_{\mathbf{A}}(\mathbf{A}_2)$ must be similar to $p_{\mathbf{A}}(\mathbf{A}_1)$ with contraction factor $r^{k_2-k_1}$.

2.3.2 Subtrees

The symmetric binary fractal trees are 'fractal' because they contain proper subsets which are similar to the whole. We will now discuss this aspect by looking at subtrees. We start by giving a precise definition of subtree.

Definition 2.3.2.1 *A* **subtree** *of a tree $T(r,\theta) \in \mathcal{T}$ is a subset of the tree $T(r,\theta)$ that is specified by a branch $\mathbf{b} = b(\mathbf{A})$, where $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 0$. We denote this subtree by $S_{\mathbf{A}}(r,\theta)$, $S_{\mathbf{A}}$, $S_{\mathbf{b}}$, or just S. The branch $\mathbf{b}$ acts as the trunk of the subtree. S is defined to be the image of T under the address map $m_{\mathbf{A}}$.*

$$S_{\mathbf{A}}(r,\theta) = m_{\mathbf{A}}(T) \tag{2.3.7}$$

The level of a subtree $S_{\mathbf{A}}$ is the level of the address map $m_{\mathbf{A}}$, so the level of the branch that forms the trunk of the subtree.

Recall that we had already shown that the image of any subset of T is also a subset of the tree (hence the name 'subtree'). The subtrees are all similar to the tree, and

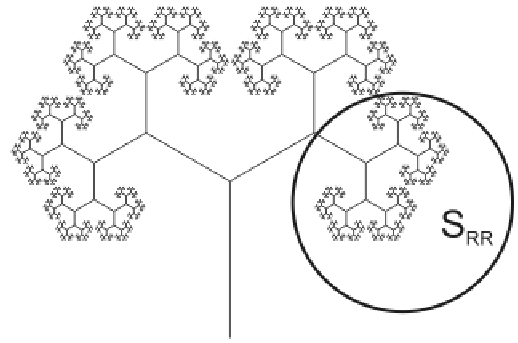

Figure 2.7: Subtree S_{RR} of $T\left(\dfrac{-1+\sqrt{5}}{2}, 60°\right)$

this is an important property of symmetric binary fractal trees. See Figure 2.7 for an example of a subtree of a specific tree.

Theorem 2.3.2.2 *Given r and θ, and a level k branch $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 1$, the subtree $S_{\mathbf{A}}$ is similar to T with contraction factor r^k, i.e., $S_{\mathbf{A}} \sim_r^k T$.*

Proof. This theorem directly follows the definition of subtree. T is a compact subset of $\mathbb{R}^2$, so applying a level k address map on T to obtain $S_{\mathbf{A}}$ yields a set which is similar to T with contraction factor r^k, as a result of the Address Map Theorem 2.1.3.2. □

Notation. For a given r and θ, and for a given non-negative integer k, we denote the collection of level k subtrees of $T(r,\theta)$ by $\mathcal{S}_k(r,\theta)$, or just $\mathcal{S}_k$. Thus $\mathcal{S}_0 = \{T(r,\theta)\}$. If $k \geq 1$, then for each $S \in \mathcal{S}_k$, there exists a level k branch which is the trunk of the subtree. That is, there exists $\mathbf{A} \in \mathcal{A}_k$ such that $S = S_{\mathbf{A}}(r,\theta)$.

2.4 Brief Chapter Summary

This chapter has introduced the generator maps m_R and m_L that are functions of a scaling ratio r and a branching angle θ, where $r \in (0,1)$ and $\theta \in (0°, 180°)$. These maps act on compact subsets of $\mathbb{R}^2$, and they are compositions of a contraction, rotation and translation. A level k address map (formed from the composition of generator maps) maps a compact subset of $\mathbb{R}^2$ to another compact subset that is similar with contraction factor r^k. Address maps enable us to give a definition of the symmetric binary fractal trees as a representation of the free monoid with two generators. Thus we have the set of trees

$$\mathcal{T} = \{T(r, \theta) \mid r \in (0, 1) \text{ and } \theta \in (0^\circ, 180^\circ)\}.$$

The address maps enable us to define other objects such as paths on a tree and subtrees of a tree, and give their addresses. The subtrees form an important class of images of a tree under address maps, and the symmetric binary fractal trees have the special property that any subtree is similar to the tree with contraction factor r^k, for some non-negative integer k.

Now that we have basic definitions and notations regarding the symmetric binary fractal trees, we can study three broad classes of trees. A tree can be self-avoiding, self-contacting or self-overlapping. Theory regarding this broad classification of trees, along with other properties of the trees, will be presented in the next chapter. This theory also provides a foundation for our analysis of the symmetric binary fractal trees using methods of computational topology, that commences in Chapter 4.

Chapter 3

Properties of Symmetric Binary Fractal Trees and Criteria for Self-Contact

In the previous chapter, we presented a new description of the symmetric binary fractal trees in terms of representations of the free monoid with two generators. This chapter presents theory about various properties of the trees. The main part of this chapter deals with the broad classification of the trees introduced by Mandelbrot and Frame in [31]. We discuss the notions of self-avoidance, self-contact and self-overlap, and give an overview of the main results of [31] that are relevant for our work. We also review criteria for determining whether a given tree $T(r, \theta) \in \mathcal{T}$ is self-avoiding, self-contacting or self-overlapping. For a given branching angle θ, there exists a unique scaling ratio r that yields a self-contacting tree (as in [31]). In fact, one is able to determine the exact value of this special scaling ratio.

In this chapter, we will not only be interested in whether a tree is self-contacting or not, but also where the self-contact occurs. If a tree is self-contacting, then the subtree S_R contains at least one point other than $(0, 1)$ that is on the y-axis, and the mirror image of such a point is on the subtree S_L and is also on the y-axis. Without loss of generality, since the trees have right-left symmetry, we consider points on S_R. There is always a vertex point or tip point involved, so we discuss the self-contacting addresses as a function of θ. For a given angle, the complete set of self-contacting addresses can be obtained from the set of non-trivial addresses on S_R whose points are on the y-axis.

We study the self-contacting trees and the self-avoiding trees, and for the study of the closed ϵ-neighbourhoods of these trees, certain sets of points are important. First we define the contact address. For a given tree $T(r, \theta)$ such that θ is not $90°$, we identify a unique contact address. It is the address for a point on S_R distinct from $(0, 1)$ that has minimal distance to the y-axis and is closest to $(0, 1)$ out of all such points on S_R. For the majority of angles, the contact address is a self-contacting address. However, for angles between 120° and 135° we shall see that the contact address for a given tree with such an angle may not be a self-contacting address for the self-contacting tree with the same angle. In addition to the contact addresses, we will identify the addresses of other vertex and tip points that are extremal in some way, such as certain canopy points.

Any self-contacting tree that is not space-filling contains holes. In this chapter, we provide constructions for the boundaries of such holes.

The last part of this chapter deals with the homeomorphism classes of non-overlapping trees. All self-avoiding trees are homeomorphic, and we discuss certain homeomorphisms. The two space-filling self-contacting trees form another homeomorphism class. For the other self-contacting trees, two trees are homeomorphic if and only if they have the same self-contacting addresses.

Although we do not yet encounter the closed ϵ-neighbourhoods of trees in this chapter, the methods developed here provide the foundation for the analysis of the closed ϵ-neighbourhoods of trees in the following chapters.

3.0.1 Preliminary Definitions

In this subsection, we provide some preliminary notations and definitions that will be useful for this chapter and the remainder of the thesis.

Angle Ranges

According to Mandelbrot and Frame [31], the branching angles 90° and 135° are topological critical angles, because the corresponding self-contacting trees are space-filling and display topological discontinuities as the branching angle of the self-contacting trees varies between 0° to 180°. Our work supports this idea, and we develop a deeper explanation for the topological criticality of these two branching angles. In this thesis, the study of symmetric binary fractal trees is divided into three main angle ranges based on these two topologically critical angles. We shall see that there are other topologically critical angles based on our analysis of the trees and their closed ϵ-neighbourhoods, and these allow for a finer classification of the symmetric binary trees based on topology.

$$\text{First Angle Range:} \quad 0^\circ < \theta < 90^\circ \tag{3.0.1}$$

$$\text{Second Angle Range:} \quad 90^\circ < \theta < 135^\circ \tag{3.0.2}$$

$$\text{Third Angle Range:} \quad 135^\circ < \theta < 180^\circ \tag{3.0.3}$$

We generally investigate trees with branching angles 90° or 135° separately, or with a range that is appropriate (*i.e.* easiest for calculations) for the specific feature under investigation.

Turning Number $N(\theta)$ and the Secondary Turning Number $N_2(\theta)$

The following number, defined as a function of the branching angle of a tree, will prove to be important for the geometry of the tree and later for the closed ϵ-neighbourhoods of the tree.

Definition 3.0.1.1 *Let $\theta \in (0^\circ, 180^\circ)$. The* **turning number** *$N(\theta)$, or just N, is defined to be the smallest integer such that $N\theta \geq 90^\circ$.*

As a result of this definition, we also have $N\theta < 180^\circ$. Note that for angles between 90° and 180°, the turning number is 1.

For certain angles, we also define the secondary turning number. This number is relevant for the secondary contact addresses discussed later in Section 3.5.

Definition 3.0.1.2 *Given an angle θ such that $45^\circ < \theta < 90^\circ$, the* **secondary turning number of the angle** *θ, denoted by $N_2(\theta)$, or just N_2, is the smallest integer N_2 such that $N_2\theta \geq 270^\circ$.*

Special Angles θ_N

When a given angle θ is such that there exists an integer N for which $N\theta = 90^\circ$, the corresponding tree $T(r, \theta)$ possesses extra symmetry, as we demonstrate later in this chapter.

Definition 3.0.1.3 *Let $N \geq 1$. Then the* **special angle** θ_N *is given by*

$$\theta_N = \frac{90°}{N} \tag{3.0.4}$$

Height, width and bounding rectangle

Any symmetric binary fractal tree is compact, so we can define the height and width of a tree. The **height** of the tree $T(r,\theta)$ is the vertical range of the tree, denoted by $h(r,\theta)$ or h. The **width** of a tree, denoted by $w(r,\theta)$, or just w, is the horizontal range of the tree $T(r,\theta)$. We discuss the actual values of h and w for non-overlapping trees later in the chapter.

Definition 3.0.1.4 *The* **maximal y-value of a tree** $T(r,\theta)$*, denoted by* $y_{\max}(r,\theta)$*, or just* $y_{\max}$*, is defined by*

$$y_{\max}(r,\theta) = \max\{y \mid \exists x \in \mathbb{R} \text{ such that } (x,y) \in T(r,\theta)\} \tag{3.0.5}$$

Definition 3.0.1.5 *The* **minimal y-value of a tree** $T(r,\theta)$*, denoted by* $y_{\min}(r,\theta)$*, or just* $y_{\min}$*, is defined by*

$$y_{\min}(r,\theta) = \min\{y \mid \exists x \in \mathbb{R} \text{ such that } (x,y) \in T(r,\theta)\} \tag{3.0.6}$$

From these definitions, it is clear that $h = y_{\max} - y_{\min}$. We are interested in the self-avoiding and self-contacting trees, and we shall see that they all have $y_{\min} = 0$, so $h = y_{\max}$.

Definition 3.0.1.6 *The* **maximal x-value of a tree** $T(r,\theta) \in \mathcal{T}$*, denoted* $x_{\max}(r,\theta)$*, or just* $x_{\max}$*, is defined by*

$$x_{\max}(r,\theta) = \max\{x \mid \exists y \in \mathbb{R} \text{ such that } (x,y) \in T(r,\theta)\} \tag{3.0.7}$$

Thus $w = 2x_{\max}$ because each tree is symmetric about the y-axis.

Based on these definitions, we have:

Definition 3.0.1.7 *For a given tree $T(r,\theta) \in \mathcal{T}$, define its* **bounding rectangle** $BR(r,\theta)$*, or just BR, to be*

$$BR = \{(x,y) \in \mathbb{R}^2 \mid x \in [-x_{\max}, x_{\max}] \text{ and } y \in [y_{\min}, y_{\max}]\} \tag{3.0.8}$$

Proposition 3.0.1.8 *Let $T(r, \theta)$ be a self-avoiding or self-contacting tree with height h and width w. Let S be a level k subtree of the tree, for some $k \geq 1$. Then S is bounded by a rectangle of length $r^k h$ and width $r^k w$, where the sides of length $r^k h$ are parallel to the trunk of the subtree and the sides of length $r^k w$ are perpendicular to the trunk of the subtree.*

Proof. This proposition is an immediate consequence of definition of height and width, because of the self-similarity of the tree and its subtrees. □

Notation for Regions of the y-axis

Since all the trees we consider are symmetric about the y-axis, we use the following concise notation.

$$\mathbf{y} = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}, \quad \text{i.e. the } y\text{-axis} \tag{3.0.9}$$

$$\mathbf{y}_I = \{(0, y) \in \mathbb{R}^2 \mid y \in I\}, \quad \text{where } I \text{ is any subset of } \mathbb{R} \tag{3.0.10}$$

3.1 Definitions of Self-Avoidance, Self-Contact and Self-Overlap; Curves and Holes of a Tree

One can classify symmetric binary fractal trees into three main categories: self-avoiding, self-contacting, and self-overlapping, see [31]. Overlap occurs when the interiors of two branches intersect. Basically, a self-avoiding tree has no self-intersection (as the name suggests), a self-contacting tree has self-intersection but no overlap, and a self-overlapping tree has overlap. This means that a self-avoiding tree is contractible and is not space-filling; a self-contacting tree has no branch overlap, and is not contractible or it is contractible and space-filling; a self-overlapping tree has branch overlap, and is not contractible or it is contractible and space-filling. The self-contacting trees with branching angles 90° or 135° are the only self-contacting trees that are space-filling, as mentioned in [31].

Definition 3.1.0.9 *A symmetric binary fractal tree $T(r, \theta)$ is* **self-avoiding** *if there is a unique path to each tip point.*

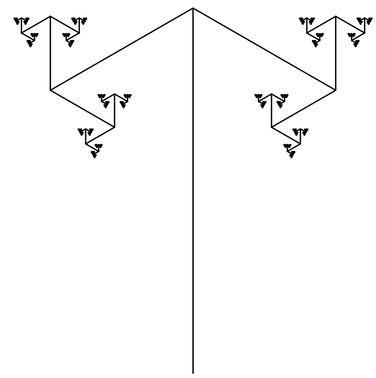

Figure 3.1: A self-avoiding tree: $T(0.45, 120°)$

Note that this definition implies that there are no non-trivial simple closed curves on the tree.See Figure 3.1 for an image of a self-avoiding tree.

Notation. $\mathcal{T}_{sa}$ denotes the collection of all self-avoiding symmetric binary fractal trees.

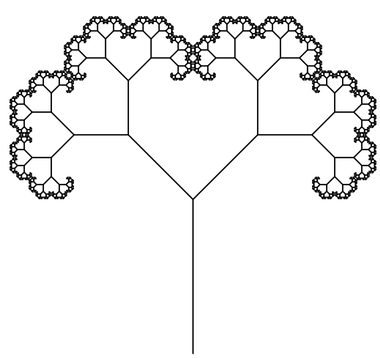

Figure 3.2: A self-contacting tree: $T(r_{sc}, 45°)$.
r_{sc} is the root of the equation $r^3 + \frac{2}{\sqrt{2}}r^2 - \frac{1}{\sqrt{2}}$ that is in $(0,1)$, and $r_{sc} \approx 0.5935$.

Definition 3.1.0.10 *A symmetric binary fractal tree $T(r, \theta)$ is* **space-filling** *if it has non-zero area.*

Definition 3.1.0.11 *A symmetric binary fractal tree $T(r, \theta)$ is* **self-contacting** *if there is no intersection at the interiors of two distinct branches and at least one of the following conditions holds:*

1. *There exists a tip point that can be reached via two distinct paths (tip-to-tip contact)*

2. *There exists a tip point that belongs to the interior of a branch (tip-to-branch contact)*

3. *There exists a branch endpoint that belongs to the interior of another branch (vertex-to-branch contact)*

See Figures 3.2 and 3.4 for images of self-contacting trees. The methods to calculate the self-contacting ratio will be discussed later in this chapter.

Notation. $\mathcal{T}_{sc}$ denotes the collection of all self-contacting, symmetric binary fractal trees. $\mathcal{T}_{tt}$ denotes the subset of $\mathcal{T}_{sc}$ consisting of trees that have tip-to-tip contact, $\mathcal{T}_{tb}$ denotes the subset of $\mathcal{T}_{sc}$ consisting of trees that have tip-to-branch contact, and $\mathcal{T}_{vb}$ denotes the subset of $\mathcal{T}_{sc}$ consisting of trees that have vertex-to-branch contact. Note that these three subsets are not disjoint, and more will be said during the discussions on criteria for self-contact and when we investigate the self-contacting points as a function of branching angle.

It was shown in [31] that 90° and 135° are the only topologically critical angles for the self-contacting trees. In our work, we will further refine this result by taking the topology not only of the trees, but also of their closed ϵ-neighbourhoods (to be defined in the next chapter), into account. We will see that 90° and 135° are critical angles for the closed ϵ-neighbourhoods of the self-avoiding trees as well. Moreover, we will show that the closed ϵ-neighbourhoods give rise to more critical angles based on the location of holes. We discuss these critical angles in Chapter 5, and identify actual values in Chapter 7. In addition, for each angle there are critical scaling ratios that are based on complexity (discussed in Chapters 6 and 7).

Definition 3.1.0.12 *A symmetric binary fractal tree $T(r,\theta)$ is* **self-overlapping** *if there exist two distinct branches that overlap at their interiors.*

See Figures 3.3 and 3.5 for images of self-overlapping trees.

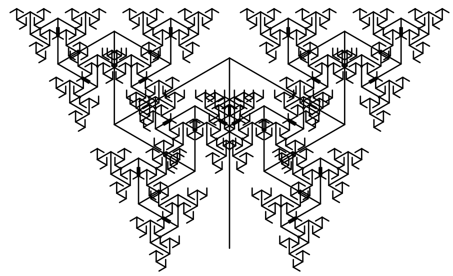

Figure 3.3: A self-overlapping tree: $T(0.7, 120°)$

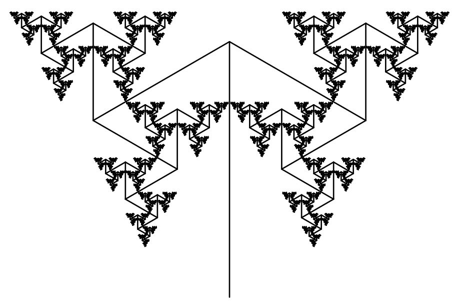

Figure 3.4: A self-contacting tree: $T\left(\frac{-1+\sqrt{5}}{2}, 120°\right)$

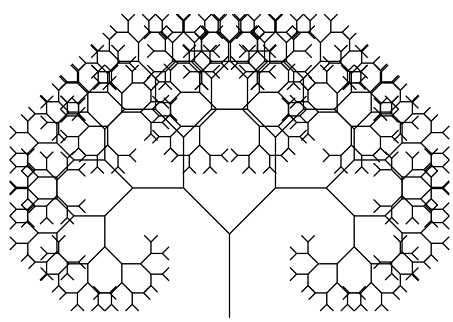

Figure 3.5: A self-overlapping tree: $T(0.78, 45°)$

Notation. $\mathcal{T}_{so}$ denotes the collection of all self-overlapping symmetric binary fractal trees.

Remark. The **contact classification** of a tree refers to whether the tree is self-avoiding, self-contacting, or self-overlapping (based on the definitions above).

Homology. In general, for self-contacting (and self-overlapping) trees, the homology is non-trivial and has infinitely many generators. In the self-contacting case, the trees with angles $90°$ and $135°$ are the only exceptions. In this section we develop some terminology to study loops in self-contacting trees, and this terminology will be generalized when we look at the closed ϵ-neighbourhoods of all trees.

Notation. Let Γ be the collection of all simple closed, non-trivial (i.e. not a point) curves in $\mathbb{R}^2$. Thus $\pi_1(\gamma) = \mathbb{Z}$ for each $\gamma \in \Gamma$.

Definition 3.1.0.13 *For any curve γ in Γ, let $O(\gamma)$ denote the unique, simply-connected, non-empty, open set in $\mathbb{R}^2$ whose boundary is γ (i.e. $O(\gamma)$ is the* **'inside'** *of γ). Note that $O(\gamma)$ is well-defined by the Jordan Curve Theorem* [38].

Definition 3.1.0.14 *A* **loop** *or* **simple, closed curve** *of the tree $T(r, \theta)$ is a curve γ in Γ that is a subset of $T(r, \theta)$. For a given $r \in (0, 1)$ and $\theta \in (0°, 180°)$, let $\Gamma(r, \theta)$ be the collection of all loops of the tree $T(r, \theta)$.*

$$\gamma \in \Gamma(r, \theta) \quad \Leftrightarrow \quad \gamma \in \Gamma, \ \gamma \subset T(r, \theta) \tag{3.1.1}$$

Definition 3.1.0.15 *Let $\gamma \in \Gamma(r, \theta)$ be such that $O(\gamma)$ is disjoint from the tree $T(r, \theta)$. Then we say that $O(\gamma)$ is a* **hole of the tree** *$T(r, \theta)$.*

Note. In Section 3.6 of this chapter we give actual constructions of simple, closed curves of self-contacting symmetric binary fractal trees that form the boundaries of holes. For now we have just provided the definition.

Notation. Let $T^C(r, \theta)$, or just T^C, denote the complement of the tree $T(r, \theta)$ in $\mathbb{R}^2$.

$$T^C(r, \theta) = T^C = \mathbb{R}^2 \backslash T(r, \theta) \tag{3.1.2}$$

Remarks. The following remarks follow directly from our definitions of self-avoiding and self-contacting trees, and from [31].

- Any self-contacting tree that is not space-filling is not contractible. For such a tree, there must exist at least one 'double' point (a point that can be reached via two distinct paths) [31]. Hence there are loops, and the tree cannot be contractible.

- If $T(r,\theta)$ is contractible, then $T^C(r,\theta)$ contains one component (i.e. it is path-connected).

- If $T(r,\theta)$ is not contractible, then $T^C(r,\theta)$ contains more than one component. A hole of the tree $T(r,\theta)$ is a component of $T^C(r,\theta)$ which is simply-connected and bounded (this will be clarified in Section 3.6 of this chapter).

- Any self-avoiding tree is contractible.

- A self-contacting, space-filling tree is contractible. The only self-contacting, space-filling trees are with angles 90° and 135° [31]. In each case, the tree completely fills a region of $\mathbb{R}^2$ (a rectangle in the case of 90° and a triangle in the case of 135°) [31].

One of the main results of [31] is the following:

Theorem 3.1.0.16 ([31]) *For any angle θ, there is a unique scaling ratio r such that the symmetric binary fractal tree $T(r,\theta)$ is self-contacting.*

We do not provide a proof for this theorem, but there are some remarks worth noting. Figure 3.6 displays a plot of the self-contacting scaling ratio as a function of branching angle. We will discuss methods for determining the self-contacting scaling ratios later in the chapter. For any pair (r,θ) below the curve, the corresponding tree is self-avoiding. For any pair above the curve, the corresponding tree is overlapping. For a given branching angle θ, the trees 'grow' as the scaling ratio increases from 0 to 1. For small values of r the trees are self-avoiding, and as r increases, eventually a scaling ratio will be reached where self-intersection first occurs, and this is the unique self-contacting ratio. Now as r continues to increase, the corresponding trees are all self-overlapping. Note that this is an important feature of the symmetric binary fractal trees. In the case of asymmetric binary fractal trees, for a given branching

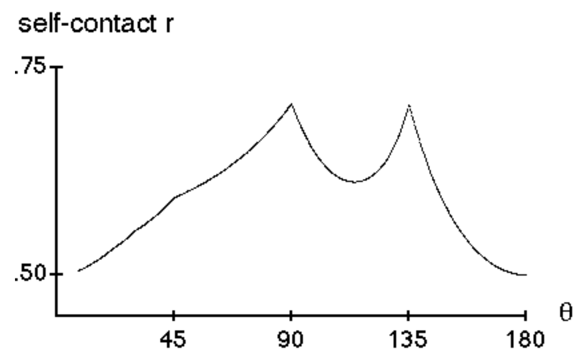

Figure 3.6: Self-contacting Scaling Ratio as a Function of Branching Angle

angle, there is more than one pair of scaling ratios that will correspond to a self-contacting tree. For general binary trees, precise criteria for self-contact have not been established to date.

Definition 3.1.0.17 *Let $\theta \in (0°, 180°)$. Then the unique scaling ratio that yields a self-contacting tree is called the* **self-contacting ratio for** θ*, and is denoted by $r_{sc}(\theta)$, or just r_{sc}.*

We can summarize the previous theorem and definitions as follows:

- $r < r_{sc}(\theta)$ implies $T(r, \theta)$ is self-avoiding
- $r = r_{sc}(\theta)$ implies $T(r, \theta)$ is self-contacting
- $r > r_{sc}(\theta)$ implies $T(r, \theta)$ is self-overlapping

Observation. For any tree, the two level 1 subtrees S_R and S_L both contain the point $(0, 1)$, so they are not disjoint. Let θ be a given branching angle. For any self-avoiding tree with branching angle θ, therefore, the self-contacting tree with branching angle θ has the smallest scaling ratio such that the two level 1 subtrees S_R and S_L have a non-trivial intersection (*i.e.*, more than just the point $(0, 1)$). The next section presents the criteria to find the smallest such scaling ratio.

NOTE: For the remainder of this chapter and the rest of the thesis, assume that the trees are self-avoiding or self-contacting. Unless otherwise stated, assume that for a given θ, $r \leq r_{sc}(\theta)$.

3.2 Criteria for Self-Contact and Self-Avoidance

According to [31], the self-similarity and left-right symmetry of the trees imply that self-avoidance is guaranteed if none of the branches that descend from $b(R)$ intersect the linear extension of the trunk. We now give a more detailed explanation for this. The ideas presented in this section and the remainder of the chapter will be discussed in detail because they provide us with the opportunity to introduce new notation and concepts.

First we show that for self-avoiding or self-contacting trees, the only vertical branch that lies on $\mathbf{y}$ is the trunk.

Proposition 3.2.0.18 *Let $\theta \in (0°, 180°)$ and $r \leq r_{sc}(\theta)$. Then there are no branches of level k, with $k \geq 1$, for which $\mathbf{y}$ is the linear extension.*

Proof. Suppose there exists a branch $\mathbf{b}$ that has $\mathbf{y}$ as its linear extension. This means that $\mathbf{b}$ is entirely contained in $\mathbf{y}$. Now consider the mirror image branch $\mathbf{b}^*$. It is also entirely contained on $\mathbf{y}$ (because it is the reflection of $\mathbf{b}$ across $\mathbf{y}$), and the two branches completely overlap, and this contradicts $r \leq r_{sc}$. □

Corollary 3.2.0.19 *Let $\theta \in (0°, 180°)$ and $r \leq r_{sc}(\theta)$. For any $\mathbf{A} \in \mathcal{A}_k$, the linear extension of branches in the subtree $S_{\mathbf{A}}$ are all distinct from the linear extension of $\mathbf{b} = b(\mathbf{A})$.*

Proof. The subtree $S_{\mathbf{A}}(r, \theta)$ is similar to the tree $T(r, \theta)$, with $\mathbf{b}$ acting as the trunk of the subtree. If some branch other than $\mathbf{b}$ on the subtree had the same linear extension, this would contradict the previous proposition. □

Remarks. Another way to phrase this corollary is that descendants cannot have the same linear extension as any of their ancestors. This corollary does not imply that there exists no other branch on the entire tree that has the same linear extension. We shall see that it is indeed possible to have more than one branch on the same linear extension. For example, any tree with branching angle $45°$ possesses branches that share the same linear extension. See Figure 3.8.

We now present some preliminary lemmas that are needed for the Self-Contact Criteria Theorem 3.2.0.27 and other results about non-overlapping trees. The first lemma is also used for results about the closed ϵ-neighbourhoods of non-overlapping trees.

Lemma 3.2.0.20 (Disjoint Lemma) *Let $\theta \in (0°, 180°)$, $r \leq r_{sc}(\theta)$, and $T(r, \theta)$ be the corresponding tree. Let S be any subtree of level 1 or higher. Then S is disjoint from one side of $\mathbf{y}$.*

Proof. Let $\theta \in (0°, 180°)$, $r \leq r_{sc}(\theta)$, and let $T(r, \theta)$ be the corresponding tree. Let S be some level k subtree, $k \geq 1$, such that S intersects both sides of $\mathbf{y}$. Now consider the mirror image subtree S^*. Then there must be branch overlap in the tree because of the overlap between S and S^*, and this contradicts the assumption that $r \leq r_{sc}$. Hence any subtree of level 1 or higher must be disjoint from one side of $\mathbf{y}$.

Corollary 3.2.0.21 *Let $\theta \in (0°, 180°)$, $r \leq r_{sc}(\theta)$. Let S and S' be level k and k' (respectively) subtrees of $T(r, \theta)$ such that $S \subseteq S'$ and $k' \leq k$. Let L be the linear extension of the trunk of S', i.e., the axis of symmetry of S'. Then S is disjoint from one side of L.*

Corollary 3.2.0.22 *Let $\theta \in (0°, 180°)$, $r \leq r_{sc}(\theta)$. Then there is no portion of the tree $T(r, \theta)$ below the line $y = 0$.*

Proof. Suppose that there is a portion of the tree below the line $y = 0$. Without loss of generality, this means that there is a portion of S_R that is below the line $y = 0$. Let L be the line that is the image of the line $y = 0$ under the address map $m_{\mathbf{R}}$. This subtree is geometrically similar to the tree, and so it must contain some portion that is to the left of the line L. So part of S is on the right side of the y-axis, and part is on the left side. This contradicts the previous lemma that S must be disjoint from one side of $\mathbf{y}$. $\square$

Lemma 3.2.0.23 *Let $\theta \in (0°, 180°)$ be such that the corresponding self-contacting tree $T(r_{sc}, \theta)$ has $y_{\max} > 1$. Suppose there exists a loop $\gamma \in \Gamma(r_{sc}, \theta)$ that intersects $\mathbf{y}_{(1,y_{\max})}$. Then any point in $\gamma \cap \mathbf{y}_{(1,y_{\max}]}$ is a tip point of the tree, and is therefore at two distinct addresses.*

Proof. Let P_γ be a point in $\gamma \cap \mathbf{y}_{(1,y_{\max}]}$. Thus $P_\gamma = (0, y_\gamma)$ for some $y_\gamma > 1$. We first wish to show that P_γ is a tip point. P_γ is necessarily on the tree, so there are three possibilities. It can be in some branch interior, it can be a branch endpoint, or it can be a tip point. We will show that the first two situations are not possible for a self-contacting tree with a portion of the tree above $y = 1$.

Suppose P_γ is in some branch interior. Then there exists a level k branch $\mathbf{b}$, for some $k \geq 1$, such that P_γ is in the interior of $\mathbf{b}$. Now consider the branch $\mathbf{b}^*$. It is distinct from $\mathbf{b}$, but they both contain the point $(0, y_\gamma)$. Hence there is branch overlap, and this contradicts the assumption that the tree is self-contacting.

Suppose P_γ is some branch endpoint. Then there exists a branch $\mathbf{b} = b(\mathbf{A})$, where $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 1$, such that $P_\gamma = P_{\mathbf{A}}$. Suppose, without loss of generality, that the starting point of the branch $\mathbf{b}$ is on the right side of the y-axis. (We have already shown it can't also be on the y-axis.) The branch $\mathbf{b}$ is the trunk of a subtree S which is geometrically similar to the tree itself. Let λ be the line that is the image of the line $y = 1$ under the address map $m_{\mathbf{A}}$. Then λ is perpendicular to $\mathbf{b}$ and contains P_γ. Just as there is a portion of the tree that is above the line $y = 1$, then there must be a portion of S that is to the left of the line λ, and also to the left of the point P_γ (so on the left side of $\mathbf{y}$). So there must be part of S that is on the left side of the tree, and this contradicts the Disjoint Lemma 3.2.0.20 which states that subtrees must be disjoint from one side of the y-axis.

Thus P_γ is a tip point, and it corresponds to some infinite address $\mathbf{A}$ and its mirror image $\mathbf{A}^*$. □

Lemma 3.2.0.24 *Let $\theta \in (0°, 180°)$ and let $T(r_{sc}, \theta) \in \mathcal{T}_{sc}$. Suppose $\gamma \in \Gamma(r, \theta)$ and the point $P_\gamma \in \gamma \cap \mathbf{y}_{[0,1)}$, then P_γ is either a tip point on the trunk or a branch endpoint on the trunk (where the branch is not the trunk).*

Proof. Let P_γ be any point that is on γ and on $\mathbf{y}_{[0,1)}$. So P_γ has coordinates $(0, y_\gamma)$ for some $0 \leq y_\gamma < 1$. As in the proof of the previous lemma, P_γ cannot be in some branch interior. We can no longer be certain that part of the tree is above the line $y = 1$, so we no longer have a restriction on branch endpoints. Thus P_γ is either a tip point or a branch endpoint. $\square$

Lemma 3.2.0.25 *Let $\theta \in (0°, 180°)$ and let $T(r_{sc}, \theta) \in \mathcal{T}_{sc}$ Let $\gamma \in \Gamma(r, \theta)$ be such that γ does not intersect $\mathbf{y}$. Then there exists $\gamma' \in \Gamma(r, \theta)$ which does intersect $\mathbf{y}$, and is such that $\gamma \sim_r^k \gamma'$, for some $k \geq 1$.*

Proof. Suppose γ is a simple closed curve on the self-contacting tree $T(r_{sc}, \theta)$ which does not intersect $\mathbf{y}$. Then it must be entirely on one side of $\mathbf{y}$, and it must therefore be entirely contained within some proper subtree (since it is disjoint from the trunk). Let S be the highest level subtree that contains γ. S is similar to the tree, and thus it also contains the self-contacting properties. γ must intersect the axis of symmetry of S, otherwise we could find a higher level subtree that contains γ. Let S have trunk $\mathbf{b} = b(\mathbf{A})$, where $\mathbf{A} \in \mathcal{A}_k$, for some $k \geq 1$. Thus $S = m_{\mathbf{A}}(T(r, \theta))$ and $S \sim_r^k T(r_{sc}, \theta)$. In particular, there must be a simple, closed curve γ' on $T(r_{sc}, \theta)$, such that $\gamma = m_{\mathbf{A}}(\gamma')$, hence $\gamma \sim_r^k \gamma'$. Now γ crosses the axis of symmetry of its tree, and therefore γ' must also cross the axis of symmetry of $T(r_{sc}, \theta)$, which is $\mathbf{y}$. $\square$

Corollary 3.2.0.26 *Any self-contacting non-space-filling tree contains simple, closed curves that intersect $\mathbf{y}_{[0, y_{\max})}$.*

Proof. A self-contacting, non-space-filling tree contains simple, closed curves and so we can apply the previous two lemmas. $\square$

We now put these results together to obtain the main theorem to determine the condition for self-contact. A similar condition was presented in [31], but a proof was not provided. Perhaps a proof was not given because the result seems intuitive. However, we offer a proof because a similar theorem will arise in the study of holes of closed ϵ-neighbourhoods, and that result is not as intuitively obvious.

Theorem 3.2.0.27 (SELF-CONTACT CRITERIA THEOREM) *As in* [31], *to determine the self-contacting ratio for a given $\theta \in (0°, 180°)$, it suffices to determine the smallest scaling ratio such that one of the following holds:*

1. *A tip point is on* $\mathbf{y}_{(1,y_{\max}]}$ *(if* $y_{\max} > 1$*)*

2. *A tip point or branch endpoint is on the trunk, i.e., on* $\mathbf{y}_{[0,1]}$

Proof. Let $\theta \in (0°, 180°)$, and let r_{sc} be the unique self-contacting scaling ratio for θ. There are two cases, depending on whether the tree is space-filling or not. If the tree is space-filling, then r_{sc} is the smallest ratio such that both 1 and 2 hold, since they must both hold if the tree is space-filling (note that this occurs for $\theta = 90°$ or $135°$). So assume that the tree is not space-filling. Then there are simple, closed curves that intersect $\mathbf{y}_{[0,y_{\max}]}$. If γ is a simple closed curve that intersects the y-axis, then it must intersect either $\mathbf{y}_{(1,y_{\max}]}$ or the trunk, but not both (except possibly at the point $(0, 1)$). If γ intersects $\mathbf{y}_{(1,y_{\max}]}$, then it suffices to determine the smallest scaling ratio such that a tip point is on the line segment $\mathbf{y}_{(1,y_{\max}]}$ (as shown in Lemma 3.2.0.23). If γ intersects the trunk, then it suffices to determine the smallest scaling ratio such that a tip point or branch endpoint is on the trunk (as shown in Lemma 3.2.0.24). In addition, if a self-contacting tree is not space-filling, then there must be curves that intersect $\mathbf{y}_{[0,y_{\max}]}$ (since any curve is similar to a curve that intersects $\mathbf{y}_{[0,y_{\max}]}$). Therefore, to determine the self-contacting scaling ratio r_{sc} for a certain branching angle θ, it suffices to determine the conditions for simple, closed curves to intersect $\mathbf{y}_{[0,y_{\max}]}$, for $\Gamma(r, \theta)$ is empty if there are no simple, closed curves that intersect the $\mathbf{y}_{[0,y_{\max}]}$. □

In a self-contacting, non-space-filling tree, the curves that intersect $\mathbf{y}_{[0,y_{max}]}$ are important, because any other curve is the image of such a curve under an appropriate address map.

Definition 3.2.0.28 *Let* $T(r_{sc}, \theta)$ *be a self-contacting tree. Let* $k \geq 0$*. Then* $\gamma \in \Gamma(r_{sc}, \theta)$ *is a* **level** k **loop** *if there exists a level* k *subtree* $S_{\mathbf{A}}$*, for some* $\mathbf{A} \in \mathcal{A}_k$*, that contains* γ*; and no subtree of level higher than* k *entirely contains* γ*. In other words, the address map* $m_{\mathbf{A}}$ *is the address map such that* γ *is equal to* $m_{\mathbf{A}}(\gamma')$*, where* γ' *is a curve that intersects* $\mathbf{y}_{[0,y_{max}]}$ *. Let* $\Gamma_k(r_{sc}, \theta)$ *denote the set of all level* k *simple, closed curves.*

Observations. The elements of $\Gamma_0(r_{sc}, \theta)$ must intersect the y-axis, and for $k \geq 1$, the elements of $\Gamma_k(0, r_{sc}, \theta)$ must intersect the linear extension of some level k branch. Given any level k curve γ, there is a level 0 curve γ' such that there is an address $\mathbf{A} \in \mathcal{A}_k$ and $\gamma = m_{\mathbf{A}}(\gamma')$, hence $\gamma \sim_r^k \gamma'$. So we have an action of the monoid M_{LR} on the set of closed curves. The level 0 curves form a kind of fundamental domain in the sense that all other curves can be obtained as images of these curves under the action of the monoid M_{LR}, and no level 0 curve is mapped to another level 0 curve under this action.

Since a hole of a self-contacting tree is $O(\gamma)$ for a simple, closed curve γ of the tree such that $O(\gamma)$ is disjoint from the tree, we can also define the level of a hole.

Definition 3.2.0.29 *If $\gamma \in \Gamma_k(r_{sc}, \theta)$ is such that $O(\gamma)$ is disjoint from the tree $T(r, \theta)$, then the hole $O(\gamma)$ is a* **level k hole**.

The notion of the level of a hole of a tree will be generalized to level of a hole of a closed ϵ-neighbourhood, and this will prove to be an important feature of a hole. Note that we still haven't provided a proof that such simple, closed curves exist. We will discuss this after we discuss self-contacting trees further.

3.3 Methods to Determine Self-Contacting Scaling Ratio r_{sc}; Comments on Self-Avoiding and Self-Contacting Trees

Now that we have the Self-Contact Criteria Theorem, we would like to know how to determine the self-contacting ratio r_{sc} as a function of the branching angle θ. We consider the three angle ranges separately, because there are different methods depending on the angle range. In the previous section, we established that r_{sc} is the smallest scaling ratio such that a tip point is on $\mathbf{y}_{(1,y_{\max}]}$ or such that either a tip point or a branch endpoint is on the trunk. When is it the case that the self-contacting tree has a tip point on $\mathbf{y}_{(1,y_{\max}]}$, and when is it the case that there is a tip point or branch endpoint on the trunk? According to [31], the former holds for branching angles less that 90°, while the latter holds for angles greater than 90°. We shall give a basic proof for this proposition. At 90°, the self-contacting tree is space-filling (as mentioned in [31]), so both cases are true. The self-contacting tree at 135° is also space-filling (again mentioned in [31]).

Proposition 3.3.0.30 *For branching angles θ where $0° < \theta \leq 90°$, self-contact occurs at the smallest scaling ratio such that a tip point is on $\mathbf{y}_{(1,y_{\max}]}$. For branching angles θ where $90° \leq \theta < 180°$, self-contact occurs at the smallest scaling ratio such that a tip point or branch endpoint is on the trunk.*

Proof. Let $T(r,\theta) \in \mathcal{T}_{sc}$. Consider the subtree S_R, i.e., the subtree whose trunk is the level 1 branch $b(R)$. It is symmetric about its linear extension L. L has slope $\cot\theta$, and it goes through the point $(0,1)$. If the slope is positive or 0, then the subtree must intersect $\mathbf{y}_{(1,y_{\max}]}$ (since this is closer than the trunk), and thus it intersects its mirror image subtree $S^* = S_L$ in at least one tip point. The slope is positive or 0 precisely when $0° < \theta \leq 90°$, so self-contact occurs at the smallest scaling ratio such that a tip point is on $\mathbf{y}_{(1,y_{\max}]}$. If the slope is negative or 0, then the subtree intersects the trunk, and thus it intersects S^* in at least one tip point or branch endpoint. The slope is negative precisely when $90° \leq \theta < 180°$, so self-contact occurs at the smallest scaling ratio such that a tip point or branch endpoint is on the trunk. □

Corollary 3.3.0.31 *For self-contacting trees with $\theta \leq 90°$, the non-trivial points of S_R that are on $\mathbf{y}$ are on S_{RL}. For self-contacting trees with $\theta \geq 90°$, the non-trivial points of S_R that are on $\mathbf{y}$ are on S_{RR}.*

In the following subsections, we discuss the criteria for self-contact in each of the three main angle ranges. The two angles 90° and 135° are discussed separately. Results about self-contacting and self-avoiding trees are presented. This analysis is more comprehensive than the analysis in [31] because we study the self-contacting and self-avoiding trees. The analysis also sheds new light on the self-contacting trees.

3.3.1 First Angle Range: $0° < \theta < 90°$

To determine the self-contacting scaling ratio r_{sc} for the first angle range, we know that it suffices to determine the smallest scaling ratio such that there is a tip point of S_{RL} on $\mathbf{y}_{(1,y_{\max}]}$. Recall that the turning number for θ, denoted $N(\theta)$, or just N, is the smallest integer such that $N\theta \geq 90°$. A result equivalent to the following proposition was presented in [31], but a complete proof was not given. We provide a proof because it sheds more light on the properties of non-overlapping trees.

Proposition 3.3.1.1 *Let $\theta < 90°$, let $r \leq r_{sc}$, and let $T(r, \theta)$ be the corresponding non-overlapping tree. The point P_c with address $RL^{N+1}(RL)^{\infty}$ is a point on S_{RL} that has minimal distance to $\mathbf{y}$, where N is the turning number for θ.*

Proof. Let $T(r, \theta)$ be a non-overlapping tree with $\theta < 90°$. The subtree S_{RL} has a vertical trunk. The subtree S_{RLL} is closer to $\mathbf{y}$ than the subtree S_{RLR}, and moreover, the subtree S_{RLR} is disjoint from the left side of $lin(RL)$ (by the Disjoint Lemma 3.2.0.20). So a point on S_{RL} that has minimal distance to $\mathbf{y}$ is necessarily on S_{RLL}.
We continue this process of determining which subtree at the next higher level is closer to $\mathbf{y}$. In general, for non-overlapping trees with $\theta < 90°$, if a branch $b(\mathbf{A})$ that forms the trunk of some subtree $S_{\mathbf{A}}$ has negative slope and its endpoint is closer to $\mathbf{y}$ than its starting point, then the next higher level subtree that descends from $S_{\mathbf{A}}$ and is closer to $\mathbf{y}$ is necessarily $S_{\mathbf{A}L}$. If the branch had positive slope, then the next higher level subtree that is closer to $\mathbf{y}$ is necessarily $S_{\mathbf{A}R}$. If the branch is horizontal, then the choice is arbitrary, because the subtrees $S_{\mathbf{A}R}$ and $S_{\mathbf{A}L}$ would be symmetric about $lin(\mathbf{A})$ and would be at the same distance from $\mathbf{y}$.
Now the slope of $b(RLL)$ is negative, because $\theta < 90°$, and so the subtree S_{RLLL} is closer to $\mathbf{y}$ than S_{RLLR}. For any integer j such that $0 \leq j < N$, the branch $b(RLL^j)$ has negative slope, so the subtree $S_{RLL^{N-1}}$ is the level $N+1$ subtree closest to $\mathbf{y}$ that is also a subtree of S_{RL}. There are two cases. The branch $b(RLL^N)$ may have positive slope or be horizontal.
In the first case, suppose the branch $b(RLL^N)$ has positive slope, so $N\theta > 90°$. Then the subtree $S_{RLL^N R}$ is closer than $S_{RLL^N L}$. The branch $b(RLL^N R)$ has slope equal to the slope of $b(RLL^N)$, so the next subtree would be $S_{RLL^N RL}$, and we would continue to alternate between right and left. That is, for any subtree $S_{RLL^N (RL)^k}$ for some $k \geq 0$, the next higher level subtree that is closer to $\mathbf{y}$ is $S_{RLL^N (RL)^k L}$, and for any subtree $S_{RLL^N R(LR)^k}$ for some $k \geq 0$, the next higher level subtree that is closer to $\mathbf{y}$ is $S_{RLL^N R(LR)^k L}$. Therefore, the tip point P_c with address $RLLN(RL)^{\infty} = RL^{N+1}(RL)^{\infty}$ is a point on S_{RL} with minimal distance to $\mathbf{y}$.

If $b(RLL^N)$ is horizontal, then the subtree $S_{RLL^N R}$ is at the same distance to $\mathbf{y}$ as $S_{RLL^N L}$. Without loss of generality, we choose the subtree $S_{RLL^N R}$. Then the next subtree would have to be $S_{RLL^N RL}$ as in the previous case. The choice for the next subtree is arbitrary again, and the pattern repeats itself, so that the point P_c with address $RL^{N+1}(RL)^\infty$ is a point on S_{RL} with minimal distance to $\mathbf{y}$. □

Corollary 3.3.1.2 *Let $\theta < 90°$, let $r \leq r_{sc}$, and let $T(r, \theta)$ be the corresponding non-overlapping tree. Let N be the turning number for θ. If $N\theta > 90°$, then there is a unique point of S_{RL} with minimal distance to $\mathbf{y}$, namely the point with address $RL^{N+1}(RL)^\infty$. If $N\theta = 90°$, there are infinitely many points on S_{RL} with minimal distance to $\mathbf{y}$, namely the points with addresses of the form $RL^{N+1}\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$ (see Equation 2.3.6).*

Proof. If $N\theta > 90°$, the proof for the previous proposition showed that the point P_c is unique, because there are no arbitrary choices. If $N\theta = 90°$, then any path where every second branch after $b(RL^{N+1})$ is horizontal leads to point on S_{RL} with minimal distance to $\mathbf{y}$. Such paths are precisely given by addresses of the form $RL^{N+1}\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$. □

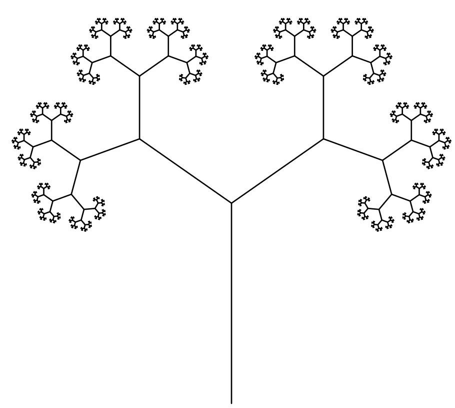

Figure 3.7: $T(0.56, 55°)$: unique point of S_{RL} with minimal distance to $\mathbf{y}$

Figure 3.7 shows a tree with a unique tip point of S_{RL} that has minimal distance to $\mathbf{y}$, and Figure 3.8 shows a tree with infinitely many.

Now we can determine the self-contacting scaling ratio for angles in the first angle range. For a given angle in the first angle range, r_{sc} will be the ratio that places the

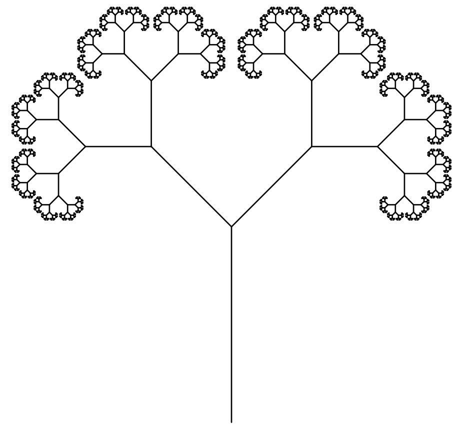

Figure 3.8: $T(0.58, 45°)$: infinitely many points of S_{RL} with minimal distance to $\mathbf{y}$

point with address $RL^{N+1}(LR)^{\infty}$ on $\mathbf{y}$.

Notation. For $\theta < 90°$ and $r \leq r_{sc}$, let $P_{c1} = (x_{c1}, y_{c1})$ denote the point on the tree $T(r, \theta)$ with address $RL^{N+1}(RL)^{\infty}$. Then

$$x_{c1} = r\sin\theta - \sum_{i=1}^{N-2} r^{i+2}\sin(i\theta) - \frac{r^{N+1}}{1-r^2}\left[\sin((N-1)\theta) + r\sin(N\theta)\right] \tag{3.3.1}$$

Details of the calculation can be found in Appendix B.

The following proposition was stated in [31], but a proof was not given. We provide the proof because similar theory will be used when dealing with closed ϵ-neighbourhoods.

Proposition 3.3.1.3 *Let $\theta \in (0°, 90°)$. The value of r_{sc} is the unique solution of $x_{c1} = 0$ that is in $(0, 1)$.*

Proof. The point P_{c1} has minimal distance to the y-axis out of all tip points of S_{RL}, provided $r \leq r_{sc}$ (by Proposition 3.3.1.1). The point P_{c1} will have the same coordinates as its mirror image P^*_{c1} precisely when $x_{c1} = 0$. By the Self-Contact Criteria Theorem 3.2.0.27 and Corollary 3.3.0.31, r_{sc} is the smallest scaling ratio such that a tip point of S_{RL} is on $\mathbf{y}$, and this must be P_{c1} since it is the closest. The point P_{c1} is on $\mathbf{y}$ when $x_{c1} = 0$, so r_{sc} must be the unique solution of $x_{c1} = 0$ that is in $(0, 1)$. $\square$

Remarks. In general, any point on a self-contacting tree that corresponds to more than one address is a **self-contacting point**. The corresponding addresses are **self-contacting addresses** or **self-contact addresses**. We can restrict our attention to self-contacting points that are on $\mathbf{y}$ and the addresses that start with R. Any self-contacting point can be obtained as the image of a self-contacting point on $\mathbf{y}$ under a suitable address map. As mentioned in Corollary 3.3.1.2, for a non-special angle θ in the first range (so $N\theta > 90°$), there is a unique self-contacting point of $T(r_{sc}, \theta)$ on S_R and $\mathbf{y}$, namely the point with address $RL^{N+1}(RL)^{\infty}$. For the special angles θ_N in the first angle range (for which $N\theta = 90°$ and $N \geq 2$), there are infinitely many self-contacting points of $T(r_{sc}, \theta)$ on S_R and $\mathbf{y}$, namely the points with addresses of the form $RL^{N+1}\mathbf{A}$ for $\mathbf{A} \in \mathcal{AL}_{\infty}$.

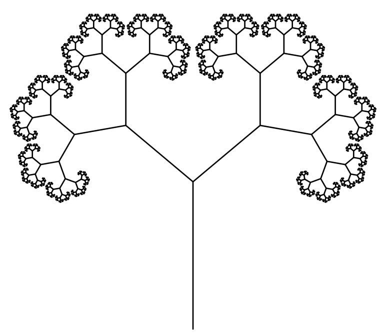

Figure 3.9: $T(0.595, 50°)$

A tree may have horizontal branches (ex. $\theta = 50°$, see Figure 3.9), but the corresponding self-contacting tree has a unique tip point on S_R that is also on $\mathbf{y}$. This is because there are no horizontal branches on the path that minimizes distance to $\mathbf{y}$, in contrast to the trees with special angles.

3.3.2 Contact for $\theta = 90°$

When $\theta = 90°$, the level 1 branches are horizontal. In fact, all odd level branches are horizontal and all even level branches are vertical. See Figure 3.10 for an image of a tree with branching angle of 90°. There are infinitely many paths on S_R that lead to points that have minimal distance to $\mathbf{y}$.

Proposition 3.3.2.1 *For $\theta = 90°$ and $r \leq r_{sc}$, the points with addresses of the form*

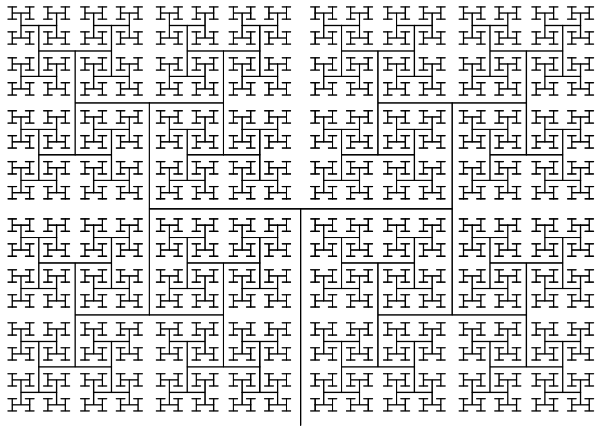

Figure 3.10: $T(0.7, 90°)$

$RLL\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$ (see Eq. 2.3.6) have minimal distance to $\mathbf{y}$ for S_{RL}; the points with addresses of the form $RRL\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$, all have minimal distance to $\mathbf{y}$ for S_{RR}; and these distances are equal.

Proof. Let $\theta = 90°$ and $r \leq r_{sc}$. As noted above, every odd level branch is horizontal and every even level branch is vertical. The subtree S_{RL} is at the same distance from $\mathbf{y}$ as S_{RR}. The subtrees S_{RLL} and S_{RRR} both have horizontal trunks with endpoints closer to $\mathbf{y}$ than their starting points. So on each subtree there are infinitely many paths leading to tip points that have minimal distance to $\mathbf{y}$, namely the points identified in the proposition. $\square$

Corollary 3.3.2.2 *The self-contacting scaling ratio for $\theta = 90°$ is $1/\sqrt{2}$.*

Proof. The x-coordinate of the points described in the previous proposition is given by

$$x = r - r^3 - r^5 \cdots = r - \frac{r^3}{1 - r^2} = \frac{r(1 - 2r^2)}{1 - r^2}$$

and setting this value equal to zero to obtain r_{sc} yields

$$r_{sc}(90°) = \frac{1}{\sqrt{2}}. \tag{3.3.2}$$

Note that this agrees with the method to determine r_{sc} in the first angle range, and it also agrees with the method in the second angle range.

The self-contacting tree for $\theta = 90°$ is space filling; it is a rectangle with corner points $(\sqrt{2}, 2), (-\sqrt{2}, 2), (-\sqrt{2}, 0), (\sqrt{2}, 0)$.

3.3.3 Contact in the Second Angle Range: $90° < \theta < 135°$

For branching angles in the second angle range, we have established that the self-contacting scaling ratio is the smallest scaling ratio such that a tip point or branch endpoint of S_{RR} reaches the trunk (see 3.2.0.27 and 3.3.1.2). This angle range is particularly interesting, because for angles between 120° and 135°, the point on S_{RR} with minimal distance to $\mathbf{y}$ is dependent on the scaling ratio.

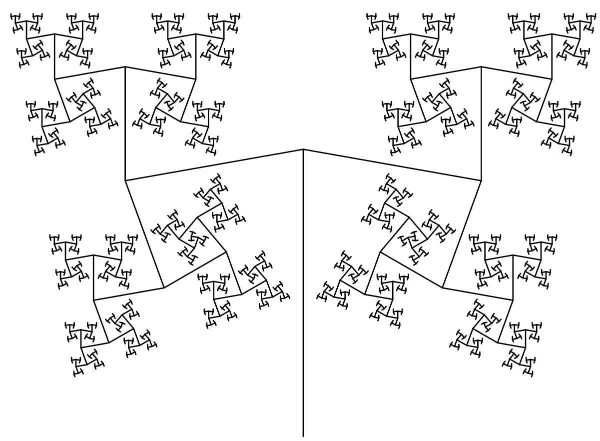

Figure 3.11: Tree with angle in second angle range: $T(0.63, 100°)$

There are two cases to consider for angles in the second angle range. We first discuss angles less than or equal to 120°, for which the address point on S_{RR} with minimal distance to $\mathbf{y}$ is the same for any scaling ratio. Then we discuss angles between 120° and 135°, where the address is not the same for any scaling ratio.

Proposition 3.3.3.1 *Let $\theta \in (90°, 120°]$ and $r \leq r_{sc}$. Then the tip point with address $RRR(LR)^\infty$ has minimal distance to* $\mathbf{y}$ *for S_{RR}.*

Proof. Let $T(r, \theta)$ be such that $\theta \in (90°, 120°]$ and $r \leq r_{sc}$. Consider the subtree S_{RR}. The branch $b(RR)$ has positive slope for such an angle, and the endpoint is closer than the starting point. Thus the subtree S_{RRR} is closer to $\mathbf{y}$ than S_{RRL}, and contains a portion that is closer than the endpoint of $b(RR)$, because the branch

$b(RRR)$ either has negative slope or is vertical. The linear extension of the branch $b(RRR)$ separates the two subtrees S_{RRRL} and S_{RRRR}, the first being on the left side, the latter on the right. Thus the subtree S_{RRRL} contains a portion that is closer to $\mathbf{y}$ than the endpoint of $b(RRR)$. The branch $b(RRRL)$ is parallel to $b(RR)$, so by a similar argument, the subtree S_{RRRLR} contains a portion that is closer than the endpoint of $b(RRRL)$. That is, we can keep alternating between right and left to continue on to higher and higher level subtrees that are closer to $\mathbf{y}$. The resulting address is $RRR(LR)^{\infty}$, and the tip point with this address has minimal distance to $\mathbf{y}$ $\square$

Proposition 3.3.3.2 *Let $\theta \in (120°, 135°)$ and $r \leq r_{sc}$.*

1. *If $r \geq -\sin(3\theta)\csc(2\theta)$, the point with address $RRR(LR)^{\infty}$ has minimal distance to $\mathbf{y}$ for S_{RR}.*

2. *If $r \leq -\sin(3\theta)\csc(2\theta)$, the point with address RR has minimal distance to $\mathbf{y}$ for S_{RR}.*

Proof. Let $T(r,\theta)$ be such that $\theta \in (120°, 135°]$ and $r \leq r_{sc}$. Consider the subtree S_{RR}. The branch $b(RR)$ has positive slope for such an angle, and the endpoint is closer than the starting point. Now the branch $b(RRR)$ has positive slope, and the subtree S_{RRR} does not necessarily contain a portion that is closer to $\mathbf{y}$ than the endpoint of $b(RR)$. Consider points with addresses of the form $RR(RL)^k$, for $k \geq 0$. Let $P_k = (x_k, y_k)$ denote the point with address $RR(RL)^k$, and let $P_{\infty} = (x_{\infty}, y_{\infty})$ denote the point with address $RR(RL)^{\infty}$. Then

$$\begin{aligned} x_0 &= r\sin\theta + r^2\sin(2\theta) && (3.3.3)\\ x_k &= x_0 + [r^3\sin(3\theta) + r^4\sin(2\theta)][1 + r^2 + \cdots r^{2(k-1)}], \quad k \geq 1 && (3.3.4)\\ x_{\infty} &= x_o + \frac{1}{1-r^2}[r^3\sin(3\theta) + r^4\sin(2\theta)] && (3.3.5) \end{aligned}$$

We are assuming the tree is non-overlapping, so necessarily we have $x_0 \geq 0$. Let

$$f(r) = r^3\sin(3\theta) + r^4\sin(2\theta)$$

If $f(r) < 0$, then $x_{\infty} < x_k$ for all $k \geq 0$, so the tip point with address $RRR(LR)^{\infty}$ is a unique point that has minimal distance to $\mathbf{y}$ for S_{RR}. This inequality is satisfied when $r > -\sin(3\theta)\csc(2\theta)$.

If $f(r) = 0$, then the points P_k for $k \geq 0$ and P_∞ all have minimal distance to $\mathbf{y}$.

If $f(r) > 0$, then $x_0 < x_k$ for all $k \geq 1$ and $x_0 < x_\infty$, so the point with address RR is a unique point of S_{RR} that has minimal distance to $\mathbf{y}$. This inequality is satisfied if $r < -\sin(3\theta)\csc(2\theta)$. □

Notation. For $\theta \in (90°, 135°)$, let P_{c2} denote the point with address $R^3(LR)^\infty$. Then the coordinates of P_{c2} in terms of r and θ are as follows. See Appendix B for more details.

$$x_{c2} = r\sin(\theta) + \frac{r^2}{1-r^2}[\sin(2\theta) + r\sin(3\theta)] \tag{3.3.6}$$

$$y_{c2} = 1 + r\cos(\theta) + \frac{r^2}{1-r^2}[\cos(2\theta) + r\cos(3\theta)] \tag{3.3.7}$$

Proposition 3.3.3.3 *Let $\theta \in (90°, 135°)$. The value of r_{sc} is the unique solution in $(0,1)$ to $x_{c2} = 0$ (see Equation 3.3.6). As in [31], the explicit expression for r_{sc} is given by*

$$r_{sc} = \frac{-\cos\theta - \sqrt{2 - 3\cos^2\theta}}{4\cos^2\theta - 2} \tag{3.3.8}$$

Proof. For $\theta \in (90°, 120°]$, the previous proposition established that P_{c2} has minimal distance to $\mathbf{y}$, so r_{sc} is the scaling ratio that places P_{c2} on $\mathbf{y}$. For $\theta \in (120°, 135°)$, there are two cases depending on the scaling ratio. If the point P_0 with address RR is on $\mathbf{y}$, then r satisfies the equation

$$x_0 = r\sin\theta + r^2\sin(2\theta) = 0$$

so $r = -1/2\cos\theta$. Any r that satisfies this equation cannot satisfy $r < \sin(3\theta)\csc(2\theta)$ (for angles between 120° and 135°), so this contradicts that the tree is self-contacting, since the point P_{c2} would be on the left side of $\mathbf{y}$.

So for self-contacting trees with angles between 90° and 135°, P_{c2} is indeed on $\mathbf{y}$, and r_{sc} is given by Equation 3.3.8. □

For self-contacting trees with angles in the second angle range, there is a unique self-contacting point, namely the tip point P_{c2} with address $R^3(LR)^\infty$.

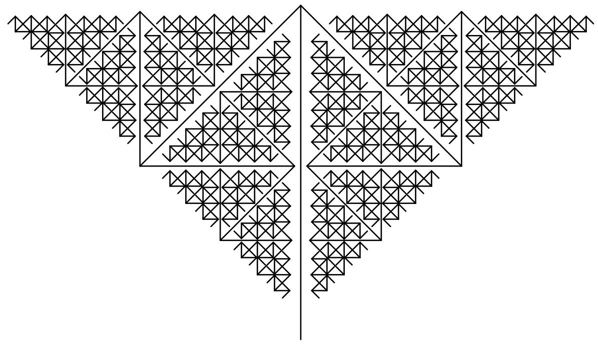

Figure 3.12: $T(0.68, 135°)$

3.3.4 Contact For $\theta = 135°$

When $\theta = 135°$, the branch $b(RR)$ is horizontal. See Figure 3.12. Consider the points $P_k = (x_k, y_k)$ with addresses $RR(RL)^k$ for $k \geq 0$. These points are endpoints of horizontal branches that descend from $b(RR)$. The x-coordinates are given by

$$x_k = \left(\frac{r}{\sqrt{2}} - r^2\right)(1 + r^2 + \cdots r^{2k}) \tag{3.3.9}$$

So assuming a tree with $\theta = 135°$ is non-overlapping, $x_0 \leq x_k$ for all k, and equality occurs when $x_0 = 0$. So for self-avoiding trees, the point with address RR is the unique point of S_{RR} with minimal distance to $\mathbf{y}$. For the self-contacting tree, there are infinitely many points of S_{RR} that are on $\mathbf{y}$, and the tree is space-filling. It is a triangle with vertices $(1, 1), (-1, 1), (0, 0)$.

Proposition 3.3.4.1 *The self-contacting scaling ratio for $\theta = 135°$ is $1/\sqrt{2}$.*

Proof. As discussed above, self-contact occurs when the point with address RR is on $\mathbf{y}$. We find r_{sc} by setting $x = 0$ for this point, and this yields

$$r_{sc}(135°) = \frac{1}{\sqrt{2}} \tag{3.3.10}$$

3.3.5 Contact in the Third Angle Range: $135° < \theta < 180°$

For self-contacting trees with branching angle θ such that $135° < \theta < 180°$, there are infinitely many self-contacting points on S_{RR}. For self-avoiding trees, there is a unique point on S_{RR} with minimal distance to $\mathbf{y}$.

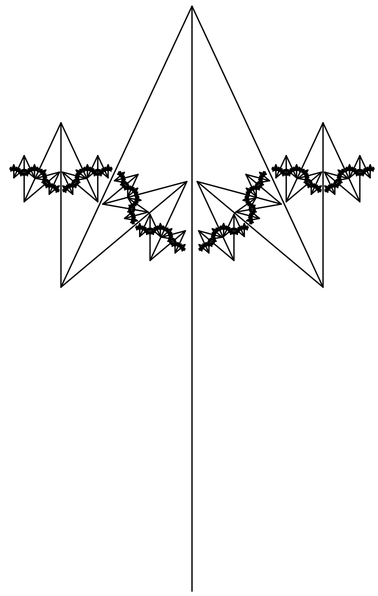

Figure 3.13: Tree with angle in the third angle range: $T(0.53, 155°)$

Proposition 3.3.5.1 *Let $\theta \in (135°, 180°)$ and $r \leq r_{sc}$. Then the point with address RR has minimal distance to $\mathbf{y}$ for S_{RR}.*

Proof. Let $T(r, \theta)$ be a non-overlapping tree with angle in the third angle range. Let $P_c = (x_c, y_c)$ denote the point with address RR. The branch $b(RR)$ has negative slope, so the subtree S_{RRL} is closer to $\mathbf{y}$ than S_{RRR}. The endpoint of the branch $b(RRL)$ is further from $\mathbf{y}$ than P_c, so we consider the branch $b(RRRL)$ (since its endpoint has smaller x-value than for the branch $b(RRRR)$). The the endpoint $P_b = (x_b, y_b)$ of this branch has $x_b = x_c(1 + r^2)$, by the scaling nature of the tree, which means that $x_b \geq x_c$. Similarly, all endpoints of branches of the form $b(RR(RL)^k)$ have x-values greater than or equal to x_c, so none of them are closer to $\mathbf{y}$ either. These points have the same x-coordinate if they are all on $\mathbf{y}$, otherwise P_c is the closest. Thus there are no vertex points closer than P_c, nor any tip points, since the closest would be with address $RR(RL)^\infty$, and the x-coordinate of this point is $x_c/(1 - r^2)$. Therefore P_c must have minimal distance to $\mathbf{y}$. $\square$

Corollary 3.3.5.2 *Let $\theta \in (135°, 180°)$. The self-contacting tree $T(r_{sc}, \theta)$ has infinitely many points on $\mathbf{y}$, namely the points with addresses $RR(LR)^k$ for $k \geq 0$ and* *the tip point with address $RR(LR)^\infty$. For $r < r_{sc}$, the self-avoiding tree $T(r,\theta)$ has a unique point on S_{RR} with minimal distance to* $\mathbf{y}$*, namely the point with address RR.*

Notation. Let $\theta \in (135°, 180°)$. The point with address RR is denoted by $P_{c3} = (x_{c3}, y_{c3})$. The coordinates of P_{c3} in terms of r and θ are:

$$\begin{aligned} x_{c3} &= r\sin\theta + r^2\sin(2\theta) \\ &= r\sin\theta[1 + 2r\cos\theta] \qquad (3.3.11) \\ y_{c3} &= 1 + r[\cos(\theta) + r\cos(2\theta)] \qquad (3.3.12) \end{aligned}$$

Proposition 3.3.5.3 *Let $\theta \in (135°, 180°)$. The value of r_{sc} is the unique solution in $(0,1)$ of $x_{c3} = 0$ (see Equation 3.3.11). As in [31], the explicit expression for r_{sc} is given by:*

$$r_{sc} = -\frac{1}{2\cos\theta} \qquad (3.3.13)$$

Proof. This follows directly from the Self-Contact Criteria Theorem 3.2.0.27 and the fact that P_{c3} has minimal distance to $\mathbf{y}$ for non-overlapping trees. Setting x_{c3} equal to 0 to find r_{sc}, we obtain 3.3.13. □

3.4 Size, Height and Width of Trees Revisited

We compare the sizes of different trees by comparing their heights and widths (which are well-defined for self-avoiding or self-contacting trees because they are bounded).

Mandelbrot and Frame discussed the height of a tree in [31]. Their results were limited to self-contacting trees, and they did not provide complete results. We now give some of the main results of their work along with our new results that will be useful later.

All self-avoiding or self-contacting trees have no portion of the tree below the line $y = 0$, so the height is equal to the maximal y-value of the tree, which is denoted by $y_{\max}$. Since the trees are symmetric about the y-axis, the width of a tree will be twice the maximal x-value of the tree, which is denoted by $x_{\max}$.

How can we find the height and width of a given tree? First we consider the height. In [31], it was stated without proof that the address $(RL)^\infty$ corresponds to a tip point that has maximal height for all tip points. We will prove that if a tree has height greater than 1, then the tip point with address $(RL)^\infty$ has y-coordinate equal to $y_{\max}$.

Proposition 3.4.0.4 *Let $T(r,\theta)$ be a self-contacting or self-avoiding tree that has height greater than 1. Then the point $P_h = (x_h, y_h)$ with address $(RL)^\infty$ is such that $y_h = y_{\max}$.*

Proof. Since T has height greater than 1, there is some portion of the tree that is above the line $y = 1$. First we show that the maximal height is reached at either a branch endpoint or a tip point. For suppose it is in some branch interior. If the branch is not horizontal, then one of its endpoints has a higher y-value, and that is a contradiction. If the branch is horizontal, then the endpoint and starting point of the branch are at the same height and so half of the subtree with this branch as trunk is above the branch, and that is also a contradiction. Without loss of generality, a path to a point with maximal height starts with R (it is not the empty address since the tree has a portion above the line $y = 1$). If $\theta \leq 90°$, then the endpoint of the branch $b(R)$ has $y > 1$. We can get higher than this point, so should the next branch on the path be $b(RL)$ or $b(RR)$? The branch $b(RL)$ is vertical, while the branch $b(RR)$ is not. The subtrees S_{RL} and S_{RR} are each contained within a rectangle of the same size, with sides parallel to the trunks of length rh and sides perpendicular to the trunks of length rw, where h and w denote the height and width of the tree. The subtree S_{RL} has a higher vertical range than the subtree S_{RR}, because it is above the linear extension of $b(R)$ and S_{RR} is below this linear extension (by the Disjoint Lemma 3.2.0.20). So a point of maximal height is on the subtree S_{RL}. Now the subtree S_{RL} is not only similar to the tree itself, but its trunk is vertical. So to find a point of maximal height of S_{RL}, we could go to the subtree S_{RLRL} by a similar argument. We can keep going to higher and higher level subtrees of the form $S_{(RL)^k}$, and at each stage the subtree will have a vertical trunk and would contain points on it that are higher than the endpoint of its trunk (by self-similarity of the tree). Thus we could keep going *ad infinitum*, and the point P_h with address $(RL)^\infty$ has $y_h = y_{\max}$. $\square$

Note. In the previous proof, the initial choice of R was arbitrary. If we had started with L, then we would go to the subtree S_{LR}. Any infinite path such that every second branch on the path is vertical and such that its endpoint is above its starting point leads to a tip point with maximal height, since the y-coordinate would be the same as for the point with address $(RL)^\infty$. So any address of the form $\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$ (see 2.3.6) corresponds to a tip point of maximal y-value (assuming the height of the tree is greater than 1).

For any tree, the y-value of a point with address $\mathbf{A}$ for $\mathbf{A} \in \mathcal{AL}_\infty$ is given by

$$\frac{1 + r\cos\theta}{1 - r^2} \tag{3.4.1}$$

See Figures 3.14 and 3.15.

Proposition 3.4.0.5 *The height of a self-contacting or self-avoiding $T(r,\theta)$ is given by*

$$y_{\max} = \begin{cases} \dfrac{1 + r\cos\theta}{1 - r^2} & \textit{if } r > -\cos\theta \\ 1 & \textit{if } r \leq \cos\theta \end{cases} \tag{3.4.2}$$

Proof. If the height of a tree is greater than 1, then we have already established that the height is reached by any point with address $\mathbf{A}$ for $\mathbf{A} \in \mathcal{AL}_\infty$, and the first case of the proposition is the y-value of such a tip point. Otherwise the height equals 1. □

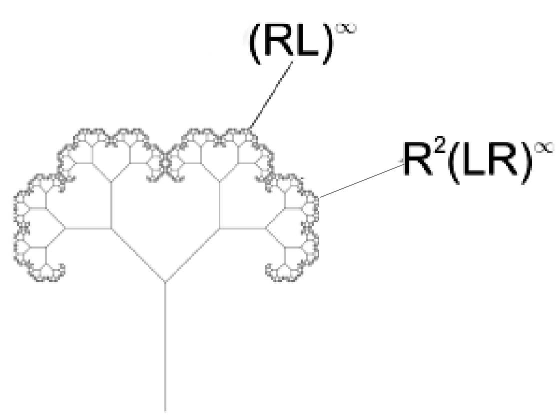

Figure 3.14: Points of maximal height and width for $T(r_{sc}, 45°)$

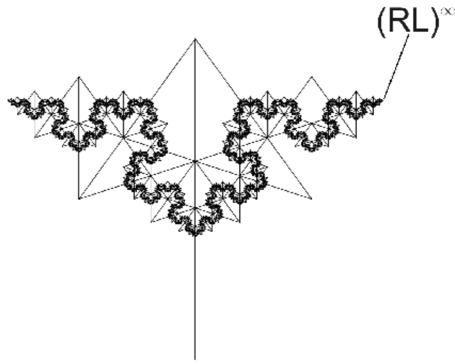

Figure 3.15: Point of maximal width for $T\left(\dfrac{-1+\sqrt{5}}{2}, 144°\right)$

Observations. For $\theta \in (0°, 90°]$, we have $r > -\cos\theta$, since $\cos\theta \geq 0$. For $\theta \in (90°, 135°)$, it is always possible to find $r \in (0, r_{sc})$ such that $r \leq -\cos\theta$. For an example of a tree with $r \leq -\cos\theta$, see $T_{10}(0.51, 165°)$ in Figure 2.6. For $\theta > 135°$, we always have $r \leq -\cos\theta$, so the trees all have height equal to 1.

Intuitively, we can see that the height increases as r increases. This fact will be useful, so we provide a proof.

Proposition 3.4.0.6 *Let $\theta \in (0°, 180°)$. Then the value of $y_{\max}$ as a function of r is a weakly increasing function.*

Proof. Let θ be given. For values of r such that $r > -\cos\theta$, recall that

$$y_{\max} = \frac{1 + r\cos\theta}{1 - r^2}$$

Let $f(r) = \dfrac{1 + r\cos\theta}{1 - r^2}$. Then $f(r)$ is positive, for all values of r such that $r > -\cos\theta$, the numerator increases as r increases and the denominator decreases as r increases. Hence $y_{\max}$ is increasing on $(0, 1)$ if $\theta \leq 90°$, or increasing on $(-\cos\theta, 1)$ if $\theta \in (90°, 180°)$. If $\theta \in (90°, 180°)$, and for values of r such that $r \leq -\cos\theta$, $y_{\max} = 1$, so $y_{\max}$ is constant on $(0, -\cos\theta)$. Therefore $y_{\max}$ as a function of r is a weakly increasing function. $\square$

Proposition 3.4.0.7 *For angles θ such that $90° < \theta < 135°$, the maximal height of the self-contacting tree $T(r_{sc}, \theta)$ is greater than 1, and as a result, the line $y = y_{max}$ is above the endpoint of the trunk. When $\theta = 135°$, the maximal height of the self-contacting tree $T(r_{sc}, 135°)$ is 1, the line $y = y_{max}$ is the line $y = 1$, and the tip points with addresses of the form $\mathbf{A}$ where $\mathbf{A} \in \mathcal{AL}_\infty$, along with all points with addresses of the form $\mathbf{A} \in \mathcal{AL}_{2k}$, $k \geq 0$ are also on the line $y = 1$.*

Proof. See Appendix B. The proof is straightforward, but a bit long.

Proposition 3.4.0.8 *For angles such that $135° < \theta < 180°$ and for scaling ratios such that $r \leq r_{sc}$, the maximal height of a tree $T(r, \theta)$ is 1, and this height is reached only at the endpoint of the trunk.*

Proof. Recall that for angles such that $135° < \theta < 180°$, the point on the subtree S_R that is closest to the trunk is the branch endpoint at RR, provided $r \leq r_{sc}$. So consider the subtree S_{RR}. The top of the trunk of this subtree is the branch endpoint at RR. Consider the line L through this branch endpoint RR that is perpendicular to the branch $b(RR)$. This line has positive slope. Suppose that the maximal height of the tree $T(r, \theta)$ was greater than 1 and it was obtained at some other point on the tree other than the top of the trunk. Suppose this point is denoted P. The mirror image P^* of P would also be at this maximal height. Then by the scaling nature of the subtrees, there would have to be two points on the subtree S_{RR} that would correspond to images of the point P and P^* under the address map m_{RR}. One of these points must be to the left of the line L, and thus would necessarily be closer to the trunk than the branch endpoint at RR, and this contradicts $r \leq r_{sc}$. Thus the maximal height of self-contacting or self-avoiding trees with angles in the third angle range is equal to 1, and occurs only at the endpoint of the trunk. $\square$

Now consider the width of a tree. We will use similar ideas to determine a path to a point of maximal x-value. For the following two lemmas, refer to Figures 3.14 and 3.15.

Lemma 3.4.0.9 *Let $\theta \in (0°, 90°)$, let N be the turning number, and let $T(r, \theta)$ be a self-avoiding or self-contacting tree. Then the x-value of the tip point with address*

$R^N(LR)^\infty$ in $T(r,\theta)$ is equal to $x_{\max}$. This value of $x_{\max}$ is given by:

$$x_{\max} = \sum_{i=1}^{N-2} r^i \sin(i\theta) \ + \frac{r^{N-1}}{1-r^2}\left[(\sin((N-1)\theta) + r\sin(N\theta)\right] \tag{3.4.3}$$

Proof. Earlier in this chapter we established that the path $RLL^N(RL)^\infty$ leads to a tip point of the tree that has minimal x-value (see Proposition 3.3.1.1). Consider the subtree S_{RL}. This subtree has a vertical trunk. So the point P_{c1} with address $RLL^N(RL)^\infty$ is $r^2w/2$ away from the linear extension of the branch $b(RL)$. This implies that the address $L^N(RL)^\infty$ leads to a tip point on the tree that is $w/2$ away from the y-axis. This point has $x < 0$ so it does not have a maximal x-value, but its mirror image will. The mirror image address is $R^N(LR)^\infty$. Some basic geometry shows that the x-value of the point at $R^N(LR)^\infty$ is:

$$x = \sum_{i=1}^{N-2} r^i \sin(i\theta) \ + \frac{r^{N-1}}{1-r^2}\left[\sin((N-1)\theta) + r\sin(N\theta)\right] \tag{3.4.4}$$

□

Lemma 3.4.0.10 *Let $\theta \in [90°, 180°)$, and let $T(r,\theta)$ be a self-avoiding or self-contacting tree. Then the x-value of the tip point at $(RL)^\infty$ is equal to $x_{\max}$. This value of $x_{\max}$ is given by:*

$$x_{\max} = \frac{r\sin\theta}{1-r^2} \tag{3.4.5}$$

Proof. Again we will choose a path that maximizes the x-value. Obviously the path will be on the subtree S_R, since the subtree S_L is disjoint from the right side of $\mathbf{y}$ by the Disjoint Lemma 3.2.0.20. Let $P_1 = (x_1, y_1)$ denote the point with address R. Now consider the subtrees S_{RL} and S_{RR}. The subtree S_{RL} has a vertical trunk, so will contain a portion that has greater x-values than x_1. The subtree S_{RR} is below the linear extension $lin(R)$ of the branch $b(R)$ (since the tree is not self-overlapping) and to the left of the the line that goes through P_1 and is perpendicular to $lin(R)$ (again because the tree is not self-overlapping). This means that there is no portion of the subtree S_{RR} that has $x > x_1$. So a point with maximal x-value is on the subtree S_{RL}. Similar to our proof in finding a tip point with maximal y-value, for each subtree

$S_{(RL)^k}$ we can go RL again to the next subtree $S_{(RL)^{k+1}}$. Thus the tip point with address $(RL)^k$ has maximal x-value. Then

$$x_{\max} = \frac{r \sin\theta}{1 - r^2} \tag{3.4.6}$$

Note. In the previous lemma, it is crucial that we have the condition that the tree is not self-overlapping. One can find examples of overlapping trees where this tip point does not have maximal y-value. For an example of such a tree, see [58].

Theorem 3.4.0.11 *Let $T(r,\theta)$ be any self-avoiding or self-contacting tree. Then the width w is given by*

$$w = \begin{cases} \displaystyle\sum_{i=1}^{N-2} r^i \sin(i\theta) \ + \frac{r^{N-1}}{1-r^2}\left[\sin((N-1)\theta) + r\sin(N\theta)\right] & \text{if } \theta < 90^\circ \\ \dfrac{2r\sin\theta}{1-r^2} & \text{if } \theta \geq 90^\circ \end{cases} \tag{3.4.7}$$

Proof. This theorem is a direct result of the previous two lemmas and the fact that $w = 2x_{\max}$. □

Proposition 3.4.0.12 *For a fixed $\theta \in (0^\circ, 180^\circ)$, the value of w as a function of r is an increasing function.*

Proof.

If $\theta < 90^\circ$, we will show that each summand in [3.4.7] is an increasing function. For each i such that $1 \leq i \leq N-2$, the function $f_i(r) = r^i \sin(i\theta)$ is increasing because $f_i'(r) = ir^{i-1}\sin(i\theta) > 0$. The function $f(r) = r/(1-r^2)$ is increasing because $f'(r) = (1+r^2)/(1-r^2)^2 > 0$. The function $g(r) = \sin((N-1)\theta) + r\sin(N\theta)$ is increasing, because $g'(r) = \sin(N\theta) > 0$. Hence $f(r)g(r)$ is increasing, and finally

$$w = 2\left[\sum_{i=1}^{N-2} f_i(r) \quad + f(r)g(r)\right]$$

must also be increasing.

If $\theta \geq 90^\circ$, then let

$$f(r) = \frac{r\sin\theta}{1-r^2}$$

Thus

$$f'(r) = \frac{\sin\theta(1 + r^2)}{(1 - r^2)^2} > 0$$

which implies that $w = f(r)$ is increasing.
Therefore, given any branching angle $\theta \in (0°, 180°)$, the value of w as a function of r is an increasing function. $\square$

A perhaps obvious feature of the symmetric binary fractal trees is that for a specific branching angle, as r increases, the size of the corresponding tree also increases. Consider the images of two trees with branching angle $50°$ given in Figures 3.4 and 3.4.

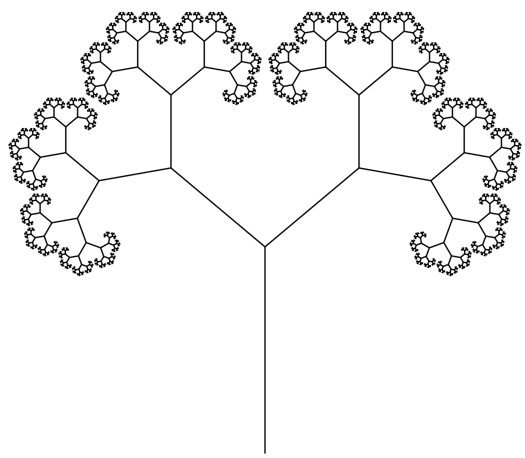

Figure 3.16: $T(0.595, 50°)$

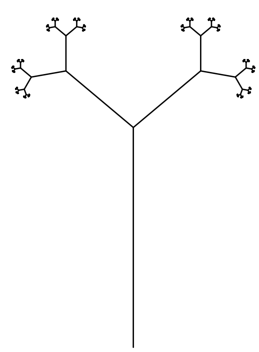

Figure 3.17: $T(0.4, 50°)$

Intuitively, for a given θ and $r_1, r_2 \in (0, 1)$ such that $r_1 < r_2$, we think of the tree

$T(r_1, \theta)$ as being 'smaller' than the tree $T(r_2, \theta)$. There are different ways to represent the 'size' of a tree. We will discuss one in terms of its height and width, and later mention other ways to characterize the relative size.

Observation. For a given tree $T(r, \theta)$, $BR(r, \theta)$ is the smallest rectangle with line segments parallel to the coordinate axes that is a superset of $T(r, \theta)$.

Proposition 3.4.0.13 *Let $\theta \in (0°, 180°)$, and let $r_1, r_2 \in (0, 1)$ such that $r_1 < r_2$. Then $BR(r_1, \theta) \subset BR(r_2, \theta)$.*

Proof. This proposition follows directly from the two previous propositions which state that the height h is a weakly increasing function of r and the width w is an increasing function of r. $\square$

There are numerous other ways to compare the relative sizes of trees with the same branching angle. For example, the trunk is the same for every tree, so one method of comparison could be based on the radius of the smallest disc centered at $(0, 1)$ that contains the portion of the tree without the trunk. Other examples include the radius of the smallest disc that covers a tree (with no fixed center) or the largest distance from the point $(0, 0)$ to any other point on the tree.

3.5 Special Types of Addresses and Points

In this section, we discuss certain classes of points of a tree, and present various results about these points. The reason for identifying and discussing these addresses and points is that will help us to locate holes in closed ϵ-neighbourhoods, and then to compare different trees by comparing the hole locations for their closed ϵ-neighbourhoods.

Recall that we defined certain sub-collections of addresses in Chapter 2 (see 2.3.5 and 2.3.6), which we repeat here because the collections are frequently referred to in this section:

$$\begin{aligned}
\mathcal{AL}_{2k} &= \{A_1A_2\cdots A_{2k} \mid A_{2i-1}A_{2i} \in \{RL, LR\},\ \ 1 \le i \le k\} \\
\mathcal{AL}_{\infty} &= \{A_1A_2\cdots \mid A_{2i-1}A_{2i} \in \{RL, LR\},\ \ \forall i\}
\end{aligned}$$

We also introduce notation for two particular addresses that are in $\mathcal{AL}_{\infty}$:

$$\mathbf{C}_R = RL(LR)^{\infty} \tag{3.5.1}$$

$$\mathbf{C}_L = LR(RL)^{\infty} \tag{3.5.2}$$

The reason for using 'C' will be explained in Subsection 3.5.4 dealing with canopy points.

3.5.1 Contact Addresses and Points

In Section 3.3, we completely identified the self-contacting points for a given angle, and also the non-trivial points of S_R of minimal distance to $\mathbf{y}$ for the self-avoiding trees. We now define a class of addresses called the contact addresses.

Definition 3.5.1.1 *For a non-overlapping, non space-filling tree $T(r,\theta)$ with $\theta \neq 90°$, we define the* **contact address** $\mathbf{A}_c(r,\theta)$ *or* $\mathbf{A}_c$*, as follows.*

1. *If $\theta \neq 90°, 135°$ and $r \le r_{sc}$, the contact address for the tree $T(r,\theta)$ is the address of a non-trivial point on S_R that has minimal distance to $\mathbf{y}$, and if there is more than one such point, it corresponds to the point that is closest to $(0,1)$.*

2. *If $\theta = 135°$ and $r < r_{sc}$, the contact address is the address of a non-trivial point on S_R that has minimal distance to $\mathbf{y}$.*

For a self-avoiding tree with $\theta = 90°$, we define two contact addresses, denoted by $\mathbf{A}_c$ and $\mathbf{A}'_c$. They correspond to the two non-trivial points of S_R that have minimal distance to $\mathbf{y}$ and are closest to $(0,1)$.

A point at the contact address for a given tree is called a **contact point**. For self-contacting trees, the contact point is the self-contacting point closest to $(0,1)$. Based on the results from Section 3.3, we summarize the contact addresses in Table 3.1.

Branching Angle	Scaling Ratio	Contact Address
$\theta \in (0°, 90°)$ and $N\theta > 90°$	$r \leq r_{sc}$	$RL^{N+1}(RL)^{\infty}$
$\theta \in (0°, 90°)$ and $N\theta = 90°$	$r \leq r_{sc}$	$RL^{N+1}(LR)^{\infty}$
$\theta = 90°$	$r < r_{sc}$	$RL^2(LR)^{\infty}$ and $R^3(RL)^{\infty}$
$\theta \in (90°, 135°)$	$-\sin(3\theta)\csc(2\theta) < r \leq r_{sc}$	$R^3(LR)^{\infty}$
$\theta \in (90°, 135°)$	$r \leq -\sin(3\theta)\csc(2\theta)$	RR
$\theta = 135°$	$r < r_{sc}$	RR
$\theta \in (135°, 180°)$	$r \leq r_{sc}$	RR

Table 3.1: Summary of Contact Addresses for Contact Points

3.5.2 Secondary Contact Addresses and Points

Now we present results about secondary contact addresses and points. As with the contact addresses, the reason for discussing such points will become clear when we discuss the location of holes in Chapters 4 and 5.

Observation. For any non-overlapping tree $T(r, \theta)$ with branching angle θ less than or equal to $45°$, there is no portion of the subtree S_R of $T(r, \theta)$ below the line $y = 1$. For any non-overlapping tree $T(r, \theta)$ with branching angle θ, there is no portion of the tree $T(r, \theta)$ above the line $y = 1$.

As a result of the previous observation, we only define secondary contact address and point for trees with angles θ such that $45° < \theta < 90°$ or $90° < 135°$.

Definition 3.5.2.1 *For non-overlapping trees with branching angles such that* $45° < \theta < 90°$ *or* $90° < 135°$, *we define the* **secondary contact address** $\mathbf{A}_s(r, \theta)$ *or* $\mathbf{A}_s$ *as follows.*

1. *If* $45° < \theta < 90°$ *and* $r \leq r_{sc}$, *the secondary contact address of* $T(r, \theta)$ *corresponds to a point on* S_{RR} *that has minimal distance to* $\mathbf{y}$, *and is closest to* $(0, 1)$ *if there is more than one.*

2. *If* $90° < \theta < 135°$ *and* $r \leq r_{sc}$, *the secondary contact address of* $T(r, \theta)$ *is the address of a point on* S_{RL} *that has minimal distance to the* y*-axis for the subtree* S_{RL}.

Branching Angle	Secondary Contact Address
$\theta \in (45°, 90°)$ and $N_2\theta > 270°$	$R^{N_2}(LR)^\infty$
$\theta \in (0°, 90°)$ and $N_2\theta = 270°$	$R^{N_2}(RL)^\infty$
$\theta \in (90°, 135°)$	$RL(LR)^\infty$

Table 3.2: Summary of Secondary Contact Addresses

Proposition 3.5.2.2 *Let θ be such that $45° < \theta < 90°$. Let N_2 be the secondary turning number of the angle. If $N_2\theta > 270°$, then there is a unique tip point of the subtree S_{RR} that has minimal distance to the y-axis out of all tip points on the subtree S_{RR}. This tip point is the point with address $R^{N_2}(LR)^\infty$. If $N_2\theta = 270°$, then there are infinitely many tip points of the subtree S_{RR} that have minimal distance to the y-axis. These are tip points with addresses of the form $R^{N_2}\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$.*

Proof. The proof of this proposition follows the same argument as the proof of Proposition 3.3.1.2, so we will not repeat it here. If $N_2\theta = 270°$, the branch $b(R^{N_2})$ is horizontal, and hence all top tip points of the subtree have minimal distance to the y-axis. $\square$

Proposition 3.5.2.3 *Let θ be such that $90° < \theta < 135°$. The point with address $RL(LR)^\infty$ has minimal distance to the y-axis out of all tip points of S_{RL}.*

Proof. For trees with such a branching angle, the line segment through the top of the subtree S_{RLL} (so through the points with addresses $RLL(RL)^\infty$ and $RLL(LR)^\infty$) has negative slope. This line segment forms a border between S_{RL} and the y-axis. So the highest tip point that is on this line segment will have the smallest x-value, *i.e.*, have minimal distance to the y-axis. The highest point has address $RLL(RL)^\infty = RL(LR)^\infty$. $\square$

A summary of the secondary contact addresses is presented in Table 3.2.

3.5.3 Collinearity; Vertex and Corner Points

Recall that a vertex is any point that is the endpoint of a branch. Certain collections of vertices are collinear and will be important for locating holes of closed ϵ-neighbourhoods.

Proposition 3.5.3.1 *(Collinearity of Vertices) Let $T(r,\theta) \in \mathcal{T}$. Let $P_k = (x_k, y_k)$ denote the point with address $(RL)^k$, where $k \geq 0$. Then the points P_k are collinear, and the slope of the line they are on is $m = \cot\theta + r\csc\theta$.*

Proof. The point P_0 is $(0,1)$. P_1 has coordinates

$$x_1 = r\sin\theta, \quad y_1 = 1 + r\cos\theta + r^2 \tag{3.5.3}$$

Using the scaling nature of the trees to find the coordinates at any point P_k, we have

$$x_k = x_1(1 + r^2 \cdots + r^{2(k-1)}) \tag{3.5.4}$$

$$y_k = 1 + (y_1 - 1)(1 + r^2 \cdots + r^{2(k-1)}) \tag{3.5.5}$$

The slope m between any two consecutive points P_k and P_{k+1} is

$$\begin{aligned} m &= \frac{[1 + (y_1 - 1)(1 + r^2 \cdots + r^{2(k)})] - [1 + (y_1 - 1)(1 + r^2 \cdots + r^{2(k-1)})]}{[x_1(1 + r^2 \cdots + r^{2(k)})] - [x_1(1 + r^2 \cdots + r^{2(k-1)})]} \\ &= \frac{r^{2k}(y_1 - 1)}{r^{2k}x_1} \\ &= \frac{y_1 - 1}{x_1} \\ &= \frac{r\cos\theta + r^2}{r\sin\theta} \\ &= \cot\theta + r\csc\theta \end{aligned} \tag{3.5.6}$$

Hence the slope between any two consecutive points is independent of k, and the points must all be collinear. □

Corollary 3.5.3.2 *Let $T(r,\theta) \in \mathcal{T}$. The points with addresses $(LR)^k$, where $k \geq 0$, are collinear.*

Corollary 3.5.3.3 *Let $T(r,\theta)$ be any tree, let $\mathbf{A} \in \mathcal{A}$ be a finite address. Then all points with addresses of the form $\mathbf{A}(RL)^k$, for $k \geq 0$, are collinear.*

For trees with $y_{\max} = 1$, certain vertex points are particularly important.

Definition 3.5.3.4 *Let $T(r,\theta)$ be any tree such that $r < -\cos\theta$. The* **top vertex points** *of the tree are the points with addresses of the form $\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$.*

Now we identify two points of a tree that are important for hole formation in closed ϵ-neighbourhoods.

Definition 3.5.3.5 *The* **corner points** *of a tree are the point with address* $(RL)^\infty$ *and its mirror image with address* $(LR)^\infty$*. The* **corner points of a subtree** $S_{\mathbf{A}}$ *are the points with addresses* $\mathbf{A}(RL)^\infty$ *and* $\mathbf{A}(LR)^\infty$*.*

The corner points of a tree, along with infinitely many other tip points, belong to a horizontal line as the following proposition states.

Proposition 3.5.3.6 *(Collinearity of Tip Points) Let* $T(r,\theta) \in \mathcal{T}$*. Then the tip points with addresses* $\mathbf{A} \in \mathcal{AL}_\infty$ *(see [2.3.6]) are collinear, and they lie on a horizontal line.*

Proof. Let $\mathbf{A} \in \mathcal{A}_\infty$. A path given such an address is such that every second branch is vertical, and the y-component of a tip point at any such address is

$$\frac{1 + r\cos\theta}{1 - r^2}$$

This was discussed in Subsection 3.4, when we discussed paths to tip points with maximal y-value. Thus they all have the same y-coordinate, and therefore lie on a horizontal line. □

In the case of a tree with $r > -\cos\theta$, the tip points in the previous proposition are all at maximal height for the tree. They also form form a generalized Cantor set on the top of the tree, with $m = 2$ and $\lambda = r^2$ (see Subsection A.2.1 in Appendix A). In the case of a tree with $r < -\cos\theta$, the tip points are not as important for the closed ϵ-neighbourhoods, because they are not extremal in terms of being at the 'top' of a tree.

3.5.4 Canopy Intervals and Canopy Points

We first need to define the notion of canopy interval. In the following discussion in this subsection about canopy intervals and canopy endpoints, assume that

$$r > -\cos\theta, \quad i.e., \quad y_{\max} > 1$$

Definition 3.5.4.1 *Define the* **degree 0 top canopy interval***, denoted* I_{tc}*, to be the closed line segment bounded by the point* $P_{\mathbf{C}_R}$ *with address* $\mathbf{C}_R = RL(LR)^\infty$ *and its mirror image* $P_{\mathbf{C}_L}$ *with address* $\mathbf{C}_L = LR(RL)^\infty$*. The point* $P_{\mathbf{C}_R}$ *is called the* **right endpoint** *of the canopy interval, and* $P_{\mathbf{C}_L}$ *is called the* **left endpoint***.*

Notes. There are no tip points of maximal height that lie between $P_{\mathbf{C}_R}$ and $P_{\mathbf{C}_L}$, since $P_{\mathbf{C}_R}$ is the leftmost tip point of maximal height on the subtree S_R. So for trees with $r > -\cos\theta$, the interval I_{tc} intersects the tree only in the points $P_{\mathbf{C}_R}$ and $P_{\mathbf{C}_L}$, so it forms a gap in the top of the tree. See Figures 3.8, 3.7, and 3.9 for examples. Because of the way address maps act on compact subsets of $\mathbb{R}^2$, the image of I_{tc} under an address map $m_\mathbf{A}$ will be a canopy interval of the subtree $S_\mathbf{A}$ in the following sense. The image $m_\mathbf{A}(I_{tc})$ is a closed line segment whose endpoints are with addresses $\mathbf{A}\mathbf{C}_R$ and $\mathbf{A}\mathbf{C}_L$, and the interior of the line segment is disjoint from the subtree (and the entire tree). If the address map were an alternating address map (as defined in 2.3.1.8), then the two endpoints would be at maximal height for the tree itself. This motivates the following definition.

Definition 3.5.4.2 *A* **degree** k **top canopy interval** *is the image of* I_{tc} *under an address map of the form* $m_\mathbf{A}$*, where* $\mathbf{A} \in \mathcal{AL}_{2k}$ *(see 2.3.1.8).*

Notes. The reason we use 'degree' instead of 'level' is because the intervals don't scale with a factor of r^k (which is an important characteristic of level k objects), but instead scale with factor r^{2k}. The 'top' refers to the fact that the interval endpoints are at the 'top' of the tree, in the sense that they have maximal height for the tree.

Definition 3.5.4.3 *Let* l_{tc} *denote the length of* I_{tc}*. The* x*-coordinate of* $P_{\mathbf{C}_R}$ *with address* $\mathbf{C}_R$ *is given by*

$$x = r\sin\theta - \frac{r^3\sin\theta}{1-r^2} = r\sin\theta\left(\frac{1-2r^2}{1-r^2}\right) \tag{3.5.7}$$

thus

$$l_{tc} = 2r\sin\theta\left(\frac{1-2r^2}{1-r^2}\right) \tag{3.5.8}$$

Proposition 3.5.4.4 *Suppose the endpoints of a degree k top canopy interval are* $\mathbf{AC}_R$ *and* $\mathbf{AC}_L$ *for some* $\mathbf{A} \in \mathcal{AL}_{2k}$*. Then the canopy interval has length* $r^{2k} l_{\mathrm{TC}}$*.*

Proof. The claim of the proposition directly follows the Address Map Lemma 2.1.3.2 and the fact that the address map $m_{\mathbf{A}}$ has level $2k$. □

Definition 3.5.4.5 *Define the* **top canopy intervals***, denoted* $\mathcal{I}_{tc}$*, to be the collection of all top canopy intervals of degree* k*, as the degree ranges over* $k \geq 0$*.*

$$\mathcal{I}_{tc} = \bigcup_{k \geq 0} \{m_{\mathbf{A}}(I_{tc}) | \mathbf{A} \in \mathcal{AL}_{2k}\} \tag{3.5.9}$$

Definition 3.5.4.6 *Define the* **top canopy points***, denoted* $\mathcal{P}_{tc}$*, to be the collection of endpoints of top canopy intervals. The degree of a top canopy point is the degree of the interval that it is an endpoint of. Right canopy points of degree* k *are at addresses* $\mathbf{AC}_R$ *for some* $\mathbf{A} \in \mathcal{AL}_{2k}$*, and left canopy points of degree* k *are at addresses of the form* $\mathbf{AC}_L$ *for some* $\mathbf{A} \in \mathcal{AL}_{2k}$*.*

Note. The degree of a top canopy point is well defined, because a tip point can be the endpoint of at most one canopy interval, as we show in the following proposition. Note that we will sometimes refer to a top canopy point as just a canopy point.

Lemma 3.5.4.7 *The top canopy point* $P_{\mathbf{C}_R} = (x_{\mathbf{C}_R}, y_{\mathbf{C}_R})$ *is such that for any tip point* $P = (x, y)$ *of the tree where* $y = y_{\mathbf{C}_R} = y_{\max}$ *and* $x > x_{\mathbf{C}_R}$*, there is another tip point of maximal height between the two.*

Proof. $P_{\mathbf{C}_R}$ is with address $\mathbf{C}_R$ and P must be with address $\mathbf{A}$, for some $\mathbf{A} \in \mathcal{AL}_\infty$ (since it is of maximal height), the first two elements of $\mathbf{A}$ are RL (since $x > 0$), and $\mathbf{A}$ is distinct from $\mathbf{C}_R$. Let $\mathbf{A} = A_1 A_2 \cdots$. Then there is an integer $k > 1$ such that $A_{2k-1} A_{2k} = RL$ (to be distinct from $\mathbf{C}_R = RL(LR)^\infty$). Consider the address $\mathbf{A}' = A_1 A_2 \cdots A_{2k-3} A_{2k-2} LRRL(LR)^\infty$. The point $P' = P_{\mathbf{A}'}$ is such that it is of maximal height and is between the other two tip points. □

Proposition 3.5.4.8 *Let* P *be a top canopy point. Then* P *is an endpoint of a unique top canopy interval.*

Proof. First consider the top canopy point $P_{\mathbf{C}_R}$. It is the right endpoint of the top canopy interval I_{tc} (by definition). $P_{\mathbf{C}_R}$ cannot be the left endpoint of any other top canopy interval by the previous lemma and the fact that the interiors of top canopy intervals are disjoint from the tree. Thus $P_{\mathbf{C}_R}$ is an endpoint of a unique top canopy interval. Any other top canopy endpoints can be obtained from $P_{\mathbf{C}_R}$ via reflection or an address map, so they must also be an endpoint of a unique top canopy interval.

Definition 3.5.4.9 *The top canopy intervals and top canopy points of a subtree $S_{\mathbf{A}}$, for some $\mathbf{A} \in \mathcal{A}_k$, are the images of the top canopy intervals and top canopy points under the address map $m_{\mathbf{A}}$.*

Remark. Given a canopy interval, there may be more than one subtree that it is a canopy interval for (which is the case for any interval that is not the degree 0 canopy interval of the tree). For example, the canopy interval with endpoint addresses $R\mathbf{C}_R$ and $R\mathbf{C}_L$ is a canopy interval of the tree but also of the subtree S_R.

The following proposition will be used for determining hole locations.

Proposition 3.5.4.10 *Let $P' = (x', y')$ be a top canopy point. Then P' is isolated from only one side (horizontally). That is, there exists $\delta' > 0$ such that for any $0 < \delta < \delta'$, the region of $\mathbb{R}^2$ specified by*

$$\{(x, y')|x \in (x' - \delta, x' + \delta)\}$$

is such that one side contains other tip points of maximal height and the other side is disjoint from the tree.

Proof. Without loss of generality, assume that P' is $P_{\mathbf{C}_R}$, the point with address $RL(LR)^\infty$. Let $\delta' = l_{tc}$, the length of the degree 0 canopy interval. Then for every δ such that $0 < \delta < \delta'$, the set

$$\{(x, y')|x \in (x' - \delta, x')\}$$

is clearly disjoint from the tree. Now there are infinitely many other tip points of maximal height to the right of P', and by Lemma 3.5.4.7, we can find a tip point of maximal height as arbitrarily close as we like. Thus the set

$$\{(x, y')|x \in (x', x' + \delta)\}$$

contains other tip points of maximal height. Thus P' is isolated from only one side. $\square$

Remark. Let $P = (x, y)$ be a point at maximal height that is not a canopy point. Then the point is not isolated from either horizontal side.

Note that it is possible for a point on a tree to be both a canopy point and a corner point of a subtree. We will use one name over another depending on the context. We will identify a class of holes by a pair of points on a tree. A point that can both be considered a canopy or a corner is referred to as canopy when it is as one element of a pair of endpoints of a canopy interval, otherwise it is referred to as a corner point. This issue will become clearer when we discuss types of holes in the following chapter.

3.6 Constructions of Level 0 Holes of Self-Contacting Trees

Now that we have discussed the self-contacting trees in detail along with special points and their properties, we are finally able to provide constructions of level 0 holes. That is, we give constructions of all level 0 simple, closed curves that do not have any portion of the tree inside them. First we need a lemma that deals with curves along the 'top' of the tree.

Lemma 3.6.0.11 *Let $T(r_{sc}, \theta)$ be a non-space-filling self-contacting tree. Let P_h be the point with address $(RL)^\infty$, and let P_h^* be its mirror image with address $(LR)^\infty$. Then there exists a path (in the sense of a subset of $\mathbb{R}^2$ between two points) on the tree from P_h to P_h^* that is simple and is such that there is no portion of the tree above the path.*

Proof. We divide the proof into three cases, one case for each angle range.

1. In the first angle range, the desired path is the canopy part of the hull of the tree, as discussed in greater detail in [31]. Following [30] and [31], the hull of a self-contacting tree is the set of points that can be reached from far away

by following a curve that does not intersect the tree. In this angle range, the canopy consists entirely of tip points.

2. In the second angle range, we construct the curve recursively. At the ith iteration, the curve is simple and there is no portion of the tree above it. These curves are all compact and have a limit as the number of iterations goes to infinity, and this limit is a curve on the tree. Start with the straight line segment from P_h to P_h^*. The first iteration is to break this line segment into 6 line segments. We describe the new curve by giving the points on the tree that the line segments go between. The first line segment is from P_h to the right canopy point of the degree 0 canopy interval, so the point with address $\mathbf{C}_R$. Then we take the line segment from this point to the left corner point of the subtree S_{RLL}. This point is on the branch $b(R)$ since the tree is self-contacting, so the next line segment we take is from this corner point to the top of the trunk (which is just a subset of $b(R)$). The rest of the curve is the mirror image of these 3 line segments. For each subsequent iteration, the line segments from the previous iteration stay the same if they are subsets of branches or they break into 6 new line segments following the same rule as for the original line segment (using the similarity of the tree and its subtrees). Then the limit of these curves as the number of iterations goes to infinity is the desired curve.

3. In the third angle range, we also construct the curve recursively. Start with two line segments, one from P_h to $(0, 1)$ and the other from $(0, 1)$ to P_h^*. This curve is simple and there is no portion of the tree above it (since the tree is self-contacting). The line segment on the right side goes through all top vertex points with addresses $(RL)^k$, for $k \geq 1$ (as discussed in the previous section dealing with top vertex points). The first iteration is to break these two line segments into 6. The new curve consists of the line segment from P_h to the point with address RL (the next highest top vertex point after $(0, 1)$), then the line segment from this top vertex point to the left corner point of the subtree S_{RL} (the point with address $RL(LR)^\infty$, which is on the branch $b(R)$ because the tree is self-contacting), then the portion of the branch $b(R)$ between this corner

point and the top of the trunk, and the rest is the mirror image of these 3 line segments. For each subsequent iteration, the line segments from the previous iteration stay the same if they are subsets of branches or they break into 6 new line segments following the same rule as for the original line segment (using the similarity of the tree and its subtrees). Then the limit of these curves as the number of iterations goes to infinity is the desired curve.

□

Now we can give explicit constructions for level 0 holes.

- **Angles in the first angle range that are not special**

  Given θ in the first angle range such that $N\theta > 90°$, there is only one tip point of S_R on the y-axis, namely the point P_{c1} with address $RL^{N+1}(RL)^{\infty}$. There is only one level 0 hole, and its boundary contains this tip point.

  The boundary of this hole consists of the path $p(RL^{N+1})$, then the path from the endpoint of the branch $S_{RL^{N+1}}$ to the left corner point of this subtree, then the path from this tip point to P_{c1} along the top of the subtree $S_{RL^{N+1}}$ (this path is just a subset of the path described in the previous lemma), and then the mirror image of these paths on the left side.

- **Angles in the second angle range that are special**

  Given θ in the first angle range such that $N\theta = 90°$, there is are infinitely many tip points on the y-axis. There is one hole whose boundary contains the lowest tip point on the y-axis, namely the point P_{c1} with address $RL^{N+1}(LR)^{\infty}$. The boundary of this hole is similar to the boundary of the hole described above. Any other hole has a boundary that is a simple, closed curve, so the boundary has two tip points on the y-axis. The only pairs of tip points that are on the y-axis and are such that there is an open interval of the y-axis between them and disjoint from the tree are the canopy endpoints of the subtree $S_{RL^{N+1}}$. Consider the hole whose boundary contains the degree 0 endpoints of this subtree. All other holes with canopy endpoints in their boundaries are similar to this hole. There is a curve on the subtree $S_{RL^{N+1}}$ between these two tip points that is simple and is such that there is no portion of the subtree $S_{RL^{N+1}}$ to the left (by the similarity of this subtree to the tree and the previous lemma). Then this curve along with its mirror image form the boundary of the hole.

- **Angles in the second angle range**

  Given any angle in the second angle range, there is a unique tip point of S_R on the trunk, the point P_{c2} with address $R^3(LR)^\infty$. Thus there is only one level 0 hole on the right side of the tree. The boundary of this hole contains P_{sc2}, the line segment from P_{c2} to $(0,1)$ (which is just a subset of the trunk), the line segment from $(0,1)$ to the right corner point P of the subtree S_{R^3}, then the curve back to P_{c2} (which is the left corner point of the subtree S_{R^3}) as described in the lemma.

- **Angles in the third angle range**

  Given any angle in the third angle range, there are infinitely many points of the subtree S_R on the trunk, namely all left top vertex points of the subtree S_{RR}, so all points with addresses of the form $RR(LR)^k$, for $k \geq 0$. First consider the hole whose boundary contains the self-contact point P_{sc3} with address RR and also the point $(0,1)$. The boundary of this hole is formed by the line segment between these two points (on the trunk), the line segment from $(0,1)$ to the right corner point of the subtree S_{RRRL} (which is on the branch $b(R)$ so this line segment is a subset of the branch $b(R)$), the curve from this right corner point to the left corner point of the subtree S_{RRRL} (as described in the previous lemma), then the line segment back from this point to P_{c3} (which is a subset of $b(RR)$). Any other hole in the self-contacting tree is similar to this hole, so the boundary would also be similar.

Now we have an idea of what the holes in self-contacting trees are, we can consider holes of closed ϵ-neighbourhoods of self-contacting and self-avoiding trees. This commences in the next chapter.

3.7 Homeomorphism Types of Non-Overlapping Symmetric Binary Fractal Trees

In this section, we discuss the homeomorphism classes of non-overlapping symmetric binary fractal trees. We consider a tree to have the subspace topology inherited from the standard topology of $\mathbb{R}^2$. We have already claimed that all self-avoiding trees are homeomorphic, here we provide a proof and the homeomorphisms. The two self-contacting trees that are space-filling, $T(1/\sqrt{2}, 90°)$ and $T(1/\sqrt{2}, 135°)$ form another homeomorphism class. The remaining classes consist of the non-space-filling self-contacting trees, and they depend on self-contact addresses.

Proposition 3.7.0.12 *The trees $T(1/\sqrt{2}, 90°)$ and $T(1/\sqrt{2}, 135°)$ are homeomorphic.*

Proof. The tree $T(1/\sqrt{2}, 90°)$ is a filled-in square, as described in Subsection 3.3.2. The tree $T(1/\sqrt{2}, 135°)$ is a filled-in triangle, as described in Subsection 3.3.4. Therefore, the two trees are homeomorphic. $\square$

Now we consider the non-space-filling trees.

Notation. In comparing two trees $T_1 = T(r_1, \theta_1)$ and $T_2 = (r_2, \theta_2)$, the notation $(\ )_1$ refers to tree T_1 and $(\ )_2$ refers to tree T_2. For example, given an address $\mathbf{A}$, $(m_{\mathbf{A}})_1$ refers to the address map $m_{\mathbf{A}}$ acting on T_1, while $(m_{\mathbf{A}})_2$ refers to the address map $m_{\mathbf{A}}$ acting on T_2.

Theorem 3.7.0.13 *All self-avoiding trees are homeomorphic.*

Proof. Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be any two self-avoiding trees. To prove the proposition, we provide a homeomorphism $f : T_1 \to T_2$. We construct the map f as follows. First of all, f restricted to the trunk of T_1 is the identity map. Let U denote the set $\mathbf{y}_{(0,1]}$. For any self-avoiding tree, given a point P on the tree that is not a tip point, there exists a unique point $P' \in U$ and a unique finite address map $m_{\mathbf{A}}$ such that $P = m_{\mathbf{A}}(P)$. A tip point is the image of any point in U under a unique address map $m_{\mathbf{A}}$, for some infinite address $\mathbf{A}$. So given a point P_1 on T_1, there exists a point P_1' in U and a unique address $\mathbf{A}$ such that

$$P_1 = (m_{\mathbf{A}})_1(P_1').$$

Then the map f is defined by imposing the condition that

$$f(P_1) = f((m_{\mathbf{A}})_1(P_1')) = (m_{\mathbf{A}})_2(f(P_1')) \tag{3.7.1}$$

for any point P_1' on the trunk. Indeed, this equation holds for any point P_1' on the tree T_1, not just points on the trunk, but knowing the action on the trunk is sufficient to completely determine the map $f : T_1 \to T_2$. Then f is a bijection, because the inverse f^{-1} is defined similarly. Given a point P_2 on T_2, there is a point P_2' in the subset U of T_2 and a unique address $\mathbf{A}$ such that

$$P_2 = (m_{\mathbf{A}})_2(P_2')$$

and so

$$f^{-1}(P_2) = f^{-1}((m_{\mathbf{A}})_2(P_2')) = (m_{\mathbf{A}})_1(f^{-1}(P_2')) \tag{3.7.2}$$

Recall that f is the identity map on the trunk, so f^{-1} is also the identity map on the trunk. The map f restricted to the trunk is clearly a homeomorphism, since it is the identity map and because the trees are self-avoiding (so there are no double points on the trunk). The address maps are continuous, so the map f and its inverse are also continuous.

The trees T_1 and T_2 are arbitrary, so there exists a homeomorphism between any two self-avoiding trees. $\square$

3.7.1 Homeomorphism Types of Non-Space-Filling Self-Contacting Trees

Now we discuss the non-space-filling self-contacting trees, where the situation is more interesting. The main result is that two self-contacting trees are homeomorphic if and only if they have the same set of self-contact addresses. To prove this result, some preliminary results first need to be presented. In this subsection, assume the trees are not space-filling, so the angle of any tree is neither $90°$ nor $135°$.

Lemma 3.7.1.1 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be homeomorphic trees, where $f : T_1 \to T_2$ is a homeomorphism between the trees. Then the image of any tip point of T_1 under f is a tip point of T_2.*

Proof. Let P be a tip point of T_1 corresponding to some address $\mathbf{A} = A_1A_2\cdots$, and let U be any open neighbourhood of P on T_1. Let $\mathbf{A}_i = A_1 \cdots A_i$. Then for every i, the point P is on the subtree $S_{\mathbf{A}_i}$. The tree T_1 is bounded, and the size of the subtrees decreases towards 0 as the level of the subtree increases. Thus there exists an integer k such that the subtree $S_{\mathbf{A}_k}$ is a subset of U, since U has non-zero diameter. By assumption, the tree T_1 is non-space-filling and self-contacting, so it has simple closed curves. By the similarity of subtrees, the subtree $S_{\mathbf{A}_k}$ also contains simple closed curves. Any point on a tree that is not a tip point is either a branch endpoint or in the interior of a branch, and it is possible to find an open neighbourhood of such a point that is contractible. For any point in a branch interior, there exists a neighbourhood on the tree that is an open line segment, and for a branch endpoint there exists a neighbourhood that contains finitely many line segments that that start at the point and are disjoint. Therefore, the image of P must be a tip point of T_2. $\square$

Lemma 3.7.1.2 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be homeomorphic trees, where $f : T_1 \to T_2$ is a homeomorphism between the trees. Then $f((0,0)) = (0,0)$.*

Proof. Any self-contacting tree is such that there exists $\delta > 0$ for which the region of the tree that is within δ of $(0,0)$ is the region $\mathbf{y}_{[0,\delta)}$ (that is, the half open line segment starting at $(0,0)$ that has length δ and is a subset of the trunk). This is because self-contact never occurs at the point $(0,0)$. The point $(0,0)$ is the only point on a self-contacting tree that has such an open neighbourhood, since any other point on a tree is a tip point, branch endpoint or in a branch interior, and cannot have such a neighbourhood. Thus a homeomorphism must send $(0,0)$ to itself. $\square$

Proposition 3.7.1.3 *If two self-contacting trees $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ are homeomorphic, then θ_1 and θ_2 must be in the same angle range, that is, they are either both less than $90°$, both strictly between $90°$ and $135°$, or both greater than $135°$.*

Proof. If two trees are homeomorphic, then the removal of one point from one tree and its homeomorphic image on the other tree will result in two sets that have the same number of components. Assume that $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ are homeomorphic self-contacting trees.

First we will show that if $\theta_1 < 90°$, then $\theta_2 < 90°$. If we remove any tip point from T_1, the resulting set is still connected, because self-contact only occurs at tip points, never on branch interiors. Suppose $90° < \theta_2 < 135°$, and consider the self-contacting tip point with address $R^3(LR)^\infty$ on T_2. Removing this point from the tree T_2 results in a set that has two components, since the part of the trunk below this point is now disconnected from the rest of the tree. Since tip points must be mapped to tip points, this implies that it is not possible for T_2 to be homeomorphic to T_1. Now suppose $135° < \theta_2$, and consider the self-contacting tip point with address $RR(LR)^\infty$ on T_2. Removing this point of T_2 also results in a set that is disconnected, so likewise T_2 could not be homeomorphic to T_1. So if θ_1 is in the first angle range (less than $90°$), then θ_2 must also be in the first angle range.

Now we distinguish between the second and third angle ranges. Let θ_1 be such that $\theta_1 > 135°$. Suppose θ_2 is such that $90° < \theta_2 < 135°$. Let P_1 denote the self-contacting point of T_1 with address RR. Then there exists a neighbourhood U_1 of P_1 which consists of P_1 along with eight half-open line segments starting from P_1. These eight line segments include two regions of the trunk above and below P_1 (since P_1 cannot be at the top of the trunk, and there is an open region of the trunk between P_1 and the next self-contacting point which has address $RRLR$), and parts of the following branches: $b(RR)$, $b(RRR)$, $b(RRL)$, $b(LL)$, $b(LLL)$, and $b(LLR)$. Now we claim that there are no neighbourhoods of T_2 that could possibly be homeomorphic to U_1. In the second angle range, self-contact occurs only at tip points, and any neighbourhood of a tip point contains entire subtrees (as shown in the proof of Lemma 3.7.1.1). So the homeomorphic image of U_1 could not contain a tip point. Any point on the tree T_2 that is not a tip point is either in a branch interior or is a branch endpoint, and in either case such a point could not be homeomorphic to P_1. Thus the angle θ_2 must also be greater than $135°$.

Therefore, self-contacting trees that are homeomorphic must be in the same angle range. ☐

Theorem 3.7.1.4 *All self-contacting trees with angles strictly between* $90°$ *and* $135°$ *are homeomorphic.*

Proof. Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be any self-contacting trees such that $90° < \theta_1, \theta_2 < 135°$. To prove the theorem, we construct a homeomorphism $f : T_1 \to T_2$. As with the self-avoiding trees, it suffices to define the map on the trunk. However, the map is not necessarily the identity map. Recall that there is one self-contacting point on the trunk of a self-contacting tree in this angle range, the point corresponding to the address $RRR(LR)^\infty$. The actual y-value of this point may be different for the two trees. Let $P_1 = (0, y_1)$ denote the corresponding point on T_1, and let $P_2 = (0, y_2)$ denote the corresponding point on T_2. We have already shown that the point $(0,0)$ of T_1 must be mapped to $(0,0)$ of T_2. There exists a neighbourhood U_1 of $(0,0)$ on T_1 that is of the form $\mathbf{y}_{[0,y_1)}$. If f is indeed a homeomorphism, then the image of U_1 under f must be of the form $\mathbf{y}_{[0,y)}$, where $0 < y \leq y_2$ (since it can't include the point P_2). Likewise, there exists a neighbourhood U_2 of $(0,0)$ that is of the form $\mathbf{y}_{[0,y_2)}$, and its pre-image under f must be of the form $\mathbf{y}_{[0,y')}$, where $0 < y' \leq y_1$. Thus $f(U_1) \subseteq U_2$ and $f^{-1}(U_2) \subseteq U_1$. For f to be a homeomorphism, we must have $f(U_1) = U_2$, and this also forces $f(P_1) = P_2$. We can define f on the trunk as follows. For points on the trunk between $(0,0)$ and P_1 inclusive, we define f to be the unique linear map that sends $(0,0)$ to $(0,0)$ and P_1 to P_2. For points on the trunk between P_1 and $(0,1)$, we define f to be the unique linear map that sends P_1 to P_2 and $(0,1)$ to $(0,1)$. Now we use the action of the address maps to define the map for the rest of the tree. For any point P of T_1, there is a point P' on the subset $U = \mathbf{y}_{(0,1]}$ of T_1 and an address map $\mathbf{A}$ such that

$$P = (m_{\mathbf{A}})_1(P')$$

If P is not a self-contacting tip point (a point corresponding to two distinct infinite addresses and also in the interior of a branch), then P' and $\mathbf{A}$ are unique. If P is a tip point that is not self-contacting, then $\mathbf{A}$ is unique. If P is a self-contacting tip point (any point with address of the form $\mathbf{A}'RRR(LR)^\infty$ or $\mathbf{A}'LLL(RL)^\infty$, for some $\mathbf{A}'$), then P is not unique, and $\mathbf{A}$ could be one of the two addresses for the tip point, or it could be the address of the branch that the tip point is on. We define the image of P as:

$$f(P) = f((m_{\mathbf{A}})_1(P')) = (m_{\mathbf{A}})_2(f(P'))$$

Then f is a homeomorphism. Clearly f is a homeomorphism on the trunk, since linear maps are homeomorphisms, and no other part of T_1 gets mapped to the trunk. The map f sends self-contacting points to self-contacting points (preserving addresses), so it is a bijection. The composition of the address map $(m_{\mathbf{A}})_2$ and the identity is continuous, so f is continuous, as is the inverse. □

Theorem 3.7.1.5 *All self-contacting trees with angles greater than* $135°$ *are homeomorphic.*

Proof. The proof of this theorem is similar to the proof of the previous theorem. Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be any self-contacting trees such that $\theta_1, \theta_2 > 135°$. We construct a homeomorphism by first defining it for the trunk, and using the action of address maps for the rest. As in the previous proof, one can show that any homeomorphism from T_1 to T_2 would have to map the lowest self-contacting point on the trunk of T_1 to the lowest self-contacting point on the trunk of T_2. This point corresponds to the address $RR(LR)^\infty$. For self-contacting trees in this angle range, there are infinitely many self-contacting points on the trunk. In addition to the point with address $RR(LR)^\infty$, other self-contacting points are the points with addresses of the form $RR(LR)^k$, for $k \geq 0$. So we define the map f for intervals of the trunk between self-contacting points, using linear maps. Let P_1 denote the point on T_1 with address $RR(LR)^\infty$, and for $k \geq 0$, let P_{1k} denote the point on T_1 with address $RR(LR)^k$. Recall that the points P_{1k} have decreasing y-values as k increases. Let P_2, P_{2k} denote the corresponding points on the tree T_2. For points between $(0,0)$ and P_1 inclusive, the map f is the unique linear map that sends $(0,0)$ to $(0,0)$ and P_1 to P_2. Given $k \geq 0$, for points between P_{1k} and $P_{1(k+1)}$ inclusive, f is the unique linear map that sends P_{1k} to P_{2k} and $P_{1(k+1)}$ to $P_{2(k+1)}$. Finally, for points between P_{10} and $(0,1)$, f is the unique linear map that sends $(0,1)$ to $(0,1)$ and P_{10} to P_{20}.

For any point P of T_1, there is a point P' on the subset $U = \mathbf{y}_{(0,1]}$ of T_1 and an address map $\mathbf{A}$ such that

$$P = (m_{\mathbf{A}})_1(P')$$

Note that such an address $\mathbf{A}$ is not unique if and only if the point is a self-contacting point. We define the image of P as:

$$f(P) = f((m_{\mathbf{A}})_1(P')) = (m_{\mathbf{A}})_2(f(P'))$$

Then f is a homeomorphism, as with the map defined in the previous theorem. Therefore, all self-contacting trees with angles greater than 135° are homeomorphic. □

Finally we have trees with angles less than 90°. For the second and third angle ranges, the self-contact addresses are constant throughout the range. This is not true in the first angle range. See Table 3.3 for a concise summary of self-contact addresses. For self-contacting trees in the first range, self-contact occurs at tip points. So the trunk is disjoint from the rest of the tree except for the starting points of the two level 1 branches.

Lemma 3.7.1.6 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be self-contacting trees such that $\theta_1, \theta_2 < 90°$. If T_1 and T_2 are homeomorphic and $f : T_1 \to T_2$ is a homeomorphism, then we have the following:*

1. $f((0,1)) = (0,1)$
2. *Any vertex point of T_1 is mapped to a vertex point of T_2*
3. *Any branch of T_1 is mapped to a branch of T_2 of the same level.*

Proof.

1. Consider the point $(0,1)$ of T_1. There is a neighbourhood U_1 of this point that consists of the entire trunk, along with part of the two branches $b(R)$ and $b(L)$ that does not include the endpoints of those branches or any either part of the tree. For example, it could be the union of the trunk and the points on the level 1 branches that are strictly less than r_1 away from $(0,1)$. The point $(0,0)$ is

in U_1, so $f(U_1)$ must also include $(0,0)$ by Lemma 3.7.1.2. The set $f(U_1)$ must also contain $(0,1)$, because it must be connected and cannot be homeomorphic to a half-open line segment (since U_1 is connected and not homeomorphic to a half-open line segment). Now $f(U_1)$ contains $(0,1)$ and it is homeomorphic to U_1. So it must contain exactly one point for which any neighbourhood of the point has 3 line segments coming out of the point. The point $(0,1)$ in $f(U_1)$ is such a point, and must be the unique one, hence $f((0,1)) = (0,1)$.

2. For self-contacting trees in the first angle range, vertex points (*i.e.*, endpoints of branches) are the only points on a tree for which there exists a 'T' shaped neighbourhood around the point. By a 'T' shaped neighbourhood around a point, we mean a point and three half-open line segments that emanate from the point. So any homeomorphism must map a vertex point to a vertex point.

3. We have that $f((0,0)) = (0,0)$ and $f((0,1)) = (0,1)$, so the image of the trunk of T_1 must be the trunk of T_2. Now we will prove the claim by using induction on the level of the branches. Let $\mathbf{b}_1$ denote the branch $b(R)$ on T_1. The starting point of $\mathbf{b}_1$ is $(0,1)$, and we have already established that $f((0,1)) = (0,1)$. Let P_1 denote the endpoint of $\mathbf{b}_1$, *i.e.*, the point with address R. Just as we proved that the image of $(0,1)$ is $(0,1)$, a similar argument shows the image of P_1 must be the endpoint of a level 1 branch. Once this endpoint is chosen, a unique level 1 branch $\mathbf{b}_2$ of T_2 is chosen, then this forces

$$f(\mathbf{b}_1) = \mathbf{b}_2$$

This also implies that the image of the branch $b(L)$ of T_1 must be the other level 1 branch of T_2. So both level 1 branches of T_1 are mapped to level 1 branches of T_2. Now let k be any integer greater than 1, and assume that for any integer i less than k, a branch of level i on T_1 is mapped to a branch of level i on T_2. Let $\mathbf{b}$ be a level k branch on T_1. Then its starting point is the endpoint of some level $k-1$ branch $b(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{A}_{k-1}$. The image of this starting point is already determined, it is the endpoint of some level $k-1$ branch $b(\mathbf{B})$ of T_2, for some $\mathbf{B} \in \mathcal{A}_{k-1}$. Let U_1 be a connected neighbourhood of the branch $\mathbf{b}$ that does not include any other vertex points besides the starting point and endpoint of $\mathbf{b}$. Thus U contains a portion of the level $k-1$ branch $b(\mathbf{A})$ that $\mathbf{b}$ descends from. $f(U)$ must contain exactly two vertex points and be connected. $f(U)$ already contains the endpoint of a level $k-1$ branch of T_2, so must contain another vertex point which is either the endpoint of a level $k-2$ branch or a level k branch. The endpoints of level $k-2$ branches have are already homeomorphic to level $k-2$ branch endpoints of T_1, so the vertex point must be the endpoint of a level k branch. This level k branch on T_2 must descend from $b(\mathbf{B})$, so its address is of the form $\mathbf{B}R$ or $\mathbf{B}L$.

□

Proposition 3.7.1.7 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be self-contacting trees such that $\theta_1, \theta_2 < 90°$. Suppose T_1 and T_2 are homeomorphic and $f : T_1 \to T_2$ is a homeomorphism. Let $k > 0$. Then the image of a level k subtree of T_1 is a level k subtree of T_2. Moreover, if the image of some level k branch $\mathbf{b}_1 = b(\mathbf{A})$ of T_1 is the level k branch $b(\mathbf{B})$ of T_2, then the image of the subtree $S_{\mathbf{A}}$ of T_1 is the subtree $S_{\mathbf{B}}$ of T_2.*

Proof. Let S_1 be a level k subtree of T_1, for some $k \geq 1$. Then there exists an address $\mathbf{A} \in \mathcal{A}_k$ such that the trunk of S_1 is $\mathbf{b}_1 = b(\mathbf{A})$, that is, $S_1 = S_{\mathbf{A}}$. The branch $\mathbf{b}_1$ is a level k branch, and by the previous lemma, its image is a level k branch of T_2. Let $\mathbf{B} \in \mathcal{A}_k$ be the address such that $\mathbf{b}_2 = b(\mathbf{B})$ is the image of $\mathbf{b}_2$. We claim that the image of S_1 is $S_2 = S_{\mathbf{B}}$. First we show that every branch of S_1 is mapped to a branch of S_2. We will prove this by induction on the levels of the branches. S_1 contains two branches of level $k+1$, given by the addresses $\mathbf{A}R$ and $\mathbf{A}L$. Consider the image of the branch $b(\mathbf{A}R)$. The starting point of the branch $b(\mathbf{A}R)$ is the endpoint of the branch $\mathbf{b}_1$, and the image of this point is the endpoint of the branch $\mathbf{b}_2$. The image of the branch $b(\mathbf{A}R)$ must be a level $k+1$ branch of T_2 that is a descendant of $\mathbf{b}_2$, so it can either be $b(\mathbf{B}R)$ or $b(\mathbf{B}L)$. If the image is $b(\mathbf{B}R)$, then the image of $b(\mathbf{A}L)$ must be $b(\mathbf{B}L)$, since it must be the other descendant of $\mathbf{b}_2$. If the image is $b(\mathbf{B}L)$, then the image of $b(\mathbf{A}L)$ must be $b(\mathbf{B}R)$. In either case, the images of the two level $k+1$ branches of S_1 are the two level $k+1$ branches of S_2. Now assume that all branches of levels k through $k+l$ of S_1 are mapped to the branches of levels k through $k+l$ of

S_2, for some $l > 0$. Consider a level $k+l+1$ branch $\mathbf{b}$ of S_1. By a similar argument as for branches of level $k+1$, there are two possible branches that could be the image of $\mathbf{b}$. The level $k+l$ branch that $\mathbf{b}$ descends from is mapped to a branch $\mathbf{b}'$ of S_2, so $\mathbf{b}$ is mapped to either one of the level $k+l+1$ branches that descend from $\mathbf{b}'$. In either case, the image of $\mathbf{b}$ is a branch on S_2. So by induction, all branches of the subtree S_1 are mapped to branches of S_2.

Now let P_1 be a tip point of S_1, then P_1 has an address of the form $\mathbf{AA}'$ for some $\mathbf{A}' \in \mathcal{A}_\infty$. We need to show that the image of P_1 is on the subtree S_2. We have already established that the image of P_1 must be a tip point (see the proof of Lemma 3.7.1.1). Let $P_2 = f(P_1)$. Suppose P_2 is not on S_2. Then there exists a neighbourhood of P_2 that is disjoint from S_2. Consider the path p_1 on the tree that starts with the branch $\mathbf{b}_1$ and goes to the point P_1, and its image under f. Every branch on the path p_1 is mapped to a branch of S_2, while the point P_2 is disjoint from S_2. This contradicts the fact that the image of the path is connected. Therefore, P_2 is indeed on S_2, and any tip point of S_1 is mapped to a tip point of S_2.

A subtree is equal to the union of its branches and tip points, so this suffices to prove that the image of S_1 is S_2. The subtree S_1 and its level were arbitrary. Therefore, given $k > 0$, the image of any level k subtree of T_1 is a level k subtree of T_2. In particular, if the image of some level k branch $\mathbf{b}_1 = b(\mathbf{A})$ of T_1 is the level k branch $b(\mathbf{B})$ of T_2, then the image of the subtree $S_\mathbf{A}$ of T_1 is the subtree $S_\mathbf{B}$ of T_2. $\square$

Proposition 3.7.1.8 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be self-contacting trees such that $\theta_1, \theta_2 < 90°$. Suppose T_1 and T_2 are homeomorphic and $f : T_1 \to T_2$ is a homeomorphism. Once the images of the level 1 branches are determined, then the images of all other branches are determined. Moreover, if the image of the branch $b(R)$ of T_1 is the branch $b(R)$ on T_2, then every branch of T_2 has the same address as its pre-image branch. If the image of the branch $b(R)$ of T_1 is the branch $b(L)$ of T_2, then every branch of T_2 has the mirror image address of its pre-image branch.*

Proof. First suppose that the image of branch $b(R)$ of T_1 is $b(R)$ of T_2. We will use induction on the levels of branches to show that this forces every branch of T_2 to have the same address as its pre-image on T_1. For the level 1 branches, the claim is trivial. By assumption, $(b(R))_1$ is mapped to $(b(R))_2$, and since every branch must be mapped to a branch of the same level, this forces the image of $(b(L))_1$ to be $(b(L))_2$. So assume that every branch of levels 1 through k on T_2 has the same address as its pre-image on T_1. From the previous proposition, this implies that subtrees of levels 1 through k also have the same address as their pre-image. Let $\mathbf{b}_2 = b(\mathbf{A})$ be a level $k+1$ branch on T_2, for some address $\mathbf{A} = A_1 A_2 \cdots A_{k+1} \in \mathcal{A}_{k+1}$. This branch descends from the level k branch $\mathbf{b}' = b(A_1 \cdots A_k)$ of T_2. Since $\mathbf{b}'$ is a level k branch, then by assumption it has the same address as its pre-image on T_1. There are two level $k+1$ branches on T_1 that descend from the pre-image of $\mathbf{b}'$, namely the branches corresponding to the addresses $A_1 \cdots A_k R$ and $A_1 \cdots A_k L$. For either tree, consider the two level k subtrees with addresses $A_1 \cdots A_k$ and $A_1 \cdots \bar{A}_k$, where $\bar{A}_k$ denotes the mirror image of A_k. These two subtrees must intersect along the linear extension of the branch $b(A_1 \cdots A_{k-1})$, because the level $k-1$ subtree with address $A_1 \cdots A_{k-1}$ is similar to the tree (which contains intersection of the two level 1 subtrees along the y-axis). This implies that exactly one of the level $k+1$ subtrees of $S_{A_1 \cdots A_k}$ must intersect the subtree $S_{A_1 \cdots \bar{A}_k}$ (since at least one must interest, and the other one is disjoint from the side of the linear extension of $b(A_1 \cdots A_k)$ where the intersection occurs). This forces the two level $k+1$ subtrees of T_2 that descend from the subtree $S_{A_1 \cdots A_k}$ to have the same address as their pre-images on T_1, since the level k subtrees have the same addresses. Then the pre-image of the branch $\mathbf{b}_2$ must have the same address.

A similar argument shows that if the image of the branch $b(R)$ of T_1 is the branch $b(L)$ of T_2, then every branch of T_2 has the mirror image address of its pre-image branch. $\square$

Corollary 3.7.1.9 *Let $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ be self-contacting trees such that $\theta_1, \theta_2 < 90°$. Suppose T_1 and T_2 are homeomorphic and $f : T_1 \to T_2$ is a homeomorphism. Then either f preserves the address of every point or f sends every point to a point with the mirror image address.*

Proof. From the previous proposition, the claim is true for all points with finite addresses (since they are all endpoints of branches). Now consider tip points. Let P_1 be a point on T_1, and let $\mathbf{A}$ be the address of P_1, for some $\mathbf{A} = A_1 \cdots \in \mathcal{A}_\infty$. Let P_2

be the point on T_2 that has address $\mathbf{A}$. Suppose f preserves the address of all vertex points. Let $\mathbf{A}_i$ denote the finite address $A_i \cdots A_i$, for $i \geq 1$. Recall that

$$P_1 = (P_{\mathbf{A}})_1 = \lim_{i \to \infty} (P_{\mathbf{A}_i})_1,$$

and

$$P_2 = (P_{\mathbf{A}})_2 = \lim_{i \to \infty} (P_{\mathbf{A}_i})_2,$$

Let U_1 be any neighbourhood of P_1, and let $U_2 = f(U_1)$. There exists an integer j such that U_1 contains branches of the form $(b(\mathbf{A}_i))_1$, for all $i \geq j$, since P_1 is the limit of the endpoints of these branches. By assumption, each branch $(b(\mathbf{A}_i))_1$ on T_1 is mapped to the branch with the same address on T_2. Thus U_2 contains every branch $(b(\mathbf{A}_i))_2$ for $i \geq j$, and hence U_2 must contain the limit as $i \to \infty$, which is the point P_2. The neighbourhood U_1 was arbitrary, so the image of the point P_1 must be P_2. So the address of tip points is preserved under the map f.
Now suppose that f sends every branch on T_1 to the branch on T_2 that has the mirror image address. Then vertex points are sent to vertex points with mirror image addresses. A similar argument as in the first case can be used to show that a tip point must be mapped to a tip point with a mirror image address, since they are both limits of vertex points. $\square$

Recall that a self-contacting point is any point on a tree that is a double point, *i.e.*, corresponds to more than one address. All self-contacting points are either on the y-axis, or are the image of a self-contacting point on the y-axis under an address map, so it suffices to consider the set of addresses that correspond to points on the y-axis.

Theorem 3.7.1.10 *Two self-contacting trees with angles both less than* $90°$ *are homeomorphic if and only if they have the same self-contact addresses.*

Proof. First suppose that $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ are two self-contacting trees with $\theta_1, \theta_2 < 90°$ such that T_1 and T_2 have different self-contact addresses. Without loss of generality, assume that there exists some address $\mathbf{A} \in (A)_\infty$ such that $\mathbf{A}$ is a self-contact address for T_1 but not for T_2, and the self-contacting point of T_1 that corresponds to $\mathbf{A}$ is on the y-axis. Let P_1 denote the point $(P_{\mathbf{A}})_1$ on T_1. Then, by assumption, P_1 is on the y-axis. In addition, the point P_1^* of T_1 that corresponds to the mirror image address $\mathbf{A}^*$ has the same coordinates as P_1. Now consider the point $P_2 = (x_2, y_2) = (P_{\mathbf{A}})_2$ of T_2. By assumption, the address $\mathbf{A}$ is not a self-contact address of T_2, so the point P_2 is not on the y-axis, it must have $x_2 > 0$. The mirror image point P_2^* that corresponds to $\mathbf{A}^*$ has coordinates $(-x_2, y_2)$. That is, P_2 and P_2^* are two distinct points of the tree T_2. If T_1 and T_2 were homeomorphic, then any homeomorphism either preserves the address of tip points, or sends a tip point to a tip point with the mirror image address. Therefore, it is not possible that T_1 and T_2 are homeomorphic, because the addresses $\mathbf{A}$ and $\mathbf{A}^*$ correspond to exactly one point of T_1 but two distinct points of T_2.

Now suppose that $T_1 = T(r_1, \theta_1)$ and $T_2 = T(r_2, \theta_2)$ are two self-contacting trees with $\theta_1, \theta_2 < 90°$ such that T_1 and T_2 have the same self-contact addresses. Then the trees are homeomorphic. We define a homeomorphism $f : T_1 \to T_2$ as follows. Let U be the open set of the tree that is equal to $\mathbf{y}_{[0,1)}$. The map f restricted to U is the identity map, and we use the action of the address maps to define f on the rest of the tree. Let P be any point of T_1. If P is not a tip point, then there exists a unique point P' on U and a unique finite address $\mathbf{A}$ such that $P = (m_{\mathbf{A}})_1(P')$. If P is a tip point, then there exists an address $\mathbf{A}$ such that $P = (m_{\mathbf{A}})_1(P')$ for any P' in U. Note that $\mathbf{A}$ is unique if and only if P is not a self-contacting tip point. Given any P on T_1, let P' and $\mathbf{A}$ be such that $P' \in U$ and $P = (m_{\mathbf{A}})_1(P')$. This homeomorphism preserves addresses. Then

$$f(P) = f((m_{\mathbf{A}})_1(P')) = (m_{\mathbf{A}})_2(P')$$

Then this map is bijective (it sends self-contacting points to self-contacting points with the same address), is continuous and has a continuous inverse, so it is a homeomorphism. Therefore, two self-contacting trees with angles both less than $90°$ are homeomorphic if and only if they have the same self-contact addresses. $\square$

Summary. In general, two self-contacting trees are homeomorphic if and only if they have the same self-contact addresses. So all self-contacting trees with angles strictly between $90°$ and $135°$ form a homeomorphism class and all self-contacting trees with angles greater than $135°$ form another class. For every $N \geq 2$, the self-contacting trees with angles strictly between θ_{N+1} and θ_N form a distinct homeomorphism class. There are infinitely many classes that consist of only one tree. These are the self-contacting trees with special angles. For every $N \geq 2$, the tree $T(r_{sc}, \theta_N)$ is the only element of its homeomorphism class. Finally, the trees $T(1/\sqrt{2}, 90°)$ and $T(1/\sqrt{2}, 135°)$ form the only class of contractible trees.

3.8 Brief Chapter Summary

This chapter has presented a discussion of various properties of symmetric binary fractal trees. The main part of this chapter provided a detailed description of self-avoiding, self-contacting and self-overlapping trees. Following and expanding the results of [31], we have presented criteria for determining the unique scaling ratio as a function of branching angle to yield a self-contacting tree. Because of the scaling nature of the trees and their subtrees, it suffices to determine when the two level 1 subtrees S_R and S_L intersect (but do not overlap). In addition, we have noted that the trees with special angles θ_N are special because they possess infinitely many tip points with minimal distance to the y-axis. Two other interesting angles are 90° and 135°, they are the only two angles whose corresponding self-contacting trees are space-filling. For all other angles besides the special angles θ_N and 90° and 135°, a tree $T(r, \theta)$ possesses a unique point on the subtree S_R with minimal distance to the y-axis. Table 3.3 summarizes the results. The height and width of self-avoiding and self-contacting trees were discussed. This chapter includes a discussion on special types of points: contact, secondary contact, top vertex, canopy and corner. The identification of different classes of points will enable us to distinguish between different kinds of holes in closed ϵ-neighbourhoods. Finally we gave explicit constructions for the boundaries of level 0 holes of self-contacting trees. The theory developed in this chapter gives a foundation for our study of trees using the closed ϵ-neighbourhoods, which begins in the next chapter.

Angle Description	**Self-Contacting Addresses**	r_{sc}
$(0°, 90°)$, not special	$RL^{N+1}(RL)^{\infty}$	Root of x_{c1} (see notes below)
$(0°, 90°)$, special	$RL^{N+1}\mathbf{A}$ for all $\mathbf{A} \in \mathcal{AL}_{\infty}$	Root of x_{c1} (see notes below)
$90°$	$RL^2\mathbf{A}$ and $R^3\mathbf{A}$ for all $\mathbf{A} \in \mathcal{AL}_{\infty}$	$\dfrac{1}{\sqrt{2}}$
$(90°, 135°)$	$R^3(LR)^{\infty}$	$\dfrac{-\cos\theta - \sqrt{2 - 3\cos^2\theta}}{4\cos^2\theta - 2}$
$135°$	$RR\mathbf{A}$ for all $\mathbf{A} \in \mathcal{AL}_{2k}$, $k \geq 0$, and $\mathbf{A} \in \mathcal{AL}_{\infty}$	$\dfrac{1}{\sqrt{2}}$
$(135°, 180°)$	$RR(LR)^k$, for $k \geq 0$	$-\dfrac{1}{2\cos\theta}$

Table 3.3: Summary Of Self-Contacting Scaling Ratios and Addresses

Notes.

- $x_{c1} = r\sin(\theta) - \left[\sum_{k=1}^{N-2} r^{k+2}\sin(k\theta)\right] - \dfrac{r^{N+1}}{1-r^2}\left[\sin((N-1)\theta) + r\sin(N\theta)\right]$
- The self-contacting addresses are all addresses on S_R (except for $\mathbf{A}_0$) that correspond to points on the y-axis in the case of the self-contacting tree.

Chapter 4

Introduction to Closed Epsilon-Neighbourhoods and Properties of Holes

4.1 Introduction

This chapter commences the computational topology analysis of symmetric binary fractal trees (introduced in Chapter 2 of this thesis). The main goal of this chapter is to develop concepts that form the foundation of our theory. Quantitative results are left for the following chapters.

Recall that the only two space-filling, self-contacting trees occur at branching angles of 90° and 135°. The homology of a self-avoiding tree is trivial, while the homology of a self-contacting, non-space-filling tree is non-trivial (in fact there are infinitely many generators for the first homology group). Homology theory provides one way to classify a symmetric binary fractal tree, but this is a very coarse classification. How can we make finer classifications? Our approach is to look at the homology not just of the tree, but also of closed ϵ-neighbourhoods (defined later in this chapter) of the trees, as ϵ ranges through the non-negative real numbers. To investigate the homology of a compact subset of $\mathbb{R}^2$, it suffices to study the holes of the subset. Generally, for $\epsilon > 0$, the closed ϵ-neighbourhoods offer a way to construct spaces with finitely generated homology groups because they contain only finitely many holes. Note that there are some interesting exceptions. For example, the self-contacting tree with branching angle 67.5° is such that there are non-zero values of ϵ where the corresponding closed ϵ-neighbourhoods have infinitely many holes.

We will further refine our study of the homology of closed ϵ-neighbourhoods by looking at different features of the holes. Various properties of the holes of the closed ϵ-neighbourhoods offer different ways to characterize the trees. We first introduce the notion of a hole class, and define the persistence interval and persistence of a hole class. We define the important concept of level of a hole, along with complexity of a hole. Both of these concepts are related to the action of the monoid M_{LR} of a tree. We also discuss the concept of a hole location. Finally, the theory in this chapter forms the foundation for the theory in the next chapter, regarding critical values of the parameters r and θ as well as critical ϵ-values, and definitions of different classes of trees that refine the topological classification described in [31].

It is important to remember that many of the constructions developed to analyze the computational topology of symmetric binary fractal trees will extend not only to more general fractal trees, but to other classes of fractals as well. With this thought in mind, many of the definitions in this chapter have been developed to be as general as possible.

4.2 Closed Epsilon-Neighbourhoods and Holes

Let $T(r, \theta) \in \mathcal{T}$. If it is self-avoiding, it is contractible. If it is self-contacting, then it contains an infinite number of loops or it is space-filling (as discussed in Chapter 4). Thus homology makes a distinction between self-avoiding trees and self-contacting trees. What about trees whose scaling ratio is close to the self-contacting ratio compared to trees whose scaling ratio is not close? Consider the images of three self-avoiding trees Figures 4.1, 4.2, and 4.3. The first two trees have the same branching angle but different scaling ratio, while the first and third trees have the same scaling ratio and slightly different branching angle.

The three trees in Figures 4.1, 4.2, and 4.3 each have trivial homology since they are contractible, and they are topologically equivalent. We will develop a kind of classification of the trees in which the first and third would be of the same class, while the second tree is in a different class.

The main idea of our characterizations of symmetric binary fractal trees is that we look not just at the homology of the trees themselves, but we also look at the homology of closed ϵ-neighbourhoods as ϵ ranges over the non-negative real numbers.

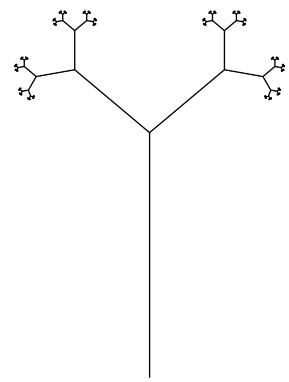

Figure 4.1: $T(0.4, 50°)$

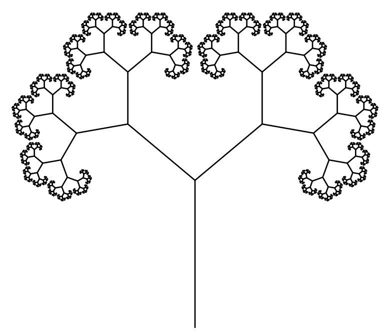

Figure 4.2: $T(0.595, 50°)$

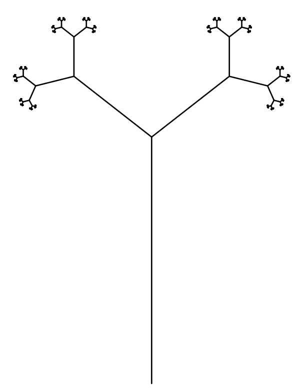

Figure 4.3: $T(0.4, 52°)$

We shall see that this homology is not always trivial because there could be holes in the closed ϵ-neighbourhoods. This chapter develops theory to answer questions such as: When can holes exist? How do they depend on r, θ, and ϵ? Can various properties of the holes characterize a tree, or a certain class of trees?

4.2.1 General Closed Epsilon-Neighbourhoods in $\mathbb{R}^2$

First we give our definition of closed ϵ-neighbourhoods of general subsets of $\mathbb{R}^2$, and then develop some notation specific to trees.

Definition 4.2.1.1 *Let $U \subset \mathbb{R}^2$. Let $\epsilon \geq 0$. The* **closed epsilon neighbourhood (ϵ-neighbourhood) of** *U, denoted by $\overline{U_\epsilon}$, is the set of all points in $\mathbb{R}^2$ that are at a distance of ϵ or less to U.*

$$\overline{U_\epsilon} = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), U) \leq \epsilon\} \tag{4.2.1}$$

Notation. We denote the boundary of a closed ϵ-neighbourhood $\overline{U_\epsilon}$ by $\partial\overline{U_\epsilon}$. Thus

$$\partial\overline{U_\epsilon} = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), U) = \epsilon\} \tag{4.2.2}$$

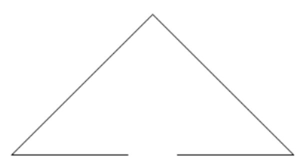

Figure 4.4: The Contractible Set U

We now consider an example to illustrate how the homology of the closed ϵ-neighbourhoods gives us information about how an object is embedded in $\mathbb{R}^2$, and how the homology varies as ϵ varies. Consider the subset U of $\mathbb{R}^2$ that is given in Figure 4.4. The set U is contractible. If ϵ is sufficiently small, the closed ϵ-neighbourhood remains contractible; see Figure 4.5. Then there is a range of values of ϵ for which the corresponding closed ϵ-neighbourhood contains a hole, and thus it is not contractible; see Figures 4.6 and 4.7. The smallest such ϵ would be equal to half the width of the gap at the bottom on the set. This value is what we will call the 'contact' value. If ϵ is sufficiently large, the whole region is covered and again the closed ϵ-neighbourhood is contractible; see Figure 4.8. The smallest ϵ for which the whole region is covered is what we will call the 'collapse' value, and it is generally more difficult to determine than the contact value. For polygonal regions, this is related to finding the largest inscribed circles and certain minimal distances (see [8]). For fractals in general, this is much more complicated, although one can use certain polygonal approximations to estimate collapse values. In the case of symmetric binary fractal trees we will use symmetry arguments and the properties of special points discussed in Section 3.5 to analyze the development of the holes as ϵ varies. In some cases we are only able to find approximations for the value of ϵ where the hole ceases to exist.

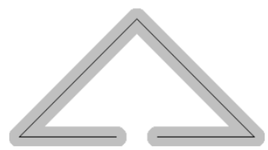

Figure 4.5: Small ϵ: A Contractible Closed ϵ-neighbourhood

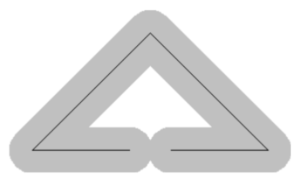

Figure 4.6: A Multiply-connected Closed ϵ-neighbourhood

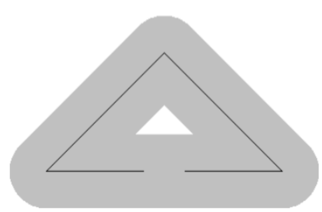

Figure 4.7: A Multiply-connected Closed ϵ-neighbourhood

For the sake of convenience, we give the following definition.

Definition 4.2.1.2 *The* **closed ϵ-neighbourhood at infinity**, *denoted by* $\overline{U_\infty}$, *is defined to be*

$$\overline{U_\infty} = \lim_{\epsilon \to \infty} \overline{U_\epsilon} \tag{4.2.3}$$

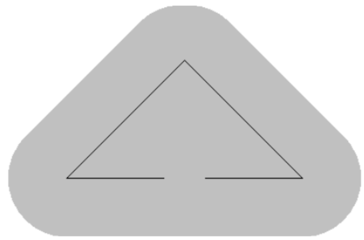

Figure 4.8: Large ϵ: A Contractible Closed ϵ-neighbourhood

Observations. For any $U \subset \mathbb{R}^2$, $\overline{U_\infty} = \mathbb{R}^2$. Also note that for any $\epsilon_1, \epsilon_2 \in [0, \infty]$ such that $\epsilon_1 \leq \epsilon_2$, we have $\overline{U_{\epsilon_1}} \subseteq \overline{U_{\epsilon_2}}$.

4.2.2 Closed Epsilon-Neighbourhoods of Symmetric Binary Fractal Trees

Notation.

1. Let $\theta \in (0°, 180°)$ and $r \in (0, 1)$ be given, and let $\epsilon \geq 0$. We denote the **closed ϵ-neighbourhood of the tree with scaling ratio r and branching angle** θ by $E(r, \theta, \epsilon)$, and it is equal to $\overline{T(r, \theta)_\epsilon}$:

$$E(r, \theta, \epsilon) = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), T(r, \theta)) \leq \epsilon\} \tag{4.2.4}$$

   We often use the notation $E(\epsilon)$ or E.

2. The boundary of $E(r, \theta, \epsilon)$ is denoted by $\partial E(r, \theta, \epsilon)$ or just ∂E. Thus

$$\partial E = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), T(r, \theta)) = \epsilon\} \tag{4.2.5}$$

The new notation for the closed ϵ-neighbourhoods of trees is used to reflect the fact that the topological and geometrical properties of a given closed ϵ-neighbourhood are a function of r, θ and ϵ.

Observation. For any r and θ, $E(r, \theta, 0) = \partial E(r, \theta, 0) = T(r, \theta)$. For any $\epsilon \geq 0$, $T(r, \theta) \subseteq E(r, \theta, \epsilon)$.

For a given tree $T(r, \theta)$, we can consider closed ϵ-neighbourhoods of subsets of the tree. Of particular interest are subsets that are subtrees, and so we give a formal definition for these.

Definition 4.2.2.1 *Let $r \in (0, 1)$ and $\theta \in (0°, 180°)$ be given. Let $\epsilon \in [0, \infty]$. Let $S = S_{\mathbf{A}}(r, \theta)$ be a subtree of $T(r, \theta)$, where $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 1$. Let $\mathbf{b} = b(\mathbf{A})$. We define the* **closed ϵ-neighbourhood of the subtree S with scaling ratio r, branching angle θ and trunk $b(\mathbf{A})$**, *denoted by $E_{\mathbf{A}}(r, \theta, \epsilon)$, $E_S(\epsilon)$, E_S, or $E_{\mathbf{b}}$, to be :*

$$E_{\mathbf{A}}(r, \theta, \epsilon) = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), S_{\mathbf{A}}(r, \theta)) \leq \epsilon\} \tag{4.2.6}$$

For a given tree, each subtree is similar to the tree, and so we would expect there to be some similarity between closed ϵ-neighbourhoods of the trees and closed ϵ-neighbourhoods of the subtrees. The following theorem clarifies this idea.

Theorem 4.2.2.2 *Let r, θ be given. Let $S = S_{\mathbf{A}}(r, \theta) \in \mathcal{S}_k$, where $\mathbf{A} \in \mathcal{A}_k$, for some $k \geq 1$. Then for any $\epsilon \in [0, \infty)$, $E_{\mathbf{A}}(r, \theta, r^k\epsilon) = m_{\mathbf{A}} E(r, \theta, \epsilon)$. Consequently $E_{\mathbf{A}}(r, \theta, r^k\epsilon) \sim_r^k E(r, \theta, \epsilon)$.*

Proof. This result follows directly from the definition of closed ϵ-neighbourhood and the Address Map Lemma 2.1.3.2. □

4.2.3 Holes in Closed Epsilon-Neighbourhoods

Intuitively, for a given tree $T(r, \theta)$ and a given $\epsilon \geq 0$, one considers a hole in $E(r, \theta, \epsilon)$ to be some open, contractible (*i.e.* simply-connected) set that is disjoint from the closed ϵ-neighbourhood, and such that the boundary of the hole is a subset of the boundary of the closed ϵ-neighbourhood.

Notation. For a given tree $T(r, \theta) \in \mathcal{T}$ with $r \in (0, 1)$ and $\epsilon \geq 0$, let $E^C(r, \theta, \epsilon)$, or E^C, denote the complement of the closed ϵ-neighbourhood in $\mathbb{R}^2$.

$$E^C(r, \theta, \epsilon) = \mathbb{R}^2 \backslash E(r, \theta, \epsilon) = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), T(r, \theta)) > \epsilon\} \tag{4.2.7}$$

In general, we use a superscript of 'C' to denote the complement of a given set in $\mathbb{R}^2$.

Observations. For any r, θ, and ϵ, E^C is an open set. So if E^C is non-empty, then the components of E^C are all open. There is one unbounded component whose first homology group is $\mathbb{Z}$, and any other components are contractible. The holes, if they exist, are the bounded components. They are also the insides of simple, closed curves of the boundary of the closed ϵ-neighbourhood.

Recall from Chapter 3, Section 3.1 that Γ denotes the set of the simple, closed curves in $\mathbb{R}^2$. For $\gamma \in \Gamma$, the inside of a curve is denoted $O(\gamma)$. We also defined the set of simple, closed curves of tree $T(r, \theta)$, denoted $\Gamma(r, \theta)$ (see Equation 3.1.1). A hole of a tree $T(r_{sc}, \theta)$ is equal to $O(\gamma)$ for some γ such that $O(\gamma)$ is disjoint from $T(r_{sc}, \theta)$. Now we define the set of simple, closed curves of the boundary of a closed ϵ-neighbourhood.

Definition 4.2.3.1 *Let $r \in (0,1)$ and $\theta \in (0°, 180°)$ be given. Let $\epsilon \in [0, \infty]$. A* **simple, closed curve of the boundary of the closed ϵ-neighbourhood** $E(r, \theta, \epsilon)$ *is a simple, closed curve γ such that γ is a subset of $\partial E(r, \theta, \epsilon)$ and $O(\gamma)$ is disjoint from the closed ϵ-neighbourhood $E(r, \theta, \epsilon)$, i.e. $O(\gamma) \subset E^C$. Let $\Gamma(r, \theta, \epsilon)$ be the collection of all simple, closed curves of the boundary of the closed ϵ-neighbourhood $E(r, \theta, \epsilon)$.*

$$\gamma \in \Gamma(r, \theta, \epsilon) \quad \Leftrightarrow \gamma \in \Gamma \;\; \textit{and} \;\; \gamma \subset \partial E \;\; \textit{and} \;\; O(\gamma) \subset E^C \tag{4.2.8}$$

Note that $\Gamma(r, \theta, 0) \subseteq \Gamma(r, \theta)$, the set of simple, closed curves of the actual tree (see 3.1.1).

Definition 4.2.3.2 *Let $r \in (0,1)$ and $\theta \in (0°, 180°)$ be given. Let $\epsilon \in [0, \infty]$. A* **hole in the closed ϵ-neighbourhood** $E(r, \theta, \epsilon)$ *is a region of the form $O(\gamma)$ for some $\gamma \in \Gamma(r, \theta, \epsilon)$. Alternately, a set H is a hole if it is an open, simply-connected subset of $\mathbb{R}^2$ such that H is a component of the complement of $E(r, \theta, \epsilon)$ in $\mathbb{R}^2$. Denote the set of all such holes by $\mathcal{H}(r, \theta, \epsilon)$, or just $\mathcal{H}(\epsilon)$. Thus*

$$H \in \mathcal{H}(r, \theta, \epsilon) \;\; \Leftrightarrow \;\; \exists \gamma \in \Gamma(r, \theta, \epsilon) \textit{ such that } H = O(\gamma) \tag{4.2.9}$$

$$\Leftrightarrow \;\; \pi_1(H) = \{0\} \textit{ and } H \subset E^C \textit{ and } \partial H \subset \partial E \tag{4.2.10}$$

Observation. For any r and θ, $\mathcal{H}(r, \theta, \infty) = \emptyset$.

Definition 4.2.3.3 *For any r and θ, let $\mathcal{H}(r, \theta)$, or just $\mathcal{H}$, denote the collection of all possible holes in the closed ϵ-neighbourhoods as ϵ ranges through $[0, \infty]$.*

$$\mathcal{H}(r, \theta) = \bigcup_{\epsilon \in [0,\infty]} \mathcal{H}(r, \theta, \epsilon) \tag{4.2.11}$$

4.2.4 Number of Holes

Notation. Let $\epsilon \geq 0$. Let $N(r, \theta, \epsilon)$ be the number of holes in the closed ϵ-neighbourhood $E(r, \theta, \epsilon)$ of the tree $T(r, \theta)$. Thus

$$N(r, \theta, \epsilon) = |\mathcal{H}(r, \theta, \epsilon)| \tag{4.2.12}$$

Often we write $N(\epsilon)$. For a given $\epsilon \geq 0$, $N(r, \theta, \epsilon)$ may be 0, a finite number, or infinite, depending on r and θ. Note that $N(\epsilon)$ is different from $N(\theta)$, the turning number for a given angle.

Proposition 4.2.4.1 *For any θ except $90°$ or $135°$, $N(r_{sc}, \theta, 0) = \infty$. That is, a self-contacting, non-space-filling tree has an infinite number of holes.*

Proof. There must exist at least one closed, simple curve in the tree $T(r_{sc}, \theta)$ such that $O(\gamma)$ is disjoint from $T(r_{sc}, \theta)$ (see Section 3.6 for actual constructions of such curves). Without loss of generality, let $\gamma \in \Gamma(r, \theta, 0)$ such that γ intersects $\mathbf{y}$ (see Lemma 3.2.0.25). For each $k \geq 1$, the set $m_{R^k}(\gamma)$ is a simple closed curve, by the similarity of the tree. Let γ_k denote $m_{R^k}(\gamma)$. Also by the similarity of the tree, we have $O(\gamma_k)$ disjoint from $T(r_{sc}, \theta)$, so $O(\gamma_k)$ is a hole of the tree. Each $O(\gamma_k)$ is distinct, because each γ_k is similar to γ with contraction factor r_{sc}^k. Therefore the corresponding holes to these simple, closed curves as k ranges through the positive integers are all distinct, and so there must be an infinite number of distinct holes. $\square$

Observation. For any θ, if $r < r_{sc}$, then $N(r, \theta, 0) = 0$. That is, the number of holes in any self-avoiding tree is 0 since there are no simple, closed curves in $\partial E(r, \theta, 0) = T(r, \theta)$, because any self-avoiding tree is contractible.

Observation. For any r and θ, $N(r, \theta, \infty) = 0$ since $\mathcal{H}(r, \theta, \infty) = \emptyset$.

We would like to study the dependence of $\mathcal{H}(r, \theta, \epsilon)$ and $N(r, \theta, \epsilon)$ on ϵ.

Definition 4.2.4.2 *A tree $T(r, \theta)$ is a* **simple tree** *if $N(r, \theta, \epsilon) = 0$ for every $\epsilon \in [0, \infty]$. Thus every closed ϵ-neighbourhood of the tree is contractible (there are no holes).*

We shall see that simple trees do exist. For example, the self-contacting, space-filling trees with branching angle 90° or 135° are contractible, and any closed ϵ-neighbourhood of such a tree is also contractible. These two trees are special cases. In general, the simple trees occur for small scaling ratios. In fact, for any branching angle, there exist scaling ratios such that the corresponding trees are simple. We discuss this in detail in the next chapter, in the section on complexity.

Definition 4.2.4.3 *A tree $T(r, \theta)$ is a* **non-simple tree** *if there exists an $\epsilon \geq 0$ such that $N(r, \theta, \epsilon) > 0$.*

To continue the study of non-simple trees, more theory regarding the holes themselves is needed.

4.3 Persistence of Holes and Hole Classes

Consider the self-contacting tree with branching angle 145°, shown in Figure 4.10. The tree itself contains infinitely many holes. For example, there is a hole whose boundary contains the vertical line segment between the point with address RR and the top of the trunk, and part of the branch $b(R)$. For values of $\epsilon > 0$ that are sufficiently small, there will still be exactly one hole in this region of the closed ϵ-neighbourhood. There will not always be a hole in this region, because for sufficiently large ϵ, the region is completely covered by the ϵ-neighbourhood. This leads us to wonder for what range of values of ϵ the holes persist.

Consider the tree $T(r_{sc}, 45°)$ shown in Figure 4.9. There is a hole H bounded in part by the point $(0, 1)$, the lowest tip point on the y-axis (with address $RL^3(LR)^\infty$), and the branches $b(R)$ and $b(L)$. In Chapter 6, we will determine the smallest ϵ for

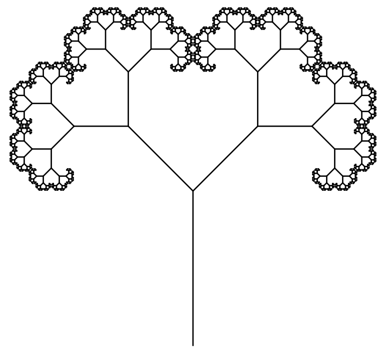

Figure 4.9: $T(r_{sc}, 45°)$.

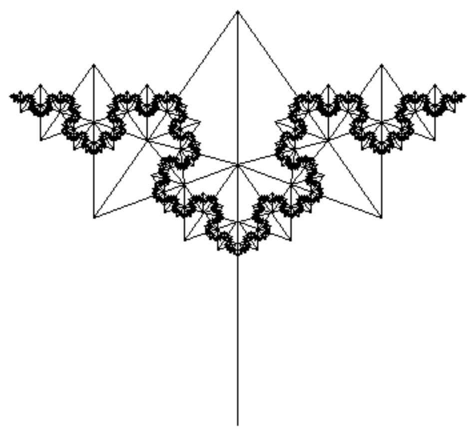

Figure 4.10: $T(r_{sc}, 145°)$

which this hole is covered by the closed ϵ-neighbourhood of the tree. For any ϵ between 0 and this 'collapse' value, there is exactly one hole in the closed ϵ-neighbourhood that has non-empty intersection with H.

This is not always the case. Consider the hole H in the tree $T(r_{sc}, 145°)$ described at the beginning of this section. Now consider the point with address $RRRL$. When ϵ is half the x-coordinate of this point, the original hole splits into two holes. Because of this possibility of a hole splitting, we have to be careful with the definition of the persistence interval. We would like the definition to be such that for any two holes, their corresponding persistence intervals are either the same or they are disjoint, because we do not want one to be a proper subset of the other (as would be the case for one of the two holes that is a result of a larger hole splitting in two).

Definition 4.3.0.4 *Let $T(r,\theta)$ be a non-simple tree, let $\epsilon_1 \geq 0$ be such that there exists a hole H_1 in $E(r,\theta,\epsilon_1)$. Then the* **persistence interval of** H_1*, denoted by $p(H_1)$, is the maximal interval I such that for any $\epsilon \in I$, there is a unique hole H of $E(r,\theta,\epsilon)$ that has non-empty intersection with H_1, and H obeys the following condition. For any $\epsilon_2 \in I$, there exists exactly one hole H_2 of $E(r,\theta,\epsilon_2)$ that has non-empty intersection with H.*

Definition 4.3.0.5 *Let $T(r,\theta)$ be a non-simple tree, let $\epsilon_0 \geq 0$ be such that there exists a hole H of $E(r,\theta,\epsilon_0)$. The* **hole class** *$[H]$ is the set of holes that have non-empty intersection with H as ϵ ranges through $p(H)$.*

Note. We often use just the symbol H to denote the hole class $[H]$ when the context is clear. For any $H' \in [H]$, we have $p(H') = p(H)$ (as a result of the definition of persistence interval), so $p(H)$ is often considered to be the persistence interval of the hole class $[H]$, and is also denoted by $p([H])$.

Definition 4.3.0.6 *The* **persistence** *of a hole class $[H]$ is defined to be the length of the persistence interval $p([H])$, and is denoted by $|p([H])|$ or $|p(H)|$.*

Note. We shall see that it is possible for a hole class to have persistence equal to 0. Various examples of such hole classes are discussed in Chapter 6.

Definition 4.3.0.7 *Let $T(r,\theta)$ be a non-simple tree, and let H be a hole of some ϵ-neighbourhood of the tree. The* **contact value** *of the hole class $[H]$ is the infimum of the persistence interval of the hole class, and it is denoted by $\underline{\epsilon_{[H]}}$ or just $\underline{\epsilon_H}$. The* **collapse value** *of the hole class $[H]$ is the supremum of the persistence interval, and it is denoted by $\overline{\epsilon_{[H]}}$ or just $\overline{\epsilon_H}$. Thus*

$$\underline{\epsilon_H} = \inf(p([H])), \quad \overline{\epsilon_H} = \sup(p([H])), \tag{4.3.1}$$

Note. The collapse value marks the end of a certain hole class. It could mean that the maximal hole of the class is now completely covered by the closed ϵ-neighbourhood, but it could also mean that the hole class has split into more than one hole class (so the region is not covered).

Definition 4.3.0.8 *Let $[H]$ be a hole class. The* **maximal hole of the hole class** *$[H]$ is denoted $[H]_{max}$, and it is the element of $[H]$ that occurs at $\underline{\epsilon_H}$. Every other hole in the hole class is a subset of this hole.*

More will be said regarding persistence intervals once we have introduced the concept of the level of a hole. In Chapter 6 we will present quantitative results about persistence intervals of hole classes for the closed ϵ-neighbourhoods of specific trees.

4.4 Symmetry and Levels of Holes

Each symmetric binary fractal tree has the y-axis, which we denote $\mathbf{y}$, as an axis of symmetry. In this section we investigate how this symmetry affects the structure of the holes. It follows immediately that those holes which intersect $\mathbf{y}$ are symmetric in $\mathbf{y}$, and other holes which do not intersect $\mathbf{y}$ have a counterpart reflected across $\mathbf{y}$. It follows from self-similarity that many holes will be symmetric in lines which form linear extensions of branches on the tree.

Thus it seems plausible that there are holes in some closed ϵ-neighbourhoods that are also symmetric, in the sense that a hole is symmetric if there is an axis of symmetry through the hole that is an axis of symmetry for the hole itself.

For self-contacting trees, we defined the notion of 'level' of a curve (see 3.2.0.28). Now we can define the notion of 'level' of a hole of a closed ϵ-neighbourhood in a similar way. The notion of level is important for developing deeper theory about a tree and its corresponding ϵ-neighbourhoods.

We now make these ideas more precise.

4.4.1 Symmetry

Definition 4.4.1.1 *For given values of r and θ and for $\epsilon \in [0, \infty)$, a hole $H \in \mathcal{H}(r, \theta, \epsilon)$ is a* **symmetric hole** *if there exists an axis of symmetry through it. Otherwise it is a* **non-symmetric** *hole.*

For example, the hole H of the tree $T(r_{sc}, 45°)$ described in the previous section (see Figure 4.9) is symmetric about $\mathbf{y}$. The hole H of the tree $T(r_{sc}, 145)$ described in the previous section (see Figure 4.10) is not symmetric. However, the mirror image of H is also a hole.

The following results are obvious based on the fact that any tree is symmetric about $\mathbf{y}$. For given values of r and θ and for $\epsilon \in [0, \infty]$:

- The closed ϵ-neighbourhood of the tree, $E(r, \theta, \epsilon)$, is symmetric about $\mathbf{y}$.
- The boundary ∂E of the closed ϵ-neighbourhood is symmetric about $\mathbf{y}$.
- Let $\gamma \in \Gamma(r, \theta, \epsilon)$ be such that γ does not intersect $\mathbf{y}$. Then the mirror image γ^* (γ reflected across $\mathbf{y}$) is an element of $\Gamma(r, \theta, \epsilon)$ distinct from γ.

Recall that $\mathbf{y}_{(1,\infty)}$ denotes the portion of the y-axis above the line $y = 1$, and $\mathbf{y}_{(-\infty,1)}$ denotes the portion below the line.

Proposition 4.4.1.2 *Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \in [0, \infty)$. If a simple, closed curve $\gamma \in \Gamma(r, \theta, \epsilon)$ is symmetric about $\mathbf{y}$ then it must be disjoint from $\mathbf{y}_{(-\infty,1)}$, and consequently γ intersects $\mathbf{y}_{(1,\infty)}$.*

Proof. Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \in [0, \infty)$. Let $\gamma \in \Gamma(r, \theta, \epsilon)$. The tree $T(r, \theta)$ does not intersect the area below $y = 0$, and therefore a loop corresponding to a hole cannot intersect that area either. It is obvious that a simple, closed curve in the boundary of the closed ϵ-neighbourhood can't intersect the trunk. Thus γ will have two intersection points with the y-axis (since it is simple). $\square$

The following lemma, which is the converse of the previous proposition, will be used for Theorem 4.4.2.7 presented in the next subsection.

Lemma 4.4.1.3 *Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \in [0, \infty)$. Let $\gamma \in \Gamma(r, \theta, \epsilon)$. If γ intersects $\mathbf{y}_{(1,\infty)}$, then γ is symmetric about $\mathbf{y}$. If γ intersects $\mathbf{y}_{(-\infty,1)}$, then γ cannot be symmetric about $\mathbf{y}$, and it is disjoint from one side of $\mathbf{y}$.*

Proof. Let θ be given, let $r \leq r_{sc}(\theta)$, and let $\epsilon \in [0, \infty)$. Let $\gamma \in \Gamma(r, \theta, \epsilon)$ be such that γ intersects $\mathbf{y}_{(1,\infty)}$. The boundary ∂E is symmetric about $\mathbf{y}$, so any element γ of $\Gamma(r, \theta, \epsilon)$ that intersects $\mathbf{y}_{(1,\infty)}$ must be such that γ is on both sides of $\mathbf{y}$, because the only part of the tree $T(r,\theta)$ that is vertical and on $\mathbf{y}$ is the trunk, since $r \leq r_{sc}$ (see Proposition 3.2.0.18). Thus γ is symmetric about $\mathbf{y}$, otherwise it could not be an element of $\Gamma(r,\theta,\epsilon)$.

Let $\gamma \in \Gamma(r,\theta,\epsilon)$ such that γ intersects $\mathbf{y}_{(-\infty,1)}$. Then γ must intersect $\mathbf{y}_{(0,1)}$, because there is no part of the tree below the line $y = 0$, since $r \leq r_{sc}$ (so a hole entirely below the line $y = 0$ is not possible). Because of the trunk, this means that γ must be disjoint from one side of $\mathbf{y}$, or it couldn't be an element of $\Gamma(r,\theta,\epsilon)$. Thus it could not be symmetric about $\mathbf{y}$. $\square$

4.4.2 Levels

It is possible to have a hole whose boundary is entirely contained in the boundary of the closed ϵ-neighbourhood of a subtree. Recall that for a given tree $T(r,\theta)$, there are subtrees of level k, for any non-negative integer k, where a level k subtree is the image of $T(r,\theta)$ under some level k address map. Note that the level 0 subtree is the tree itself. This motivates the definition of the 'level' of a hole. Before we give the definition, we present some preliminary results about holes and subtrees.

Theorem 4.4.2.1 (SUBTREE THEOREM) *Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \geq 0$ be such that $\mathcal{H}(r,\theta,\epsilon)$ is non-empty. Let $H \in \mathcal{H}(r,\theta,\epsilon)$. Then there exists a unique integer $k \geq 0$ such that:*

1. *For every $m \leq k$, there exists a unique level m subtree S such that $\partial H \subset \partial E_S$.*
2. *For all integers $l > k$, there are no level l subtrees S' such that $\partial H \subset \partial E_{S'}$.*

Proof. Suppose θ is given, $r \leq r_{sc}$, and $\epsilon \geq 0$ such that $\mathcal{H}(r,\theta,\epsilon)$ is non-empty. Let $H \in \mathcal{H}(r,\theta,\epsilon)$.

If ∂H was completely included in the boundary of more than one subtree, then these two subtrees must have been overlapping at branch interiors, and this contradicts that $r \leq r_{sc}$. So if ∂H is contained in the boundary of the closed ϵ-neighbourhood of some level m subtree, then this subtree is the unique such subtree of level m.

Trivially, ∂H is a subset of the boundary of the closed ϵ-neighbourhood of the level 0 subtree (which is just the tree itself). Now for any $m \geq 0$ such that there exists a unique level m subtree whose closed ϵ-neighbourhood boundary contains ∂H, then the same is true for all integers l such that $0 \leq l \leq m$. Let l be such that $0 \leq l \leq m$. Then there is a unique level l subtree S' that is a superset of S (since there is a unique level l branch that is an ancestor of the trunk of the subtree S), and hence $\partial H \subset \partial E_{S'}$.

Finally, there exists an integer m such that there are no level m subtrees whose closed ϵ-neighbourhood boundaries entirely contain ∂H. If $\epsilon = 0$, then $\partial E = T(r, \theta)$, and ∂H must partially consist of branches of the tree. Any integer that is higher than the level of a branch that is part of ∂H could be taken as m. If $\epsilon > 0$, then there exists an integer m such that the closed ϵ-neighbourhood of any subtree of level m or higher is contractible (since the size of the subtrees is decreasing as the levels increase), and so such an m would be such that there are no level m subtrees whose closed ϵ-neighbourhood boundaries entirely contain ∂H.

Thus there must exist a maximal non-negative integer k such that there exists a level k subtree S such that $\partial H \subset \partial E_S$, and this level k subtree is unique and for all integers $l > k$, there are no level l subtrees S' such that $\partial H \subset \partial E_{S'}$. $\square$

Definition 4.4.2.2 *Let θ and $r \leq r_{sc}$ be given, let $\epsilon \geq 0$ be such that $\mathcal{H}(r, \theta, \epsilon) \neq \emptyset$. Let $H \in \mathcal{H}(r, \theta, \epsilon)$, let $H' = [H]_{max}$ and $\epsilon' = \underline{\epsilon_H}$. Then H is a* **level** k **hole***, or a* k**th level hole***, for some $k \geq 0$, if k is the the largest integer such that there exists a subtree $S \in \mathcal{S}_k$, whose closed ϵ-neighbourhood boundary $\partial E_S(\epsilon')$ contains the boundary $\partial H'$ of H'. We write $l(H) = k$. For a given non-negative integer k, denote the set of all level k holes by $\mathcal{H}_k(r, \theta, \epsilon)$, or just $\mathcal{H}_k(\epsilon)$. Thus*

$$\begin{aligned} H \in \mathcal{H}_k(r, \theta, \epsilon) \Leftrightarrow & \\ & [\exists S \in \mathcal{S}_k \mid \partial H' \subset \partial E_S(\epsilon') \text{ and } \forall_{j>k} \forall_{S \in \mathcal{S}_j} \partial H' \not\subseteq \partial E_S] \end{aligned} \tag{4.4.1}$$

Notes. An immediate consequence of this definition is that the elements of any hole class are all at the same level. In addition, because $\partial E_S(\epsilon')$ contains the boundary $\partial H'$ of H' we also have that $\partial E_S(\epsilon)$ contains the boundary ∂H of H.

Notation. Let θ and $r \leq r_{sc}$ be given, and let $\epsilon \geq 0$. Let $k \geq 0$. Then $N_k(r, \theta, \epsilon)$, or just $N_k(\epsilon)$, is the number of level k holes of the corresponding closed ϵ-neighbourhood.

$$N_k(r, \theta, \epsilon) = |\mathcal{H}_k(r, \theta, \epsilon)| \tag{4.4.2}$$

Definition 4.4.2.3 *Let r and θ be given. Then $\mathcal{H}_k(r, \theta)$, or just $\mathcal{H}_k$, is the collection of all level k holes as ϵ ranges through $[0, \infty]$.*

$$\mathcal{H}_k(r, \theta) = \bigcup_{\epsilon \in [0,\infty]} \mathcal{H}_k(r, \theta, \epsilon) \tag{4.4.3}$$

The following theorems will be extremely useful for analyzing the characteristics of the elements of $\mathcal{H}(r, \theta)$ for the trees in $\mathcal{T}$.

Theorem 4.4.2.4 (HOLE LEVEL THEOREM) *Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$ such that $\mathcal{H}_k(r, \theta, \epsilon)$ is non-empty, for some $k \geq 1$. Let $H \in \mathcal{H}_k(r, \theta, \epsilon)$. Then for $\epsilon' = r^{-k}\epsilon$, there exists a hole $H' \in \mathcal{H}_0(r, \theta, \epsilon')$ and a level k address map $m_{\mathbf{A}}$ such that $H = m_{\mathbf{A}}(H')$, and thus $H \sim_r^k H'$.*

Proof. Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \geq 0$. Let k be a positive integer. Let $H \in \mathcal{H}_k(r, \theta, \epsilon)$. Let $\epsilon' = r^{-k}\epsilon$. There exists a subtree $S = S_{\mathbf{A}}(r, \theta) \in \mathcal{S}_k$, for some $\mathbf{A} \in \mathcal{A}_k$, such that $\partial H \subset \partial E_S$ (by definition of a level k hole). Now S is a level k subtree, so $E_{\mathbf{A}}(r, \theta, \epsilon) \sim_k E(r, \theta, \epsilon')$ since $E_{\mathbf{A}}(r, \theta, \epsilon) = m_{\mathbf{A}}(E(r, \theta, \epsilon'))$. Now ∂H is a simple, closed curve that is a subset of the boundary ∂E_S of E_S. $\partial E_S \sim_r^k \partial E$, via the address map $m_{\mathbf{A}}$, so there exists $\gamma \in \partial E(r, \theta, \epsilon')$ such that $m_{\mathbf{A}}(\gamma) = H$ and so $\partial H \sim_r^k \gamma$. Let $H' = O(\gamma) \in \mathcal{H}(r, \theta, \epsilon')$. Then $m_{\mathbf{A}}(H') = H$ and $H \sim_r^k H'$. Note that the boundary of E_S may intersect the boundaries of other level k subtrees, but the intersection can't affect ∂H because that would contradict H being a level k hole. $\square$

Recall that if one set is the image of another set under an address map, we refer to the image as a descendant and the pre-image as an ancestor. So in the previous theorem, the hole H is a descendant of H', and H' is a level 0 ancestor of H. The converse of the previous theorem is also true and is proved below in the Descendant Theorem 4.4.2.9.

Notes. The Hole Theorem may seem like an obvious corollary to the earlier Theorem 4.2.2.2 which stated that the closed ϵ-neighbourhood of a subtree is the image of a suitable closed ϵ-neighbourhood of the tree under a suitable address map. However, the closed ϵ-neighbourhoods of subtrees may overlap in the formation of the closed ϵ-neighbourhood of the tree, so it is more complicated than just considering the closed ϵ-neighbourhoods of subtrees as disjoint sets.

The Hole Theorem tells us that any hole in any closed ϵ-neighbourhood is the image of a level 0 hole under a level k address map, *i.e.*, it is a result of an action of the free monoid. This is a very special property, because this implies that the level 0 holes yield information about holes at *any level*. We can restrict our attention to the level 0 holes, and this is the same thing as taking the fundamental domain under the action of the free monoid, though it is interesting to see how the critical values and scaling ratios interact.

Notation. Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \geq 0$. Let k be a positive integer. Let $H \in \mathcal{H}_k(r, \theta, \epsilon)$. We write H_{-k} for the level 0 hole H' in $\mathcal{H}_0(r, \theta, r^{-k}\epsilon)$ identified in the previous theorem, that is, the unique level 0 ancestor of H'.

We now direct our attention to the properties of the level 0 holes, and then derive more general results about higher level holes. Recall the Self-Contact Criteria Theorem from Chapter 3 (Theorem 3.2.0.27), and its corollary. Self-contact for a given branching angle θ occurs at the smallest scaling ratio such that either a tip point intersects $\mathbf{y}_{(1,y_{\max}]}$, or a tip point or branch endpoint intersects the trunk. This implies that r_{sc} is the smallest scaling ratio such that either S_{RL} and S_{LR} intersect or S_{RR} and T_0 intersect. In terms of holes of the tree $T(r, \theta) = E(r, \theta, 0)$, self-contact occurs for the smallest scaling ratio such that there is a level 0 hole in the tree. For $\epsilon > 0$, the conditions for a hole to be level 0 are similar.

Theorem 4.4.2.5 (LEVEL 0 HOLE CRITERIA) *Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$. A hole $H \in \mathcal{H}(r, \theta, \epsilon)$ is level 0 if and only if one of the following conditions holds:*

1. *The boundary of H has non-empty intersection with both $E_{RL}(\epsilon)$ and $E_{LR}(\epsilon)$.*

2. *The boundary of H has non-empty intersection with both $E_{RR}(\epsilon)$ and $T_0(\epsilon)$.*

3. *The boundary of H has non-empty intersection with both $E_{LL}(\epsilon)$ and $T_0(\epsilon)$.*

Proof. Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$. Let $H \in \mathcal{H}(r, \theta, \epsilon)$.

Suppose one of the three conditions holds. Then the tree $T(r, \theta)$ is the highest level subtree whose closed ϵ-neighbourhood boundary contains ∂H. In the first case of the Theorem, we need one subtree on each side of $\mathbf{y}$, so we need the whole tree. In the second and third case, we need the trunk. The tree is a level 0 subtree, so H is level 0 by definition of level of a hole.

Now suppose H is a level 0 hole. Then ∂H cannot be entirely contained in the closed ϵ-neighbourhood boundary of either of the level 1 subtrees. This implies that one of the three conditions must hold, otherwise there would be a level 1 subtree whose closed ϵ-neighbourhood boundary contained ∂H. □

Corollary 4.4.2.6 *A hole is level 0 if and only if it intersects $\mathbf{y}_{(1,\infty)}$ (in the first case of Theorem 4.4.2.5) or it does not intersect $\mathbf{y}$ but needs the trunk for formation (in the second and third cases).*

Theorem 4.4.2.7 (SYMMETRY OF HOLES) *Let θ be given, let $r \leq r_{sc}$, and let $\epsilon \geq 0$. Let $H \in \mathcal{H}_0(r, \theta, \epsilon)$. If H intersects $\mathbf{y}_{[1,\infty)}$, then H is symmetric about the y-axis. If H does not intersect $\mathbf{y}_{[1,\infty)}$, then it is disjoint from one side of the y-axis, and the mirror image H^* is another element of $\mathcal{H}_0(r, \theta, \epsilon)$.*

Proof. This theorem is a direct result of the previous corollary and Lemma 4.4.1.3. □

Lemma 4.4.2.8 *Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$ such that there exists a level 0 hole H. Then $m_R(H)$ and $m_L(H)$ are distinct level 1 holes in $(r, \theta, r\epsilon)$.*

Proof. Let θ be given, and let $r \leq r_{sc}$. We need to show that the images $m_R(H)$ and $m_L(H)$ are level 1 holes and that they are distinct. The fact that they are distinct requires the assumption that the tree is not self-overlapping. There are two cases to consider. If $\epsilon = 0$, then $r = r_{sc}$, otherwise there would be no holes whatsoever. Then the lemma is a direct result of the scaling nature of the symmetric binary fractal trees, because $r\epsilon = 0$. Suppose $\epsilon > 0$ and $H \in \mathcal{H}_0(r, \theta, \epsilon)$. Then by the Level 0 Hole Criteria Theorem 4.4.2.5, we know that either H intersects $\mathbf{y}_{(1,\infty)}$ or H does not intersect $\mathbf{y}$, and the trunk is necessary for the formation of ∂H. Consider the closed ϵ-neighbourhoods of the two level 1 subtrees S_R and S_L, the trunk T_0, and the actual tree $T(r, \theta)$, for the ϵ-value $r\epsilon$. These are denoted $E_R(r, \theta, r\epsilon)$, $E_L(r, \theta, r\epsilon)$, $E(T_0, r\epsilon)$ and $E(r, \theta, r\epsilon)$ respectively. We clearly have

$$E(r, \theta, r\epsilon) = E_R(r, \theta, r\epsilon) \cup E_L(r, \theta, r\epsilon) \cup E(T_0, r\epsilon)$$

since $T(r, \theta) = S_R \cup S_L \cup T_0$. Our claim is that $m_R(H) \in \mathcal{H}_1(r, \theta, r\epsilon)$ and $m_L(H) \in \mathcal{H}_1(r, \theta, r\epsilon)$. Because of the symmetry of closed ϵ-neighbourhoods, it suffices to prove the claim for the right side. Let $\gamma = \partial H$. Then $\gamma \in \partial E$, and so $m_R(\gamma) \subset \partial E_R(r, \theta, r\epsilon)$. Suppose $m_R(\gamma) \nsubseteq \partial E(r, \theta, r\epsilon)$. This would mean that either $E_L(r, \theta, r\epsilon)$ or $E(T_0, r\epsilon)$ would have to overlap with $E_R(r, \theta, r\epsilon)$ in the region of $\mathbb{R}^2$ where $m_R(H)$ is. But this would contradict that $r \leq r_{sc}$ and $H \in \mathcal{H}_0(r, \theta, \epsilon)$. Thus $m_R(\gamma) \subset \partial E(r, \theta, r\epsilon)$, and also $m_L(\gamma) \subset \partial E(r, \theta, r\epsilon)$ (since it is the mirror image). $\square$

Theorem 4.4.2.9 (Descendant Theorem) *Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$ be such that there exists a level 0 hole $H \in \mathcal{H}(r, \theta, \epsilon)$. For any positive integer j, there exist 2^j corresponding distinct level j holes in $\mathcal{H}(r, \theta, r^j\epsilon)$, namely the holes of the form $m_{\mathbf{A}}(H)$, where $\mathbf{A} \in \mathcal{A}_j$.*

Proof. We prove the theorem by induction. We can apply the previous lemma to prove the claim for $j = 1$.

Now assume the claim is true for some $j \geq 1$, we need to show it is true for $j+1$. Let $\epsilon \geq 0$ such that there exists a level 0 hole $H \in \mathcal{H}(r, \theta, \epsilon)$. By assumption, there are 2^j distinct level j holes. Let H' be a hole of the form $m_{\mathbf{A}}(H)$, where $\mathbf{A} \in \mathcal{A}_j$. From H', we can form two new distinct holes in $\mathcal{H}(r, \theta, r^{j+1}\epsilon)$ of level $j+1$ by taking the image of H' under m_R and m_L. We can do this for each of the 2^j holes. Since these holes were all distinct, so will the new 2^{j+1} holes. They will all be in $\mathcal{H}(r, \theta, r^{j+1}\epsilon)$ and will be of the form $m_{\mathbf{A}}(H)$, where $\mathbf{A} \in \mathcal{A}_{j+1}$.

Therefore, we have proved the claim by induction. $\square$

Corollary 4.4.2.10 *Let θ be given, and let $r \leq r_{sc}$. Let $\epsilon \geq 0$ be such that there exists a level k hole H, where $k \geq 0$. Then for any positive integer j, there exist 2^j holes of level $k+j$ in $\mathcal{H}(r, \theta, r^j\epsilon)$, where each such hole is $m_{\mathbf{A}}(H)$ for some $\mathbf{A} \in \mathcal{A}_j$.*

Notation. For a hole $H \in \mathcal{H}_0(r, \theta)$ and a level k address map $m_{\mathbf{A}}$, $H_{\mathbf{A}}$ denotes the hole $m_{\mathbf{A}}(H)$ in $\mathcal{H}_k(r, \theta)$.

Proof. Let θ be given and let $r \leq r_{sc}$. If $\epsilon = 0$, then $r = r_{sc}$, and the theorem is a result of the Address Map Lemma. If $r < r_{sc}$, then the previous theorem applies to $k = 0$. If $k > 0$, then the hole H is equal to $m_{\mathbf{A}'}(H')$ for some $H' \in \mathcal{H}_0(r, \theta)$ and some $\mathbf{A}' \in \mathcal{A}_k$. Then we can apply the theorem to H', and the 2^j holes will be of the form $m_{\mathbf{A}'\mathbf{A}}(H')$, where $\mathbf{A} \in \mathcal{A}_j$. $\square$

Due to the nice scaling properties of holes, we have some immediate results about hole classes. To determine the persistence interval of any hole, it suffices to determine the persistence interval of its corresponding level 0 hole.

Proposition 4.4.2.11 *Let $T(r, \theta)$ be a tree. Let H be a level 0 hole, and let $H_{\mathbf{A}}$ be the corresponding level k hole for some level k address map $m_{\mathbf{A}}$. Then $\epsilon \in p([H])$ if and only if $r^k\epsilon \in p([H_{\mathbf{A}}])$.*

Proof. Let $T(r, \theta)$ be any tree. Let $H \in \mathcal{H}_0(r, \theta, \epsilon')$ for some $\epsilon_0 \geq 0$. Let $m_{\mathbf{A}}$ be a level k address map. Then $H_{\mathbf{A}} \in \mathcal{H}_k(r, \theta, r^k\epsilon_0)$. Hole classes are independent of the element chosen. Thus $H' \in [H]$ if and only if $m_{\mathbf{A}}(H') \in [H_{\mathbf{A}}] = [m_{\mathbf{A}}(H)]$ (because of the way address maps work on level 0 holes). Therefore $\epsilon \in p([H])$ if and only if $r^k\epsilon \in p([H_{\mathbf{A}}])$.

Corollary 4.4.2.12 *Let $T(r, \theta)$ be any tree, and let $H \in \mathcal{H}_0(r, \theta, \epsilon)$ for some $\epsilon \geq 0$. Let $\mathbf{A} \in \mathcal{A}_k$ for some $k \geq 1$. Let $H_{\mathbf{A}} = m_{\mathbf{A}}(H) \in \mathcal{H}_k(r, \theta, r^k\epsilon)$. Then*

$$\underline{\epsilon_{H_{\mathbf{A}}}} = r^k \underline{\epsilon_H} \quad \textit{and} \quad \overline{\epsilon_{H_{\mathbf{A}}}} = r^k \overline{\epsilon_H} \tag{4.4.4}$$

The Use of Levels and Symmetry: To study the holes in $\mathcal{H}(r, \theta, \epsilon)$ as ϵ ranges through $[0, \infty)$, it suffices to determine the critical values for the level 0 holes as a result of Theorems 4.4.2.4 and 4.4.2.9. Any level k hole is the image of a level

0 hole under a suitable level k address map. Conversely, any level 0 hole has 2^k corresponding level k holes via the address maps. So to get the complete picture for the persistent homology, we need to know how the critical values and the scaling ratios interact. Another important factor regarding holes is symmetry. The closed ϵ-neighbourhoods all possess left-right symmetry, so it suffices to consider holes that are not disjoint from the right side of the y-axis. This will suffice to give information about holes on both sides.

4.5 Complexity of a Hole Class

Now we introduce another aspect of a hole class, called the complexity. For level 0 hole class that has contact value equal to 0, there are infinitely many levels of corresponding hole classes that have the same contact value. So for $\epsilon = 0$, the hole class has a representative at every possible level. Now we consider hole classes that have non-zero contact value.

Definition 4.5.0.13 *Let $T(r, \theta)$ be a non-simple tree, let $H \in \mathcal{H}_0(r, \theta, \epsilon')$ for some $\epsilon' \geq 0$. The* **complexity of the hole class** *$[H]$, denoted by $C([H])$ or just $C(H)$, is the maximum number of levels of holes that correspond to the hole class $[H]$ and its descendants that are possible for any given $\epsilon \geq 0$.*

Conventions. If H is a level 0 hole such that $\underline{\epsilon_H} = 0$, then there is no maximum number of levels of holes that correspond to the hole class $[H]$ for $\epsilon = 0$, so we consider the complexity of such a hole class to be infinite. If H is a level 0 hole such that $|p([H])| = 0$ and $\underline{\epsilon_H} > 0$, then the complexity of the hole class is equal to 1.
The complexity of a hole class is related to its persistence interval and the scaling ratio in the following way.

Proposition 4.5.0.14 *Let $T(r, \theta)$ be a non-simple tree, and let H be a level 0 hole such that $\underline{\epsilon_H} > 0$. Then the hole class has finite complexity, and $C(H)$ is equal to j, where j is the smallest integer such that*

$$j \geq \frac{\ln \underline{\epsilon_H} - \ln \overline{\epsilon_H}}{\ln r} \tag{4.5.1}$$

Proof. Let $T(r, \theta)$ be a non-simple tree, and H be a level 0 hole such that $\underline{\epsilon_H} > 0$, and let $p([H])$ be its persistence interval. Then for any $k \geq 0$, the corresponding level k descendant hole classes obtained via the level k address maps have persistence interval $r^k p([H])$. The complexity of H is equal to the number of values of $k \geq 0$ such that the persistence intervals $r^k p([H])$ have non-empty intersection with $[\underline{\epsilon_H}, \overline{\epsilon_H})$. When $k = 0$, we just have the original persistence interval, so the complexity is at least 1. Let j be the smallest integer such that

$$j \geq \frac{\ln \underline{\epsilon_H} - \ln \overline{\epsilon_H}}{\ln r}.$$

Then

$$\begin{aligned}
& j \geq \frac{\ln\left(\underline{\epsilon_H}/\overline{\epsilon_H}\right)}{\ln r} \\
\Rightarrow \quad & j \ln r \leq \ln\left(\underline{\epsilon_H}/\overline{\epsilon_H}\right) \\
\Rightarrow \quad & \ln(r^j) \leq \ln\left(\underline{\epsilon_H}/\overline{\epsilon_H}\right) \\
\Rightarrow \quad & r^j \leq \underline{\epsilon_H}/\overline{\epsilon_H} \\
\Rightarrow \quad & r^j \overline{\epsilon_H} \leq \underline{\epsilon_H}
\end{aligned}$$

Thus $j-1$ is the highest integer k such that $[\underline{\epsilon_H}, \overline{\epsilon_H})$ has non-empty intersection with $[r^k \underline{\epsilon_H}, r^k \overline{\epsilon_H})$. Hence for a given $\epsilon \in [\underline{\epsilon_H}, \overline{\epsilon_H})$, there can be holes in at most j levels, namely level 0 through level $j-1$. Because of the scaling relationship between any level hole class that corresponds to $[H]$ and the original level 0 hole class, this suffices to show that there can be holes in at most j levels for any value of ϵ. Therefore, the complexity of $[H]$ is equal to j as given in 4.5.1. □

In the next chapter we will define the complexity of a tree. The notion is similar to complexity of a hole class, but it will just be the maximum number of levels of any holes possible for *any* $\epsilon \geq 0$. This will enable us to define a relation that is one way to compare and distinguish between self-avoiding trees. To develop this theory we need to introduce the concept of a hole location.

4.6 Hole Location

Now we know that we can restrict our attention to level 0 holes that are not disjoint from the right side of $\mathbf{y}$. Where are these holes? To try to answer this question, we start by looking at level 0 holes of self-contacting trees. Results about these holes help to motivate theory about holes in closed ϵ-neighbourhoods of any tree.

4.6.1 Self-Contacting Hole Classes

For any tree we have $E(r, \theta, 0) = T(r, \theta)$. $\mathcal{H}(r, \theta, 0)$ is non-empty if and only if $r = r_{sc}$ and θ is not equal to $90°$ or $135°$. From Theorem 4.4.2.5, a hole in a self-contacting tree is level 0 if its boundary intersects $\mathbf{y}_{[0,y_{\max}]}$. Now we discuss a systematic method to locate a level 0 hole class of a self-contacting tree.

Definition 4.6.1.1 *A* **self-contacting hole class** *is any hole class* $[H]$ *such that* $\underline{\epsilon_{[H]}} = 0$.

Let H be a level 0 hole of the tree $T(r_{sc}, \theta)$ that is not disjoint from the right side of $\mathbf{y}$, and let γ be the curve in $\Gamma(r_{sc}, \theta)$ that is the boundary of the hole. The curve γ is either symmetric about $\mathbf{y}$ if it is above the trunk, or it contains a portion of the trunk (and is disjoint from the left side of $\mathbf{y}$). In the first case, there are 2 distinct points of γ on $\mathbf{y}$, and in the second case, there are two distinct points that are the upper and lower bound of the intersection of γ and the trunk. These two points correspond to two different addresses of the subtree S_R and they can be used to identify the hole class. What are the addresses? Any address of a point with $x = 0$ is such an address. These addresses were summarized in Table 3.3 at the end of the previous chapter.

We now define 'hole locators' of a self-contacting tree. Instead of using points to identify and locate holes, we use their addresses, because the addresses will allow us to compare hole locators for different trees. Because of the left-right symmetry of trees, closed ϵ-neighbourhoods and holes, and the the fact that all hole classes descend from level 0 hole classes, our discussion of hole location is restricted to level 0 holes not disjoint from the right side of $\mathbf{y}$.

Definition 4.6.1.2 *A pair of addresses* $(\mathbf{A}_1, \mathbf{A}_2)$ *is said to be a* **level 0 hole locator pair for a tree** $T(r_{sc}, \theta)$, *if the two corresponding points* $P_{\mathbf{A}_1}$ *and* $P_{\mathbf{A}_2}$ *of the tree are on the y-axis and they form the upper and lower vertical bounds of the intersection of the boundary of a level 0 hole and* $\mathbf{y}$. *That is, they are the highest and lowest points of a level 0 curve* $\gamma \in \Gamma(r_{sc}, \theta)$ *intersected with* $\mathbf{y}$.

Definition 4.6.1.3 *If* $(\mathbf{A}_1, \mathbf{A}_2)$ *is a hole locator pair, then* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are referred to as* **hole locator addresses** *and the two points* $P_{\mathbf{A}_1}$ *and* $P_{\mathbf{A}_2}$ *are* **hole locator points**.

We say that $\mathbf{A}_1$ and $\mathbf{A}_2$, or $P_{\mathbf{A}_1}$ and $P_{\mathbf{A}_2}$, 'locate' a hole.

Summary of Hole Locator Pairs of Self-Contacting Trees

1. **Special Angles** $\theta = \theta_N$, $N \geq 2$. The first case we consider are the special angles. These trees are interesting because they possess infinitely many level 0 holes, due to the existence of infinitely many tip points on the y-axis. The holes can also be thought of as a result of gaps in the top of the subtree $S_{RL^{N+1}}$. These gaps are precisely the top canopy intervals as discussed in Section 3.5 of Chapter 3.

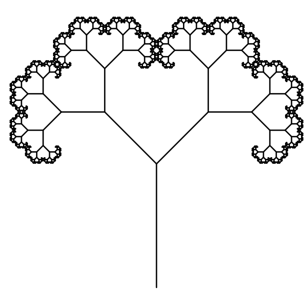

Figure 4.11: $T(r_{sc}, 45°)$.

For example, consider the self-contacting tree with special angle $\theta_2 = 45°$, shown in Figure 4.11. The obvious hole is between the top of the trunk and the lowest tip point of the subtree S_R that is on $\mathbf{y}$. The other holes are related to the canopy intervals of the subtree S_{RL^3}. The holes are all above the trunk, and the hole locator addresses are the following pairs:

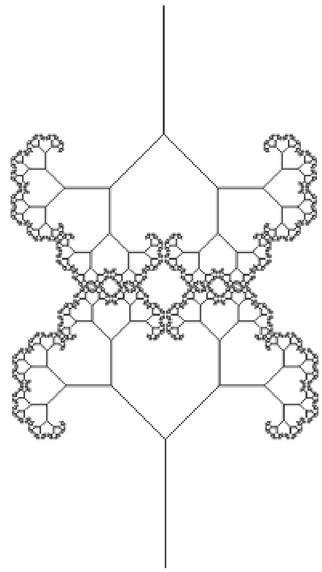

Figure 4.12: Double tree of $T(r_{sc}, 45°)$.

- $(\mathbf{A}_0, RL^3(LR)^\infty)$, which correspond to the top of the trunk and the lowest tip point of S_R with $x = 0$
- $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$, i.e., the addresses of the endpoints of the degree 0 canopy interval of the subtree S_{RL^3}
- Any pair $(RL^3\mathbf{AC}_L, RL^3\mathbf{AC}_R)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 1$ (see 2.3.5), i.e., the addresses of the endpoints of canopy intervals of degree 1 or higher of the subtree S_{RL^3}

To get a better idea of what the holes resulting from the canopy intervals look like, consider the double tree in Figure 4.12. The subtrees S_{RL^3} and S_{LR^3} meet together along the y-axis in the same way that the two trees in the double tree meet. We will use this double tree in Chapter 6 to determine the collapse values for holes located by canopy pairs.

For general special angles, we present the following proposition:

Proposition 4.6.1.4 *For any self-contacting tree with special angle θ_N, the pairs of level 0 hole locators are:*

- $(\mathbf{A}_0, RL^{N+1}(LR)^\infty)$
- *Any pair* $(RL^{N+1}\mathbf{AC}_L, RL^{N+1}\mathbf{AC}_R)$*, where* $\mathbf{A}$ *is the empty address or* $\mathbf{A} \in \mathcal{AL}_{2k}$ *for some* $k \geq 1$ *(see 2.3.6)*

Proof. It is clear that each pair in the proposition corresponds to a pair of points that are on the y-axis. We claim that for each pair, the region of the y-axis between the two points is disjoint from the tree, and hence they are hole locators.

For the first pair, this claim is valid because the point with address $RL^{N+1}(LR)^\infty$ is the lowest tip point of S_R with $x = 0$.

Any other pair consists of a pair of endpoints of a canopy interval of the subtree $S_{RL^{N+1}}$. In Chapter 3, Section 3.5, we discussed various properties of the canopy intervals, and one of the properties discussed is that the interval is disjoint from the tree if $y_{\max} > 1$, which is always the case if $\theta < 90°$. Since the subtree is similar to the tree, the same can be said about canopy intervals of the subtree. Any other tip point of the subtree S_R that has $x = 0$ but is not the endpoint of a canopy interval of the subtree $S_{RL^{N+1}}$ is such that for any open vertical neighbourhood (i.e. region on $\mathbf{y}$) of the point, the neighbourhood contains other top tip points of the subtree $S_{RL^{N+1}}$, both above and below. This property was also discussed in Chapter 3. So there are no other hole locator pairs.

2. **Non-special Angles in the First Angle Range ($0° < \theta < 90°$ and θ not special).** As discussed in Section 3.3 of Chapter 3, for any self-contacting tree in this category, there is a unique tip point of S_R that is on $\mathbf{y}$. This is the point P_{c1} with address $RL^{N+1}(RL)^\infty$. Thus the only hole locator pair is $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$.

3. $\theta = 90°$ **or** $135°$ There are no holes in either self-contacting tree (because they are both space-filling), so there are no hole locators.

4. **Second Angle Range ($90° < \theta < 135°$).** For any self-contacting tree in this category, there is a unique tip point of S_R that is on $\mathbf{y}$, as discussed in Section

3.3 of Chapter 3. This is the point P_{c2} with address $R^3(LR)^\infty$. Thus the only hole locator pair is $(\mathbf{A}_0, R^3(LR)^\infty)$.

5. **Third Angle Range ($135° < \theta < 180°$).** Any self-contacting tree in this angle range has infinitely many points of S_R on the trunk, namely $(0,1)$, any point with address $RR(LR)^k$ for $k \geq 0$, and the point with address $RR(LR)^\infty$. As k increases, the y-coordinate of the point decreases. Thus any pair of addresses $(RR(LR)^k, RR(LR)^{k+1})$, for $k \geq 0$, is a hole locator pair, in addition to the pair $(\mathbf{A}_0, RR)$. The only point of S_R that has $x = 0$ that is not part of a pair of hole locators is the point with address $RR(LR)^\infty$. This is because any open vertical neighbourhood of this point contains infinitely many other points of S_R, since we can find an integer K such that all points with addresses $RR(LR)^k$, for $k \geq K$, are in the neighbourhood.

The previous remarks are sufficient to locate every level 0 hole of a self-contacting tree, and thus any hole of a self-contacting tree. To locate the level 0 hole corresponding to a pair of hole locator addresses, find the two corresponding hole locator points that are the global minima, and there will be a unique level 0 hole, not disjoint from the right side of $\mathbf{y}$, and whose boundary contains the two points. Since every level k hole is the image of a level 0 hole under a level k address map, the level k hole locators are just the images of hole locators under the level k address maps.

The locations of self-contacting hole classes are thus completely determined. Now we need to consider hole classes whose contact values are non-zero.

4.6.2 General Hole Classes

We would like to generalize the ideas of the previous subsection to develop a systematic way to identify and locate holes in the closed ϵ-neighbourhood of any tree. Consider the image of a self-avoiding tree with branching angle 105° shown in Figure 4.13.

It seems reasonable that there are holes in certain closed ϵ-neighbourhoods of this tree. For example, consider the point $P_c = (x_c, y_c)$ with the contact address $\mathbf{A}_c = R^3(LR)^\infty$. When $\epsilon = x_c/2$, the point $P = (\epsilon, y_c)$ will be ϵ away from the

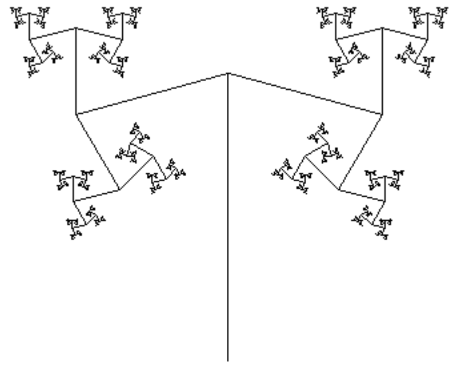

Figure 4.13: $T(0.55, 105°)$.

trunk and the subtree S_{RR}, and it will be on the boundary of a level 0 hole. Recall that $R^3(LR)^\infty$ and $\mathbf{A}_0$ are hole locators for the self-contacting tree $T(r_{sc}, 105°)$, so it seems as if this pair may also be a hole locator for the tree $T(0.55, 105°)$. For the tree $T(.55, 105°)$, the point P_c with address $R^3(LR)^\infty$ is the point on S_{RR} closest to the trunk. So it is a 'local minimum' in some sense. This motivates us to give a definition of a local minimum.

Definition 4.6.2.1 *A point $P_{\mathbf{A}} = (x, y)$, for some address $\mathbf{A}$, on a tree is said to be a* **local minimum on the right** *if it is on S_R and there exists an open neighbourhood in $\mathbb{R}^2$ of the form*

$$(0, a) \times (b, c)$$

that contains P, and any other point $P' = (x', y')$ of the tree that is also in this neighbourhood is such that $x \leq x'$, i.e., the point P' is not at a smaller distance to $\mathbf{y}$ *than P is.*

Using this definition, the point P_{c2} of $T(0.55, 105°)$ is indeed a local minimum.

We will also see points that are one-sided local minima, so we give those definitions now. Note that a point can be a one-sided local minimum without being a point with an address, if it is in some branch interior. For local minima we only consider points that have a corresponding address.

Definition 4.6.2.2 *A point $P' = (x', y')$ on a tree is said to be a* **one-sided local minimum on the right from above** *if it is on S_R and there exists a set in $\mathbb{R}^2$ of* *the form*

$$(0, a) \times [y', c)$$

that contains P', and any other point $P = (x, y)$ of the tree that is also in this set is such that $x' \leq x$.

A point $P' = (x', y')$ on a tree is said to be a **one-sided local minimum on the right from below** *if it is on S_R and there exists a set in $\mathbb{R}^2$ of the form*

$$(0, a) \times (b, y']$$

that contains P', and any other point $P = (x, y)$ of the tree that is also in this set is such that $x' \leq x$.

How can we relate general hole classes and local minima? For any hole class that is not a self-contacting hole class, we shall prove that the class identifies a local minimum. To show this, we first define two points of any level 0 maximal hole of the class, called P_{top} and P_{bot}. Theorem 4.4.2.5 and its corollary imply that H is either above the line $y = 1$ and symmetric about $\mathbf{y}$, or H is below the line $y = 1$ and is disjoint from the left side of $\mathbf{y}$. In the first case, the boundary of H intersects $\mathbf{y}$ in two points. In the second case, the boundary of H intersects the boundary of the closed ϵ-neighbourhood of the trunk in a vertical line segment.

Definition 4.6.2.3 *Let H be a maximal hole not disjoint from the right side $\mathbf{y}$, let ∂H be its corresponding boundary and $\epsilon = \underline{\epsilon_H}$. The* **top point of** *H and* **bottom point of** *H, denoted by P_{top} and P_{bot} respectively, are defined as follows. If H is above the line $y = 1$, then P_{top} and P_{bot} are the two points that are the elements of the set $\partial H \cap \mathbf{y}$, with P_{top} being the point with the greater y-value. If H is below the line $y = 1$, then P_{top} and P_{bot} are the upper and lower endpoints, respectively, of the line segment formed by $\partial H \cap E(T_0)$ (where $E(T_0)$ denotes the closed ϵ-neighbourhood of the trunk).*

Now there must be two corresponding points on the subtree S_R that are at a distance of ϵ to each of P_{top} and P_{bot}.

Definition 4.6.2.4 *Let H be a level 0 maximal hole, and let $P_{top} = (x_{top}, y_{top})$ and $P_{bot} = (x_{bot}, y_{bot})$ be the top and bottom points of H. Then $P_{TOP} = (x_{TOP}, y_{TOP})$*

denotes the point on the subtree S_R that is at a distance of ϵ from P_{top}. If there is more than one such point, then we take P_{TOP} to be the one with the lowest y-value. $P_{BOT} = (x_{BOT}, y_{BOT})$ denotes the point on the subtree S_R that is at a distance of ϵ from P_{bot}, and if there is more than one such point, we take P_{BOT} to be the one with the greatest y-value.

Note that the points P_{TOP} and P_{BOT} do not both have to be addressed points of the tree. One can be in some branch interior. However, we will show that at least one of them is a local minimum (and hence has an address) and the other must at least be a one-sided local minimum. This will help us to define the hole locators of general hole classes. From the definitions of these points, we have $x_{top} = x_{bot}$, $x_{top} \leq x_{TOP}$, and $x_{bot} \leq x_{BOT}$. We don't necessarily have $x_{TOP} = x_{BOT}$.

Lemma 4.6.2.5 *Let H be a level 0 maximal hole not disjoint from the right side* $\mathbf{y}$ *such that $\epsilon = \underline{\epsilon_H} > 0$. Let $P_{top} = (x_{top}, y_{top})$, $P_{TOP} = (x_{TOP}, y_{TOP})$, $P_{bot} = (x_{bot}, y_{bot})$ and $P_{BOT} = (x_{BOT}, y_{BOT})$ be the points defined in the previous two definitions. Then*

1. *$y_{top} \leq y_{TOP}$ and $y_{bot} \geq y_{BOT}$,*

2. *either $y_{top} = y_{TOP}$ or $y_{bot} = y_{BOT}$,*

3. *P_{TOP} is a one-sided local minimum from below and P_{BOT} is a one-sided local minimum from above.*

Proof. Let H be a level 0 maximal hole not disjoint from the right side $\mathbf{y}$ such that $\epsilon = \underline{\epsilon_H} > 0$.

1. Suppose $y_{top} < y_{TOP}$. Then consider the points in $\mathbb{R}^2$ that are of the form $P = (x, y)$ where $x_{top} \leq x \leq x_{TOP}$ and $y_{TOP} \leq y \leq y_{top}$. Any such point is within ϵ to the point P_{TOP}, by the triangle inequality. This means that they are not in the hole H. This contradicts how P_{top} was defined. Hence $y_{top} \leq y_{TOP}$. Similarly we have that $y_{bot} \geq y_{BOT}$.

2. Suppose $y_{top} < y_{TOP}$ and $y_{bot} > y_{BOT}$. Let

$$a = \min\{y_{TOP} - y_{top}, y_{bot} - y_{BOT}\}$$

Consider the points $P_1 = (x_{top}, y_{top} + a)$ and $P_2 = (x_{bot}, y_{bot} - a)$. All points on the line $x = x_{top} = x_{bot}$ between P_1 and P_{top} or between P_{bot} and P_2 have a distance to S_R that is strictly less than ϵ. Then we could find $\delta > 0$ such that for any $\epsilon' \in (\epsilon - \delta, \epsilon)$, there is exactly one hole in the closed ϵ-neighbourhood at ϵ' that has non-empty intersection with H. This contradicts that H is the maximal hole of its class. The contradiction came from assuming that $y_{top} < y_{TOP}$ and $y_{bot} > y_{BOT}$. Thus $y_{top} = y_{TOP}$ or $y_{bot} = y_{BOT}$.

3. P_{top} is the top point of a hole, so there exists $\delta > 0$ such that all points $P = (x_{top}, y)$ where $y \in (y_{top} - \delta, y_{top})$ are at a distance of at least ϵ to the tree. Hence P_{TOP} must be a one-sided local minimum from below, otherwise there would be a point P that has a distance to the tree that is strictly less than ϵ, and that would contradict the previous statement. Similarly, P_{BOT} is a one-sided local minimum from above.

$\square$

Proposition 4.6.2.6 *Let H be a level 0 maximal hole not disjoint from the right side* $\mathbf{y}$ *such that $\epsilon = \underline{\epsilon_H} > 0$. Let $P_{top} = (x_{top}, y_{top})$, $P_{TOP} = (x_{TOP}, y_{TOP})$, $P_{bot} = (x_{bot}, y_{bot})$ and $P_{BOT} = (x_{BOT}, y_{BOT})$. Then at least one of P_{TOP} or P_{BOT} must be a local minimum.*

Proof. By the previous lemma, we know that $y_{top} \leq y_{TOP}$ and $y_{bot} \geq y_{BOT}$ and $y_{top} = y_{TOP}$ or $y_{bot} = y_{BOT}$. Suppose $y_{top} = y_{TOP}$. Then we claim that P_{TOP} is a local minimum. We have already proven that P_{TOP} is a one-sided local minimum from below. We need to show that P_{TOP} is an addressed point and that it is a local minimum. If P_{TOP} were not an addressed point, then it would be in the interior of a branch. If the branch is positively sloped, then this contradicts that P_{TOP} is the closest point to P_{top}. If the branch is vertical or negatively sloped, then this contradicts that P_{top} is the top point on the boundary of a maximal hole. Hence P_{TOP} must be $P_{\mathbf{A}}$ for some address $\mathbf{A}$. If $y_{top} < y_{TOP}$, then $y_{bot} = y_{BOT}$, and by a similar argument the point P_{BOT} must be $P_{\mathbf{A}}$ for some address $\mathbf{A}$.

Suppose $y_{top} = y_{TOP}$ and $y_{bot} > y_{BOT}$. Let $\mathbf{A}$ be the address such that $P_{TOP} = P_{\mathbf{A}}$.

To show that $P_{\mathbf{A}}$ is a local minimum, it suffices to show that it is a one-sided local minimum from above. Suppose it is not. Then there is a portion of the line $x = x_{top}$ right above the point P_{top} that is at a distance to the tree that is strictly less than ϵ. Since $y_{bot} > y_{BOT}$, there is a portion of the line $x = x_{bot}$ right below the point P_{bot} that is at a distance to the tree that is strictly less than ϵ. This contradicts that H is the maximal hole of a hole class. Thus $P_{\mathbf{A}} = P_{TOP}$ must be a local minimum. Similarly, if $y_{top} < y_{TOP}$ and $y_{bot} = y_{BOT}$, then the point P_{BOT} is a local minimum. If both $y_{top} = y_{TOP}$ and $y_{bot} = y_{BOT}$, then both points P_{TOP} and P_{BOT} are addressed and must be local minima, by the same argument that H is the maximal hole of a hole class and that P_{top} and P_{bot} are the top and bottom points of the hole. $\square$

Conditions for local minima to correspond to a level 0 hole class

1. **Holes Above The Trunk** A level 0 hole above the trunk is symmetric about the y-axis. For any point $P = (x, y)$ in the hole H, the distance between P and the tree is less than or equal to the distance between the point $(0, y)$ and the subtree S_R. If $\mathbf{A}$ is such that $P_{\mathbf{A}} = (x_{\mathbf{A}}, y_{\mathbf{A}})$ is a local minimum, then $P_{\mathbf{A}} = P_{TOP}$ for some level 0 hole class above the trunk if and only if:

   (a) $P_{\mathbf{A}}$ is the closest point on the subtree S_R to the point $(0, y_{\mathbf{A}})$,

   (b) there is a point $P_b = (0, y_b)$ with $1 \leq y_b < y_{\mathbf{A}}$ such that P_b is at a distance of $x_{\mathbf{A}}$ to the subtree S_R and all points $P = (0, y)$ with $y_b < y < y_{\mathbf{A}}$ are at a distance of strictly more than $x_{\mathbf{A}}$.

   Similarly, $P_{\mathbf{A}} = P_{BOT}$ for some level 0 hole class above the trunk if and only if:

   (a) $P_{\mathbf{A}}$ is the closest point on the subtree S_R to the point $(0, y_{\mathbf{A}})$,

   (b) there is a point $P_t = (0, y_t)$ with $y_{\mathbf{A}} < y_t$ such that P_t is at a distance of $x_{\mathbf{A}}$ to the subtree S_R and all points $P = (0, y)$ with $y_{\mathbf{A}} < y < y_t$ are at a distance of strictly more than $x_{\mathbf{A}}$.

2. **Holes Involving The Trunk** Suppose H is a level 0 hole involving the trunk on the right side of $\mathbf{y}$. Then for any point $P = (x, y)$ in the hole, the distance between the point and the subtree S_R is less than the distance between $P' =$

(x', y) and the subtree, where $0 \leq x' < x$. If $\mathbf{A}$ is such that $P_{\mathbf{A}} = (x_{\mathbf{A}}, y_{\mathbf{A}})$ is a local minimum, then $P_{\mathbf{A}} = P_{TOP}$ for some level 0 hole class involving the trunk if and only if:

(a) $P_{\mathbf{A}}$ is the closest point on the subtree S_R to the point $(x_{\mathbf{A}}/2, y_{\mathbf{A}})$,

(b) there is a point $P_b = (x_{\mathbf{A}}/2, y_b)$ with $y_b < y_{\mathbf{A}}$ such that P_b is at a distance of $x_{\mathbf{A}}/2$ to the subtree S_R and all points $P = (x_{\mathbf{A}}/2, y)$ with $y_b < y < y_{\mathbf{A}}$ are at a distance of strictly more than $x_{\mathbf{A}}/2$.

Similarly, $P_{\mathbf{A}} = P_{BOT}$ for some level 0 hole class involving the trunk if and only if:

(a) $P_{\mathbf{A}}$ is the closest point on the subtree S_R to the point $(x_{\mathbf{A}}/2, y_{\mathbf{A}})$,

(b) there is a point $P_t = (x_{\mathbf{A}}/2, y_t)$ with $y_{\mathbf{A}} < y_t \leq 1$ such that P_t is at a distance of $x_{\mathbf{A}}/2$ to the subtree S_R and all points $P = (x_{\mathbf{A}}/2, y)$ with $y_{\mathbf{A}} < y < t$ are at a distance of strictly more than $x_{\mathbf{A}}/2$.

In the discussion above, if P_b or P_t exists, then they are P_{BOT} or P_{TOP} respectively.

If a local minimum satisfies one of the above four conditions, then the point locates a hole class. It could be that it locates more than one hole class, so we need to associate another point such that the pair of points identifies a unique level 0 hole class not disjoint from the right side of $\mathbf{y}$. We already have the pair P_{TOP} and P_{BOT}. However, it is possible that one is not an addressed point. The main reason for defining hole locators is to have a way to compare the locations of holes for different trees, so to do this we need the addresses of the points, not the actual points. To solve this problem, if one if the points does not have an address, the address we associate with it is the empty address $\mathbf{A}_0$. The following lemma gives an explanation.

Lemma 4.6.2.7 *Let H be a level 0 maximal hole not disjoint from the right side of $\mathbf{y}$. If P_{BOT} does not have an address, then $y_{BOT} > 1$, so the hole is above the trunk, and as ϵ ranges through the persistence interval of the hole, there is only one hole whose top and bottom points are between P_{top} and $(0, 1)$. On the other hand, if P_{TOP}*

does not have an address, then $y_{TOP} < 1$, so the hole involves the trunk, and as ϵ ranges through the persistence interval of the hole, there is only one hole whose top and bottom points are between P_{bot} and $(0,1)$.

Proof. Suppose P_{BOT} does not have an address, so it is in the interior of some branch **b**. The branch must necessarily be positively sloped since P_{BOT} is a one-sided local minimum from above. First we claim that the hole must be above the trunk. Suppose the hole involved the trunk, so below the line $y = 1$. Thus the angle must be greater than $45°$ and there is a portion of the subtree S_R that is below the line $y = 1$ (since P_{TOP} is a local minimum below the line $y = 1$). If the starting point of **b** were closer to the y-axis than the endpoint, then there would have to be a portion of the subtree S_R that is between P_{BOT} and P_{bot}. All subtrees are similar to the tree, so either the subtree with the branch **b**, or the subtree whose trunk is the branch whose endpoint is the starting point of **b**, would contain a point between P_{BOT} and P_{bot}, and that is a contradiction. Likewise if the endpoint of **b** were closer to the y-axis. Thus the hole is above the trunk.

The only one-sided local minima from above that are above the trunk on a positively sloped branch are on the branch $b(R)$. Any other positively sloped branch of the subtree S_R is too far away from the y-axis to contain any one-sided local minima from above. Thus as ϵ ranges through the persistence interval of the hole, there is only one hole whose top and bottom points are between P_{top} and $(0,1)$, because there are no local minima between P_{BOT} and $(0,1)$ since P_{BOT} is on $b(R)$.

Now suppose P_{TOP} is not addressed, then it is in the interior of some branch **b**. This branch must be negatively sloped since P_{TOP} is a one-sided local minimum from below. The hole must be below the line $y = 1$. If not, then the tree has $y_{\max} > 1$. For angles greater than $90°$ and less than $135°$, every negatively sloped branch that has a portion of the branch above the line $y = 1$ is such that there is some other part of the subtree S_R between the branch and the y-axis. For $\theta = 90°$, there are no negatively sloped branches. For angles under $90°$, the only negatively sloped branches that could possibly contain one-sided local minima from below on the subtree S_R are the branches $b(RLL^j)$ for $1 \leq j < N$ (where N is the turning number), and there are no local minima below these branches so there could be no P_{BOT}. Thus the hole involves the trunk.

The only one-sided local minima from below that involve the trunk are on the branch $b(R)$. Any other negatively sloped branch of the subtree S_R is too far away from the trunk to contain any one-sided local minima from below. Thus as ϵ ranges through the persistence interval of the hole, there is only one hole whose top and bottom points are between P_{bot} and $(0, 1)$, because there are no local minima between P_{TOP} and $(0, 1)$ since P_{TOP} is on $b(R)$. □

Definition 4.6.2.8 *Let H be a level 0 maximal hole that is not disjoint from the right side of $\mathbf{y}$. Let $P_{top} = (x_{top}, y_{top})$, $P_{TOP} = (x_{TOP}, y_{TOP})$, $P_{bot} = (x_{bot}, y_{bot})$ and $P_{BOT} = (x_{BOT}, y_{BOT})$. Then the* **hole locator addresses***, denoted $\mathbf{A}_{TOP}$ and $\mathbf{A}_{BOT}$, are the addresses defined as follows. If both P_{TOP} and P_{BOT} are addressed, then $\mathbf{A}_{TOP}$ and $\mathbf{A}_{BOT}$ are such such that $P_{TOP} = P_{\mathbf{A}_{TOP}}$ and $P_{BOT} = P_{\mathbf{A}_{BOT}}$. If one point does not have an address, then take the address to be $\mathbf{A}_0$. The hole locator pair is $(\mathbf{A}_{TOP}, \mathbf{A}_{BOT})$. The pair is not ordered, but often the first address corresponds to the point that is closer to the point $(0, 1)$.*

Note. As a result of the previous definition, a pair of hole locator addresses locates a unique class of level 0 holes that are not disjoint from the right side of $\mathbf{y}$. For a given tree, any hole class can be obtained from a level 0 hole class not disjoint from the right side of $\mathbf{y}$, either by reflection across $\mathbf{y}$ (if the hole class is disjoint from the right side of $\mathbf{y}$) or via a level k address map (if the hole class is level k) or a combination of reflection and address map. For this reason, it is justified to restrict our attention to hole locator pairs of level 0 hole classes not disjoint from the right side of $\mathbf{y}$. Throughout the thesis, a hole locator pair will always refer to such a hole class, so we often leave out the phrase "level 0 hole class not disjoint from the right side of $\mathbf{y}$".

4.7 Brief Chapter Summary

This chapter has introduced fundamental concepts for this thesis. We defined the closed ϵ-neighbourhoods of arbitrary subsets of $\mathbb{R}^2$, and in particular we defined the closed ϵ-neighbourhoods of a symmetric binary fractal tree and its subtrees. As ϵ

ranges through the non-negative numbers, the topology of the closed ϵ-neighbourhoods of a given tree may change if the tree is non-simple. A simple tree is such that every closed ϵ-neighbourhood is contractible. The non-simple trees and the corresponding closed ϵ-neighbourhoods are interesting objects to study. We introduced the idea of a 'hole' of a closed ϵ-neighbourhood, along with notions of persistence, hole class, complexity and level of a hole. We developed a method of locating self-contacting classes of holes, and generalized these ideas to general classes of holes. The various features of holes developed in this chapter form the foundation of deeper theory about critical values and classifications that will begin in the next chapter. As well, these features will become more tangible when we look at specific examples of trees in Chapter 6.

Chapter 5

More Theory Regarding ϵ-Neighbourhoods: Classifications and Critical Values

In the previous chapter, we introduced theory about individual hole classes of closed ϵ-neighbourhoods for a specific tree. Now we look at a more general picture of the closed ϵ-neighbourhoods, from different points of view. We develop theory to characterize and classify the trees based on the parameters (branching angle and scaling ratio) and critical values of ϵ. First we continue with theory regarding hole locations, following the theory developed in Section 4.6 of the previous chapter. For each tree, *i.e.*, for each pair r and θ, we can associate a hole location set that characterizes where the holes are in relation to the tree. This is one characteristic that we can use to compare and distinguish between trees. A type class is based on what type of points the hole locators are. This is another characteristic of a tree, and it provides a coarser classification than the location classification. Based on hole location, there are critical angles that identify a change in hole location. We also define types of holes. A different kind of characteristic of a tree is also defined. This is the hole partition and sequence, which is based on the critical ϵ-values for a tree. We can use the hole sequence to compare trees, and this comparison is very different from the location classification. Finally we discuss the complexity of a tree, a characteristic that provides yet another distinct way to classify the trees. The complexity classification is not comparable to any other classification. However, complexity is related to the hole sequence, persistence and scaling. For a given angle, we can define critical scaling ratios based on complexity.

This chapter contains mainly qualitative details, but there are some quantitative details that lead into the next chapter on specific examples. Further discussion about the theory developed in this chapter will be included in Chapter 7, following the presentation of a collection of specific examples of trees and their closed

ϵ-neighbourhoods.

5.1 Location and Type Classifications

In this section, we discuss two different classifications of parameter pairs (r, θ). The first is based on the addresses that are hole locators for the tree $T(r, \theta)$. The second is based on the types of points that the hole locator addresses correspond to. The type classification is a coarser classification than the location classification.

5.1.1 Hole Location Sets and Location Classification

For any level 0 hole class not disjoint from the right side of the y-axis, recall that we associate a pair of addresses that can be used to identify and locate the hole class (see 4.6.2.8). The set of hole locator pairs for a given tree characterizes the level 0 hole classes not disjoint from the right side of $\mathbf{y}$, and hence we have information about all hole classes. This characterization can be used to compare different trees.

Definition 5.1.1.1 *For a tree $T(r, \theta)$, the* **hole location set of the pair** (r, θ)*, denoted by $\mathcal{HL}(r, \theta)$ is the set of pairs of addresses that are locators of level 0 hole classes not disjoint from the right side of* $\mathbf{y}$.

Definition 5.1.1.2 *For a given θ, the* **hole location set of the angle** θ*, denoted by $\mathcal{HL}(\theta)$, is the union of all hole sets $\mathcal{HL}(r, \theta)$ as r ranges through $(0, r_{sc}]$.*

$$\mathcal{HL}(\theta) = \bigcup_{r \in (0, r_{sc}]} \mathcal{HL}(r, \theta) \tag{5.1.1}$$

What are the elements of the hole location set of a given pair or a given angle? We have already determined $\mathcal{HL}(r_{sc}, \theta)$ for any θ. How does $\mathcal{HL}(r_{sc}, \theta)$ relate to $\mathcal{HL}(\theta)$? In the case of the angles 90° and 135°, the self-contacting hole location sets are empty, because the self-contacting trees are space-filling. However, we shall see that it is possible for either angle to have a scaling ratio such that the corresponding tree is not simple (and so the hole location set of the angle is not empty). So to find the elements of $\mathcal{HL}(\theta)$ for a given θ, it is not as straightforward as looking at the self-contacting hole classes.

If a pair of addresses are hole locators of a self-contacting hole class of a tree $T(r_{sc}, \theta)$, then it seems reasonable that this pair could be hole locators for a tree $T(r, \theta)$ for certain values of r, especially for values of the scaling ratio that are close to r_{sc}. It also seems reasonable that the pair could be hole locators for $T(r_{sc}(\theta'), \theta')$ for some θ' that is close to θ. This motivates us to define two relations, one for pairs (r, θ), and one for angles.

Definition 5.1.1.3 *We define a relation* $\sim_{Loc}$ *on the set of pairs* $\{(r, \theta)\}$ *as follows.* $(r_1, \theta_1) \sim_{Loc} (r_2, \theta_2)$ *if* $\mathcal{HL}(r_1, \theta_1) = \mathcal{HL}(r_2, \theta_2)$.

Definition 5.1.1.4 *We define a relation* $\sim_{Loc}$ *on the set of branching angles as follows.* $\theta_1 \sim_{Loc} \theta_2$ *if* $\mathcal{HL}(\theta_1) = \mathcal{HL}(\theta_2)$.

Observation. The relation $\sim_{Loc}$ is an equivalence relation on the set of pairs or on the set of branching angles.

5.1.2 Types of Holes and Type Classification

In Chapter 3 we identified different types of points: contact, secondary contact, canopy and top vertex. Refer to Section 3.5 for complete details. We can define a new characteristic of hole classes called the 'type' based on what type of points correspond to a hole locator pair. This leads to a type classification on pairs (r, θ).

Definition 5.1.2.1 *Let* $T(r, \theta)$ *be a non-simple tree. Let* H *be a level 0 hole not disjoint from the right side of* $\mathbf{y}$. *Let* $\mathbf{A}_{TOP}$ *and* $\mathbf{A}_{BOT}$ *be the hole locator addresses of* $[H]$ *as defined in 4.6.2.8, with* P_{TOP} *and* P_{BOT} *the corresponding points on the tree. Then we define the* **type** *of the hole class* $[H]$ *(or the* **type of the pair***) based on the types of the points* P_{TOP} *and* P_{BOT} *as follows:*

1. *If one of* $\mathbf{A}_{TOP}$ *or* $\mathbf{A}_{BOT}$ *is the contact address for* θ*, and the other address is the empty address* $\mathbf{A}_0$*, then the type is the* **main type**.

2. *If one of* $\mathbf{A}_{TOP}$ *or* $\mathbf{A}_{BOT}$ *is the secondary contact address for* θ*, and the other address is the empty address* $\mathbf{A}_0$*, then the type is the* **secondary contact type**.

3. *If* $\mathbf{A}_{TOP}$ *and* $\mathbf{A}_{BOT}$ *are a pair of addresses that correspond to endpoints of a canopy interval of some subtree of* $T(r,\theta)$*, then the type is the* **canopy type**.

4. *If* $\mathbf{A}_{TOP}$ *and* $\mathbf{A}_{BOT}$ *are addresses of top vertex points of some subtree of* $T(r,\theta)$*, then the type is the* **vertex type.**

5. *If* $\mathbf{A}_{TOP}$ *and* $\mathbf{A}_{BOT}$ *do not correspond to any of the four previous types, then the type is the* **mixed type**. *In addition, if either* $\mathbf{A}_{TOP}$ *or* $\mathbf{A}_{BOT}$ *is* $\mathbf{A}_0$*, then the type is often referred to as the* **main mixed type**.

Definition 5.1.2.2 *For a pair* (r,θ) *such that* $r \leq r_{sc}$*, the* **type set of the pair** (r,θ)*, or the* **type set of the tree** $T(r,\theta)$*, denoted by* $\mathcal{TY}(r,\theta)$*, is the set of types of hole classes.*

For example, a tree that is said to have holes of only the main type is such that there is only one level 0 hole class not disjoint from the right side of $\mathbf{y}$, namely the class that corresponds to the contact address for the tree. We often refer to these holes as **main holes**. We shall see that many trees have only the main holes, and there are some nice properties that these trees share. It is not true that every non-simple tree has main holes. For example, we shall see that for non-simple, self-avoiding trees with branching angle 90°, the hole locator pairs are all canopy pairs. The type set of a tree is often the easiest characteristic of the tree to determine. However, it is still a useful characteristic, because it can be used to distinguish between trees and angles.

Definition 5.1.2.3 *For an angle* θ*, the* **type set of** θ*, denoted by* $\mathcal{TY}(\theta)$*, is the union of all type sets* $\mathcal{TY}(r,\theta)$ *as* r *ranges through* $(0, r_{sc}]$.

$$\mathcal{TY}(\theta) = \bigcup_{r\in(0,r_{sc}]} \mathcal{TY}(r,\theta) \tag{5.1.2}$$

Definition 5.1.2.4 *We define the* **type relation** *on the collection of pairs* (r,θ) *such that* $T(r,\theta)$ *are non-overlapping, denoted by* $\sim_{Type}$*, as follows. We say* $(r_1,\theta_1) \sim_{Type} (r_2,\theta_2)$ *if* $\mathcal{TY}(r_1,\theta_1) = \mathcal{TY}(r_2,\theta_2)$.

Definition 5.1.2.5 *We define a relation* $\sim_{Type}$ *on the set of branching angles as follows.* $\theta_1 \sim_{Type} \theta_2$ *if* $\mathcal{TY}(\theta_1) = \mathcal{TY}(\theta_2)$.

The type relation is clearly an equivalence relation and it is coarser than the location relation $\sim_{Loc}$.

Proposition 5.1.2.6 *Let (r_1, θ_1) and (r_2, θ_2) be pairs such that $(r_1, \theta_1) \sim_{Loc} (r_2, \theta_2)$. Then $(r_1, \theta_1) \sim_{Type} (r_2, \theta_2)$.*

Proof. This proposition is a direct consequence of the definition of type of a pair and the relations $\sim_{Loc}$ and $\sim_{Type}$. □

To show that the type relation is indeed coarser, we just need to provide an example of two pairs that have the same type set but different location set. Consider the self-contacting trees $T(r_{sc}, 45°)$ and $T(r_{sc}, 30°)$. We shall see that the only hole locator pairs for either tree are the ones for the self-contacting hole classes, so main pairs and canopy pairs. Thus they have the same type set. The locations of the pairs are distinct, however, because they have different contact addresses.

5.1.3 Hole Locator Pairs

In this subsection, we discuss hole locator pairs according to whether they are above or below the line $y = 1$ and also according to type. The goal of this subsection is to provide a broad sample of the ideas used to find hole locator pairs, both of self-contacting and non-self-contacting hole classes. For the main and secondary contact types, the list here is complete (because of how we defined these types). For the other types, we present the most common pairs (as will be demonstrated in the examples in the next chapter).

What portion of S_R is relevant for finding hole locators? The simple answer is that it is the 'left' part of the subtrees S_{RL} (for holes above the trunk) and S_{RR} (for holes below the line $y = 1$).

Above the Line $y = 1$

Here we discuss which descendant subtrees of S_{RL} are relevant for hole formation above the line $y = 1$. Since non-overlapping trees with $\theta \geq 135°$ all have height equal to 1, we only need to consider angles less than 135°.

- For angles less than 90°, consider the subtrees $S_{RL^2}, \cdots, S_{RLL^M}$, where M is the largest integer such that $M\theta < 180°$. For each of these subtrees $S_{\mathbf{A}}$, let $L_{\mathbf{A}}$ be the line segment through the two corner points with addresses $\mathbf{A}(RL)^\infty$ and $\mathbf{A}(LR)^\infty$ (forming a sort of top of the subtree). The collection of these line segments form a border between the subtree S_R and $\mathbf{y}_{(1,y_{\max})}$, in the following sense. For any point $P(x, y)$ on such a line segment, there is no point of the tree that has the same y but smaller x.

- When $\theta = 90°$, the self-contacting tree is space-filling. For self-avoiding trees, the only relevant subtree for holes above the trunk is the subtree S_{RLL}. This subtree has a horizontal trunk, so all canopy pairs are possible hole locators. Both subtrees S_{RL} and S_{RL^3} are vertical, so their tops are horizontal, and the canopy points could not be hole locators above the trunk.

- For angles θ such that $90° < \theta < 135°$, the only relevant subtree for holes above the trunk is S_{RLL}.

Below the Line $y = 1$

Here we need to consider angles greater than 45°, since non-overlapping trees with $\theta \leq 45°$ are such that no part of S_R is below the line $y = 1$. We summarize which descendant subtrees of S_{RR} are such that part of their tops could form a border between the left portion of S_R and the y-axis.

- For angles between 45° and 90°, the relevant subtrees are S_{R^j} where $3 \leq j \leq 6$.

- For $\theta = 90°$, the only relevant subtree is S_{R^3}.

- For angles between 90° and 135°, the relevant subtrees are S_{RR} and S_{R^3}.

- For angles greater than or equal to 135°, the only relevant subtree is S_{RR}.

Now we discuss hole locator pairs according to type. More details and explanations will be in the next chapter when we discuss specific examples and also in Chapter 7 which discusses theory in light of the examples.

Main Hole Locator Pairs

The main holes are located by pairs of the form $(\mathbf{A}_0, \mathbf{A}_c)$, where $\mathbf{A}_c$ denotes the contact address for a particular tree $T(r, \theta)$. We have already determine the contact address for any self-avoiding or self-contacting tree (see Table 3.1), so we do not repeat the details here.

Secondary Contact Hole Locator Pairs

The secondary contact holes are located by pairs of the form $(\mathbf{A}_0, \mathbf{A}_s)$, where $\mathbf{A}_s$ is the secondary contact address of a particular tree. Not every tree has a secondary contact address associated with it. See Table 3.2 for a summary of secondary contact addresses.

Canopy Hole Locator Pairs

In the previous chapter, we established that the canopy pairs of subtrees of the form $S_{RL^{N+1}}$, where $N \geq 2$, are hole locator pairs. Such a subtree has a horizontal trunk for $\theta = \theta_N$, and there is no portion of the tree between the top of the subtree and the y-axis. Now we investigate if there are any other subtrees whose canopy endpoints could be hole locator pairs.

- **Canopy Holes Above the Line** $y = 1$

  We shall see that canopy pairs of S_{RL^2} are hole locator pairs for $\theta = 90°$ (discussed in the next chapter). There are other subtrees whose canopy pairs may be hole locators. For example, in Chapter 7 we show that for the tree $T(r_{sc}, 30°)$, the subtree S_{RL^3RLL} contains hole locator pairs. However, for the specific examples of trees discussed in the next chapter, we will focus on canopy pairs of subtrees S_{RL^j} for $j \geq 2$ because they are the most straightforward to work with, and for the vast majority of angles they are the only relevant subtrees in terms of canopy pairs.

- **Canopy Holes Below the Line** $y = 1$

  For a tree to have canopy holes in some closed ϵ-neighbourhood, the tree must have some portion of S_R below the line $y = 1$, so must have branching angle greater than 45°. To have canopy intervals, the branching angle must be less than 135°. The primary subtrees to consider are S_{R^j} for $2 \le j \le 6$. Given j such that $2 \le j \le 5$, consider the angle $\theta_j = 270°/j$. The self-contacting tree with branching angle θ_j is such that the subtree S_{R^j} has a horizontal trunk and the top contains local minima. For example, consider 90° and the subtree S_{R^3}. We shall see that the canopy pairs of this subtree can be hole locator pairs.

Consider the self-contacting tree with branching angle 67.5°, shown in Figure 5.1. The subtree S_{R^4} has a horizontal trunk. So the top tip points of this

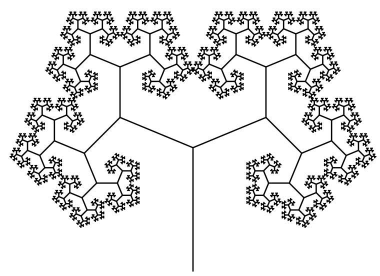

Figure 5.1: $T(r_{sc}, 67.5°)$

tree are all local minima because the highest top tip point of this subtree, with address $R^4(RL)^\infty$, can be shown to be below the line $y = 1$.

Now consider the canopy points of the degree 0 canopy interval of S_{R^4}. Let $P_T = (x_T, y_T)$ denote the point with address $R^4\mathbf{C}_R$. Then the point $P_t = (x_T/2, y_T)$ is at a distance of $x_T/2$ to the subtree S_R (one could show this with actual calculations, but this is easy to see just from Figure 5.1). There is an interval below this point on the vertical line through $x_T/2$ which is more than $x_T/2$ from the subtree S_R The maximal such interval is in fact the open interval where y ranges between Y_T and the y-coordinate of the other endpoint of the canopy interval (with address $R^4\mathbf{C}_L$). Thus this pair of addresses locates a hole. In fact, possible hole locator pairs are the pairs of addresses for the endpoints of any degree canopy interval of this subtree S_{R^4}. It turns out that not every such pair are hole locators, because if they are sufficiently high, the branch $b(R)$ starts to interact, and the calculations get quite complicated.

Now consider the self-contacting tree with angle 54°, shown in Figure 5.2.

Similar to the tree with angle 67.5° and its subtree S_{R^4}, the canopy points

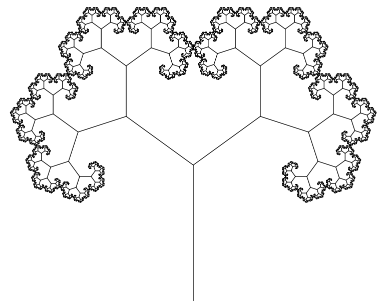

Figure 5.2: $T(r_{sc}, 54°)$

of the subtree S_{R^5} are local minima. However, we shall see in Chapter 7 that they are not hole locator pairs.

Mixed Types

The mixed types of holes are most common for angles in the second angle range. In this angle range, the only subtrees we need to consider are S_{RR} and S_{RRR}. At $\theta = 120°$, the subtree S_{RRR} has a vertical trunk, so the top is horizontal (see Figure 5.3). So for $\theta \geq 120°$, the subtree S_{RRR} is no longer relevant.

As θ gets closer to 135°, the top vertex points of the subtree S_{RR} become more important, and the canopy pairs become less important because the height of the trees is getting smaller (so the canopy interval gets closer to the top of the trunk). The point with $RR\mathbf{C}_L$ may still be a hole locator, but as part of the mixed pair $(RR, RR\mathbf{C}_L)$. In general, the mixed types are pairs of the form $(RR\mathbf{A}\mathbf{C}_R, RR\mathbf{A})$ or $(RR\mathbf{A}, RR\mathbf{A}\mathbf{C}_L)$ for $\mathbf{A} \in \mathcal{AL}_{2k}$, $k \geq 0$.

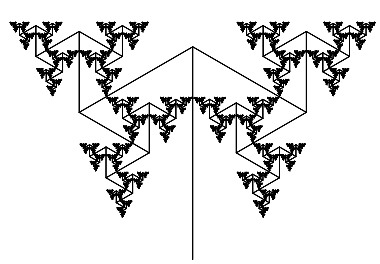

Figure 5.3: $T(r_{sc}, 120°)$.

Vertex Types

For the self-contacting tree $T(r_{sc}, 135°)$, there are infinitely many points of the subtree S_{RR} on the trunk. These correspond to finite addresses of the form $RR\mathbf{A}$ where $RR\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$, and infinite addresses of the form $RR\mathbf{A}'$ for some $\mathbf{A}' \in \mathcal{AL}_\infty$. For self-avoiding trees, the top of the trunk is the only part of the tree with $y = 1$. The local minima are the vertex points of the subtree S_{RR}, with addresses $RR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$. For a given $k \geq 0$, the top vertex points with addresses $RR\mathbf{A}$, for $\mathbf{A} \in \mathcal{A}_{2k}$, all have the same x-coordinate. The value of this x-coordinate increases as k increases. Points with addresses of the form $RR\mathbf{A}RL(LR)^\infty$ or $RR\mathbf{A}LR(RL)^\infty$ where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$ cannot be local minima because every open neighbourhood around such a point contains a vertex point, which has a smaller x-value. Hole locator pairs are consecutive top vertex points of the subtree S_{RR}, so they are pairs of the form $(RR\mathbf{A}, RR\mathbf{A}RL)$ or $(RR\mathbf{A}, RR\mathbf{A}LR)$ for some $\mathbf{A} \in \mathcal{A}_{2k}$. For trees with $\theta > 135°$, we still have vertex pairs (since they locate the self-contacting hole classes), but if θ is large enough there are no hole locator points above the point with address RR. If there is no hole locator above this point, then any pair is of the form $(RR(LR)^k, RR(LR)^{k+1})$ for some $k \geq 0$. Consider the images of two trees in Figures 5.4 and 5.5.

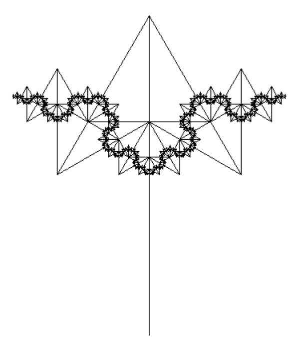

Figure 5.4: $T(r_{sc}, 150°)$.

5.1.4 Critical Angles Based on Location

Now that we have discussed hole locator pairs in general, we can discuss the equivalence classes based on location. For example, the pair $(\mathbf{A}_0, R^3(LR)^\infty)$ locates the

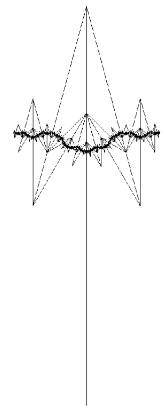

Figure 5.5: $T(r_{sc}, 165°)$.

main hole in $T(r_{sc}, 120°)$ (see Figure 5.3), but not for the tree $T(r_{sc}, 150°)$ (see Figure 5.4). So given a pair, there is a range of angles for which the pair locates a hole.

In this subsection, we define critical angles. We will discuss the actual values of some critical angles in Chapter 7, after looking at examples in Chapter 6.

Definition 5.1.4.1 *Let* $(\mathbf{A}_1, \mathbf{A}_2)$ *be any pairs of addresses. Then we define the* **angle range of the pair** $(\mathbf{A}_1, \mathbf{A}_2)$ **with respect to location**, *denoted by* $AR(\mathbf{A}_1, \mathbf{A}_2)$, *to be the range of values of* θ *such that* $(\mathbf{A}_1, \mathbf{A}_2) \in \mathcal{HL}(\theta)$.

Remark. The hole location set is restricted to addresses that are level 0 hole locators that locate a hole class not disjoint from the right side of $\mathbf{y}$. So for many pairs, the angle range is empty.

We can relate the angle ranges of pairs with equivalence classes with respect to the relation $\sim_{Loc}$ on branching angles. Let θ be any branching angle. For each pair $(\mathbf{A}_1, \mathbf{A}_2)$ in $\mathcal{HL}(\theta)$, we clearly have $\theta \in AR(\mathbf{A}_1, \mathbf{A}_2)$. Consider the collection $\mathcal{C}$ of angle ranges as the pairs range through the elements of $\mathcal{HL}(\theta)$. Then

$$\mathcal{C} = \{AR(\mathbf{A}_1, \mathbf{A}_2) | (\mathbf{A}_1, \mathbf{A}_2) \in \mathcal{HL}(\theta)\}$$

Then the equivalence class $[\theta]$ is equal to the intersection of all elements of $\mathcal{C}$, since $\theta' \in [\theta]$ if and only if $\theta' \in AR(\mathbf{A}_1, \mathbf{A}_2)$ for each $(\mathbf{A}_1, \mathbf{A}_2) \in \mathcal{HL}(\theta)$.

Conjecture. For any pair $(\mathbf{A}_1, \mathbf{A}_2)$, the angle range of the pair is a connected set. Moreover, for any θ, the equivalence class $[\theta]$ with respect to $\sim_{Loc}$ is a connected set.

Remark. In the previous conjecture, the second statement follows the first, since the intersection of connected sets is connected. We do not have a complete proof for the first statement in the conjecture, but our examples and discussion in Chapter 7 support the claim.

Definition 5.1.4.2 *A* **critical angle with respect to location** *is any branching angle that is the infimum or supremum of a non-empty angle range of a pair* $(\mathbf{A}_1, \mathbf{A}_2)$.

It will turn out that there are infinitely many critical angles based on location. Some critical angles are straightforward to determine, while others are more complicated. Some critical angles are more interesting than others, when they indicate a major shift in the locations of holes. We will discuss the critical angles in more detail in Chapter 7.

5.2 Critical Values of ϵ for a Specific Tree and The Hole Partition and Sequence

In the previous section, we discussed two way to characterize a tree, based on hole location or type. Now we investigate a different way to characterize a tree, based on what can happen in the closed ϵ-neighbourhoods of a specific tree as ϵ ranges through $[0, \infty]$. First we discuss the critical values of ϵ for a tree, that is, for a pair r and θ. Then we will define a relation on the ϵ-values, and this will enable us to define a hole partition and sequence. The hole sequence is the new characteristic of a tree that is a kind of 'topological barcode' in the sense of Carlsson *et al.* [7].

5.2.1 Critical Values of ϵ

Recall that for a given hole class $[H]$, we have the contact value $\underline{\epsilon_H}$ and the collapse value $\overline{\epsilon_H}$.

Notation. Let r, θ be such that $T(r,\theta)$ is not a simple tree. The set $Con(r,\theta)$ denotes the set of contact values for all possible hole classes (of any level).

$$Con(r,\theta) = \{\underline{\epsilon_H} | H \in \mathcal{H}(r,\theta)\} \tag{5.2.1}$$

The set $Col(r,\theta)$ denotes the set of collapse values for all possible hole classes (of any level).

$$Col(r,\theta) = \{\overline{\epsilon_H} | H \in \mathcal{H}(r,\theta)\} \tag{5.2.2}$$

Because of the scaling nature of the contact and collapse values and because we can restrict our attention to holes that are the maximal hole of their class, we have

$$Con(r,\theta) = \{r^k \underline{\epsilon_H} | k \geq 0, H \text{ is a maximal level 0 hole}\} \tag{5.2.3}$$

and

$$Col(r,\theta) = \{r^k \overline{\epsilon_H} | k \geq 0, H \text{ is a maximal level 0 hole}\} \tag{5.2.4}$$

Definition 5.2.1.1 *For a pair (r,θ), a* **critical ϵ-value of** (r,θ) *is any contact or collapse value. The set of all critical values is denoted by $Crit(r,\theta)$. If a tree is simple, then we say $Crit(r,\theta)$ is empty.*

Remark. If $Crit(r,\theta)$ is non-empty, then it has infinitely many elements. Given a level 0 hole class H, its collapse value $\overline{\epsilon_H}$ is non-zero. For each $k \geq 1$, there is a level k hole H_k (corresponding to some level k address map) whose collapse value is $r^k\overline{\epsilon_H}$. Thus there are infinitely many distinct critical values.

It is straightforward to determine $\underline{\epsilon_H}$ for a given hole class. Recall that for a level k hole class, the contact value is just r^k times the contact value of the corresponding level 0 hole class.

Observation. Let H be a level 0 hole not disjoint from the right side of the y-axis. Let $(\mathbf{A}_1, \mathbf{A}_2)$ be the hole locator pair, and without loss of generality assume that $\mathbf{A}_1$ is a local minimum. Let $P = (x, y)$ denote the point with address $\mathbf{A}_1$. Then, following the results from Section 4.6,

$$\underline{\epsilon_H} = \begin{cases} x & \text{if } H \text{ is above the line } y = 1 \\ x/2 & \text{if } H \text{ is below the line } y = 1 \end{cases} \tag{5.2.5}$$

Unfortunately, $\overline{\epsilon_H}$ is generally not as easy to determine. Often a lower and upper bound are sufficient for determining other features that we consider (like complexity). We now present notation for the bounds that we will consider.

Notation. A subscript of 'gh' refers to a lower bound for a critical value of ϵ. The 'gh' stands for 'guaranteed hole'. A subscript of 'gc' refers to an upper bound for a critical value of ϵ. The 'gc' stands for 'guaranteed collapse'.

The lower bounds we consider are denoted by ϵ_H^{gh}, where we require that there exists a hole of $[H]$ at ϵ_H^{gh}. Thus

$$\underline{\epsilon_H} \leq \epsilon_H^{gh} < \overline{\epsilon_H} \tag{5.2.6}$$

The upper bounds we consider are denoted by ϵ_H^{gc}, where we require that there are no holes of $[H]$ for any $\epsilon \geq \epsilon_H^{gc}$. Thus

$$\overline{\epsilon_H} \leq \epsilon_H^{gc} \tag{5.2.7}$$

We will use these bounds in Chapter 6 when dealing with specific examples. Note that we try to make ϵ_H^{gh} as large as possible, and ϵ_H^{gc} as small as possible

5.2.2 The Hole Partition and Sequence

For any tree, if ϵ is sufficiently large, then the entire closed ϵ-neighbourhood is simply-connected. As ϵ decreases to 0, the number of holes may become non-zero. We now investigate the number of holes as a function of ϵ. However, we are concerned not just with the number of holes, but also the classes of holes.

Definition 5.2.2.1 *Let* $\theta \in (0°, 180°)$ *and let* $r \leq r_{sc}$. *The* **hole congruence relation of the pair** (r, θ), *denoted by* $\sim_{r,\theta}$, *is defined on the set* $[0, \infty]$ *of* ϵ*-values as follows.* $\epsilon_1 \sim_{r,\theta} \epsilon_2$ *if the number of holes and the location of holes remains constant for* ϵ *between* ϵ_1 *and* ϵ_2 *(inclusively).*

The hole congruence relation $\sim_{r,\theta}$ is obviously an equivalence relation.

Definition 5.2.2.2 *Let $\theta \in (0°, 180°)$ and let $r \leq r_{sc}$. The* **hole congruence partition of the pair** *r, θ, denoted $\mathcal{HP}(r,\theta)$, is the partition of $[0,\infty]$ into the equivalence classes with respect to the hole congruence relation $\sim_{r,\theta}$. An equivalence class is denoted by $[\epsilon]_{r,\theta}$, or just $[\epsilon]$ when r and θ are understood.*

We can order the equivalence classes of the hole partition as follows. We say that $[\epsilon_1] \leq [\epsilon_2]$ if and only if $\epsilon_1 \geq \epsilon_2$. This reverse ordering is used because when we consider actual trees, we consider what happens as ϵ decreases from ∞, we don't start by looking at small values of ϵ. In addition, we use this reverse ordering to define the hole sequence.

Lemma 5.2.2.3 *Let $\epsilon \in Crit(r,\theta)$. Then ϵ is the infimum of $[\epsilon]$.*

Proof. Let $\epsilon \in Crit(r,\theta)$. If ϵ is a contact value of some hole class $[H]$, then ϵ is the infimum of the persistence set of $[H]$, and so there are no elements of $[H]$ for any $\epsilon' < \epsilon$, hence ϵ is not hole congruent to ϵ' for any $\epsilon' < \epsilon$. Thus $\epsilon = \inf([\epsilon])$. On the other hand, if ϵ is the collapse value of some hole class $[H]$, then it is the supremum of the of the persistence set of $[H]$, and there is no element of $[H]$ is the closed ϵ-neighbourhood. There exists $\delta > 0$ such that for $\epsilon' \in [\epsilon - \delta, \epsilon)$, there is an element of $[H]$ in the corresponding closed ϵ-neighbourhood, so ϵ' is not hole congruent to ϵ. Thus $\epsilon = \inf([\epsilon])$.

Theorem 5.2.2.4 *Let $\epsilon \in Crit(r,\theta)$. Then $[\epsilon] = [\epsilon, \epsilon')$ if ϵ' is the next highest element of $Crit(r,\theta)$ (so there are no elements of $Crit(r,\theta)$ between ϵ and ϵ'); $[\epsilon] = [\epsilon, \infty]$ if ϵ is the largest value of $Crit(r,\theta)$; otherwise $[\epsilon] = \{\epsilon\}$ (if there are critical values higher than ϵ, but no next highest).*

Proof. Let $\epsilon \in Crit(r,\theta)$. Then by the previous lemma, ϵ is the infimum of its equivalence class. If it is also the supremum, then $[\epsilon] = \{\epsilon\}$.

So suppose that ϵ is not equal to the supremum of its equivalence class. If ϵ is the largest value of $Crit(r,\theta)$, then there are no critical values higher than ϵ. Hence ϵ must be the largest possible collapse value for a hole class. So for any $\epsilon' > \epsilon$, the corresponding closed ϵ-neighbourhood has no holes. Thus $\epsilon \sim_{r,\theta} \infty$ and $[\epsilon] = [\epsilon, \infty]$.

Now suppose that there are critical values that are greater than ϵ. Let ϵ' be the supremum of $[\epsilon]$. By the definition of hole congruence, the supremum of $[\epsilon]$ is a value that indicates a change in the location of the hole classes. So it must either be a contact value for a new hole class $[H]$, where $\epsilon \notin p([H])$, or it is a collapse value for a hole class $[H]$, where $\epsilon \in p([H'])$. In either case, ϵ' is an element of $Crit(r, \theta)$. Now we need to show that there are no other elements of $Crit(r, \theta)$ that could be between ϵ and ϵ'. If ϵ_1 is an element of $Crit(r, \theta)$ between ϵ and ϵ', then ϵ_1 is the infimum of its equivalence class, and this would contradict that the number and location of hole classes remains constant over $[\epsilon, \epsilon')$. Hence there can be no critical ϵ-values between ϵ and ϵ', so ϵ' is the next highest element.

Therefore, $[\epsilon]_{r,\theta} = [\epsilon, \epsilon')$ if ϵ' is the next highest element of $Crit(r, \theta)$; otherwise $[\epsilon] = \{\epsilon\}$. $\square$

Proposition 5.2.2.5 *For any non-simple tree $T(r, \theta)$, the equivalence class of 0 with respect to the hole congruence relation is the singleton set $\{0\}$.*

Proof. Suppose the tree is a self-contacting tree. Then 0 is an element of $Crit(r, \theta)$, since it is the contact value of infinitely many hole classes. Let $[H]$ be an level 0 hole class, and $\overline{\epsilon_H}$ its collapse value. Then $r^k\overline{\epsilon_H}$ is an element of $Crit(r, \theta)$ for every $k \geq 0$. So for any $\delta > 0$, there is k such that $r^k\overline{\epsilon_H} < \delta$. Thus there is no next highest element in $Crit(r, \theta)$, since δ was arbitrary. Thus $[0] = \{0\}$.

Suppose the tree is not self-contacting. Then $0 \notin Crit(r, \theta)$. The tree is non-simple, by assumption, so there is at least one level 0 hole class $[H]$. Then the limit of the contact values of the corresponding level k hole classes, given by $r^k\underline{\epsilon_H}$, goes to 0 as k goes to infinity. For any $\delta > 0$, there is an element of $Crit(r, \theta)$ in $(0, \delta)$, hence there are no positive real numbers that are hole congruent to 0. Thus $[0] = \{0\}$. $\square$

Finally we can define the hole sequence of a tree.

Definition 5.2.2.6 *For a pair (r, θ) such that $r \leq r_{sc}$, the* **hole sequence of the pair** *(r, θ), or the* **hole sequence of the tree** *$T(r, \theta)$, is the ordered set of numbers*

$$\{N_\epsilon(r, \theta)\} = \{N(r, \theta, \epsilon) | \epsilon \in Crit(r, \theta) \text{ or } \epsilon = 0\}, \tag{5.2.8}$$

where the set is ordered according to decreasing values of ϵ. Hence the reason for calling it a sequence.

We shall see that for many trees the hole sequence is order isomorphic to the natural numbers (*i.e.*, can be indexed by the natural numbers). For example, this happens when a tree has only the main type of holes. However, is not true in general. The complications arise when there are non-zero values of ϵ that have singleton equivalence classes. An example of how this can occur is with the tree $T(r_{sc}, 67.5°)$, as we shall see in the next chapter.

Definition 5.2.2.7 *We define the* **hole sequence relation** *on pairs* (r, θ)*, denoted by* $\sim_{HS}$*, as follows:*

$$(r_1, \theta_1) \sim_{HS} (r_2, \theta_2) \Leftrightarrow \{N_\epsilon(r_1, \theta_1)\} = \{N_\epsilon(r_2, \theta_2)\} \tag{5.2.9}$$

5.3 The Complexity Classification of Trees

In this section we introduce the notion of the complexity of a tree, based on the levels of holes in the closed ϵ-neighbourhoods of the trees. Recall that we have already defined the complexity of a hole class (see Definition 4.5.0.13 in Section 4.5). We can use complexity to define critical scaling ratios for a specific branching angle and to compare different trees.

5.3.1 Complexity

A self-contacting tree has an infinite number of holes, and in fact it has holes at all possible levels. A self-avoiding tree has no holes at any level. What about the closed ϵ-neighbourhoods of self-contacting or self-avoiding trees? What is the range of levels of holes that occur as ϵ ranges over the non-negative real numbers?

Definition 5.3.1.1 *Let* θ *be given, let* $r \leq r_{sc}$*. Let* $\epsilon \geq 0$*. Define the* **level set of** ϵ*, denoted* $LS(r, \theta, \epsilon)$*, to be the set of non-negative integers such that for each* $i \in LS(r, \theta, \epsilon)$*, there exists a level* i *hole in* $\mathcal{H}(r, \theta, \epsilon)$*.*

$$i \in LS(r, \theta, \epsilon) \Leftrightarrow \mathcal{H}_i(r, \theta, \epsilon) \neq \emptyset \tag{5.3.1}$$

Definition 5.3.1.2 *Let* θ *be given, let* $r \leq r_{sc}$*. Let* $\epsilon \geq 0$*. Define the* **level range of** ϵ*, denoted* $LR(r, \theta, \epsilon)$*, by*

$$LR(r, \theta, \epsilon) = \max\{LS(r, \theta, \epsilon)\} - \min\{LS(r, \theta, \epsilon)\} \tag{5.3.2}$$

Definition 5.3.1.3 *Let θ be given, let $r \le r_{sc}$. The* **complexity of the tree** *$T(r,\theta)$, denoted $C(r,\theta)$, or just C, is defined by*

$$C(r,\theta) = [\max_{\epsilon \ge 0}\{LR(r,\theta,\epsilon)\}] + 1 \tag{5.3.3}$$

We define the complexity of a simple tree to be 0. If a tree can have holes in an infinite number of levels for some ϵ, then we say the tree has **infinite complexity***.*

For example, the closed ϵ-neighbourhood of a tree with complexity 1 has holes in at most one level for any given ϵ. A tree with complexity k can have holes in levels j through $j+k$ for some $j \ge 0$.

Lemma 5.3.1.4 *Let θ be given, and let $r \le r_{sc}$. Let $\epsilon > 0$. Then the level set of ϵ, $LS(r,\theta,\epsilon)$, is a finite set. In other words, for any $\epsilon > 0$, the corresponding closed ϵ-neighbourhood can have holes in only a finite number of levels.*

Proof. Given θ and r, the tree $T(r,\theta)$ is compact. Let D be the diameter of the tree. For the given $\epsilon > 0$, there exists an integer m such that $r^m D < \epsilon$. So all subtrees of level m or higher are covered by the closed ϵ-neighbourhood of the tree. Thus there can only be holes in levels 0 through $m-1$, i.e., $LS(r,\theta,\epsilon)$, is a finite set.

Proposition 5.3.1.5 *Let $\theta \ne 90°, 135°$, and $r \le r_{sc}$ be given. The tree $T(r,\theta)$ has infinite complexity if and only if $r = r_{sc}$. That is, a tree is a non-space-filling, self-contacting tree if and only if it has infinite complexity.*

Proof. If a tree $T \in \mathcal{T}_{sc}$ is not space-filling, then the tree itself contains at least one level 0 hole $H_0 \in \mathcal{H}_0(r,\theta,0)$. Let $k \ge 1$. For any $\mathbf{A} \in \mathcal{A}_k$, the hole $m_{\mathbf{A}}(H_0)$ is a level k hole. Thus the tree contains holes at any level, so it has infinite complexity.

If a tree has infinite complexity, then there exists $\epsilon \ge 0$ such that $\mathcal{H}(r,\theta,\epsilon)$ has holes at every level. If $\epsilon = 0$, this implies that the tree itself has simple, closed curves at any level. This is only true for self-contacting trees. If $\epsilon > 0$, there can only be holes at a finite number of levels (for any tree), so this contradicts the assumption that the tree has infinite complexity. Thus the tree must be self-contacting (and non-space-filling).

Therefore, a tree is a non-space-filling, self-contacting tree if and only if it has infinite complexity. □

Remarks. Suppose $T(r,\theta) \in \mathcal{T}$ such that $r \leq r_{sc}$. and $\epsilon > 0$ is such that $\mathcal{H}(r,\theta,\epsilon)$ has holes at levels j and k, where $j < k$. Then one might claim that there are holes in $\mathcal{H}(r,\theta,\epsilon)$ at all levels i such that $j \leq i \leq k$. However, this claim is not true, since we do find a counterexample, the tree $T(r_{sc}, 108°)$ discussed in the next chapter.
Now suppose $r \leq r' < r_{sc}$ for some θ. Then we might claim that $C(r,\theta) \leq C(r',\theta)$. If $r, r' < r_{sc}$, then both $C(r,\theta)$ and $C(r',\theta)$ are finite numbers. The level k subtrees are related to the original trees as $S(r,\theta) \sim_{r^k} T(r,\theta)$ and $S(r',\theta) \sim_{(r')^k} T(r,\theta)$. Because $r < r'$, any given level k subtree $S(r,\theta)$ is similar to $T(r,\theta)$ with a contraction factor of r^k, which is strictly less than the contraction factor $(r')^k$ that a level k subtree $S(r',\theta)$ is similar to $T(r',\theta)$. That is, the subtrees are decreasing in size at a faster rate in the tree $T(r,\theta)$ compared to the tree $T(r',\theta)$. These remarks might make the claim seem probable, but again it is not true because we have a counter-example, the angle 90° discussed in the next chapter.

5.3.2 Critical Scaling Ratios of the Branching Angle θ

Definition 5.3.2.1 *Let θ be given. Define the* **complexity relation $\sim_\theta$ of the branching angle** θ *on the set* $(0, r_{sc}(\theta)]$ *by* $r \sim_\theta r'$ *whenever* $C(r,\theta) = C(r',\theta)$.

Remark. The complexity relation $\sim_\theta$ is an equivalence relation.

Notation. For $r \in (0, r_{sc}(\theta)]$, let $C([r],\theta)$ denote the complexity of the equivalence class of r, i.e., for all $r' \in [r]$, $C(r',\theta) = C([r],\theta)$.

Observation. For any $\theta \neq 90°, 135°$, the equivalence class $[r_{sc}]$ is just the singleton set $\{r_{sc}\}$. This is because r_{sc} is the only scaling ratio that yields a tree with infinite complexity.

So what happens with the equivalence classes with respect to the complexity relation of θ as we look at the entire interval $(0, r_{sc}]$? For any angle, can we find an interval that has complexity equal to k, for any non-negative integer k?

Definition 5.3.2.2 *Let θ be given. Let k be any non-negative integer. We define the k-**complexity class of** θ, denoted $C_k(\theta)$, to be the equivalence class $[r]$ such that $C([r], \theta) = k$.*

Theorem 5.3.2.3 *Let θ be given. Then $C_0(\theta)$ is non-empty. Moreover, there exists a scaling ratio r' such that $T(r, \theta)$ is simple for all $r \leq r'$.*

Proof. The main idea of this proof is that for any θ, we can find a scaling ratio r' small enough that for any $\epsilon > 0$, the corresponding closed ϵ-neighbourhood is contractible. It suffices to restrict our attention to the level 0 holes. A level 0 hole is the result of intersection between the closed ϵ-neighbourhoods of the two level 1 subtrees, $E_R(r, \theta, \epsilon)$ and $E_L(r, \theta, \epsilon)$. We need to show that for a given θ, there exists a scaling ratio r such that there are no level 0 holes for any ϵ. This would mean that for any ϵ large enough so that $E_R(r, \theta, \epsilon)$ and $E_L(r, \theta, \epsilon)$ intersect each other, i.e large enough for contact, they overlap in such a way that the entire closed ϵ-neighbourhood $E(r, \theta, \epsilon)$ is contractible.

There are three cases to consider, based on the three main angle ranges.

1. First angle range: $0^\circ < \theta \leq 90^\circ$. Recall that the closest point to contact, P_c, is with address $RLL^N(RL)^\infty$, where N is the turning number. From Chapter 3, we know that $x_{c1} = r \sin\theta - r^2 x_{\max}$. See Equations 3.4.3 and 3.3.1 for more details. Consider the subtree S_{RL}. In this angle range, it is the level 2 subtree on the right side of $\mathbf{y}$ that is closest to $\mathbf{y}$ (since it contains the point at $RLL^N(RL)^\infty$). The highest point on this subtree is at a height of $1 + r\cos\theta + r^2 h$, where h is the height of the tree. See Section 3.4 for more information about height. For the closed ϵ-neighbourhood of the subtree S_{RL} to reach $\mathbf{y}$, to yield a level 0 hole, we need $\epsilon \geq r \sin\theta - r^2 x_{\max}$. There definitely can be no hole if the closed ϵ-neighbourhood of the branch $b(R)$ covers the portion of $\mathbf{y}$ between the origin and the point at $(0, 1 + r\cos\theta + r^2 h)$. Hence there can be no hole

if $\epsilon \geq (r\cos\theta + r^2h)\sin\theta$, the distance from the branch $b(R)$ to the point $(0, 1 + r\cos\theta + r^2h)$. There are no holes for any value of ϵ if the following inequality is true:

$$r\sin\theta - r^2x_{\max} \geq (r\cos\theta + r^2h)\sin\theta$$

When the above inequality is satisfied, as soon as ϵ is large enough for contact, it is large enough for collapse. Simplifying the inequality yields:

$$\sin\theta(1 - \cos\theta) \geq rx_{\max} + rh\sin\theta$$

In this angle range, the left side of the above inequality is always positive. Now recall that both $x_{\max}$ and h are increasing functions, so they decrease as r decreases towards 0. The limit of the right side of the inequality as r tends to 0 is 0, and so there must be a positive value of r' where the inequality is true for all $r \leq r'$, i.e., where there can be no holes for any value of ϵ. Thus, for any branching angle such that $0 < \theta \leq 90°$, there exists a scaling ratio r' such that $T(r,\theta)$ is simple for all $r \leq r'$.

2. Second Angle Range: $90° < \theta < 135°$. Consider the subtree S_{RR}, and the bounding rectangle that contains it. The point on the bounding rectangle that is closest to the trunk has x-value of

$$x_{min} = r\sin\theta - r^2h|\sin(2\theta)| - r^2x_{\max}|\cos(2\theta)|$$

and the lowest point on the rectangle has y-value given by

$$y_{min} = 1 - r|\cos\theta| - r^2h|\cos(2\theta)| - r^2x_{\max}|\sin(2\theta)|$$

Because the actual subtree S_{RR} is contained within in its bounding rectangle, we now know that there can be no hole if ϵ is less than half the distance from the bounding rectangle to the trunk, i.e., $x_{min}/2$. There can also be no hole if ϵ is large enough so that the closed ϵ-neighbourhood of the branch $b(R)$ covers the point at (ϵ, y_{min}) (the trunk will cover the region to the left of the line $x = \epsilon$. The closed ϵ-neighbourhood of the branch $b(R)$ is a line segment of the line $y = 1 + \cot\theta - \csc\theta$. We find the value of y on this line when $\epsilon = x_{min}/2$, and there can be no hole if this y is less than or equal to y_{min}. As with the first angle range, we obtain an inequality that will guarantee no holes for any ϵ:

$$\begin{aligned} & 1+\frac{1}{2}\left(\frac{\cos\theta-1}{\sin\theta}\right)(r\sin\theta-r^2h|\sin(2\theta)|-r^2x_{\max}|\cos(2\theta)|) \\ \leq \quad & 1-r|\cos\theta|-r^2h|\cos(2\theta)|-r^2x_{\max}|\sin(2\theta)| \end{aligned}$$

Simplifying the inequality yields:

$$\begin{aligned} & \sin\theta(1-|\cos\theta|) \\ \geq \quad & |\sin(2\theta)|(rh(1-\cos\theta)+2rx_{\max}\sin\theta) \\ & \qquad +|\cos(2\theta)|(rx_{\max}(1-\cos\theta)+2rh\sin\theta) \end{aligned}$$

The left side of the inequality is always positive for $90° < \theta < 135°$. Using a similar argument as in the first angle range, the right side of the inequality tends to 0 as r goes to 0, and so there must be a positive value of r_0 where the inequality is true for all $r \leq r_0$. Thus, for any branching angle such that $90° < \theta < 135°$, there exists a scaling ratio r' such that $T(r,\theta)$ is simple for all $r \leq r'$.

3. Third Angle Range: $135° \leq \theta < 180°$. Here can we use a similar argument as for the second angle range. The only difference is that the lowest point of the bounding rectangle of the subtree S_{RR} is at

$$y_{min} = 1-r|\cos\theta|-r^2x_{\max}|\sin(2\theta)|$$

The inequality now becomes

$$\begin{aligned} & 1+\frac{1}{2}\left(\frac{\cos\theta-1}{\sin\theta}\right)(r\sin\theta-r^2h|\sin(2\theta)|-r^2x_{\max}|\cos(2\theta)|) \\ \leq \quad & 1-r|\cos\theta|-r^2x_{\max}|\sin(2\theta)| \end{aligned}$$

Simplifying the inequality yields:

$$\begin{aligned} & \sin\theta(1-|\cos\theta|) \\ & \geq |\sin(2\theta)|(rh(1-\cos\theta)+2rx_{\max}\sin\theta)+|\cos(2\theta)|(rx_{\max}(1-\cos\theta)) \end{aligned}$$

Again there must be a positive value of r' where the inequality is true for all $r \leq r'$. Thus, for any branching angle such that $135° \leq \theta < 180°$, there exists a scaling ratio r' such that $T(r,\theta)$ is simple for $r \leq r'$.

Therefore, given any branching angle θ such that $0° < \theta < 180°$, there exists a scaling ratio r' such that $T(r,\theta)$ is simple for all $r \leq r'$. and thus $C_0(\theta)$ is non-empty. □

Note. In the proof of the previous theorem, we used the bounding rectangles of subtrees to approximate the trees, so the values of r' obtained by solving the inequalities are not necessarily the suprema of the classes $C_0(\theta)$, they are just a lower bound.

5.3.3 General Complexity

The previous subsections introduced the complexity of a tree. For a fixed branching angle, the complexity can be used to determine critical scaling ratios based on the complexity classes. Now we are going to discuss a complexity relation on all trees.

Definition 5.3.3.1 *The* **general complexity relation** *is a relation on the collection of pairs* (r,θ). *The pair* (r,θ) *is related to the pair* (r',θ'), *denoted by* $(r,\theta) \sim_C (r',\theta')$, *if* $C(r,\theta) = C(r',\theta')$.

Notation. For $k \geq 0$, C_k denotes the equivalence class that has complexity equal to k. Thus

$$C_k = \bigcup_{\theta\in(0°,180°)} \bigcup_{r\in C_k(\theta)} (r,\theta) \tag{5.3.4}$$

C_∞ denotes the equivalence class that has infinite complexity.

Observation. The only self-contacting trees that are space-filling and contractible are $T(r_{sc}, 90°)$ and $T(r_{sc}, 135°)$. Thus $C_\infty = \bigcup_{\theta\in(0°,180°)\setminus\{90°,135°\}} (r_{sc}(\theta),\theta)$.

For now we have just presented the definition of the general complexity classes. We will discuss the complexity of symmetric binary fractal trees in greater detail in Chapter 6 while looking at specific examples, and the complexity classification in Chapter 7 after looking at the examples of trees.

5.4 Brief Chapter Summary

This chapter has continued with the development of theory regarding symmetric binary fractal trees and their closed ϵ-neighbourhoods. We began the chapter with a characterization of a tree based on the location of the level 0 holes not disjoint from the right side of $\mathbf{y}$, called the hole location set. We also defined the notion of a hole location set of a branching angle. Based on these hole location sets, we define the relation $\sim_{Loc}$ on pairs (r, θ) and on branching angles. We discussed possible hole locator pairs. Another characterization of a tree was defined, based on the types of holes that the closed ϵ-neighbourhoods of the trees may have. This yields another relation, $\sim_{Type}$, which is a coarser relation than $\sim_{Loc}$. Critical angles based on location were defined. We discussed the critical ϵ-values of a tree, and also the corresponding hole partition and hole sequence. The hole sequence is yet another characterization, and yields a very different relation $\sim_{HS}$. Finally we discussed the complexity of a tree. For each branching angle, we can use complexity to define critical scaling ratios. We can also use complexity to compare trees, and the fourth relation we defined is $\sim_C$.

We shall investigate each of these characterizations and classifications in the next chapter dealing with specific examples. Following the examples in Chapter 6, Chapter 7 will contain further discussion on theoretical and quantitative aspects of these characterizations and classifications.

Chapter 6

Specific Examples

We continue the computational topology analysis of the symmetric fractal binary trees with a quantitative study of the closed ϵ-neighbourhoods and corresponding features such as complexity, persistence, location and hole sequences, by looking at specific examples of trees. For some trees we provide complete details and explanations, while for others we just present the quantitative results. A more thorough discussion of classifications of fractal trees is saved for the next chapter.

Given a tree, we first try to determine what the hole locator pairs are. Then we determine the critical values for the level 0 holes. The contact values are easy to determine, but the collapse values are not. So sometimes we use polygonal regions to find upper and lower bounds for a collapse value. For a polygon, finding the collapse value amounts to finding the radius of the maximal inscribed circle and possibly the medial axis and Voronoi diagram. See [8] for more information. The situation for fractal trees is not as straightforward as for polygons, because many of the boundaries are not polygonal, because they contain part of the tipset. However, symmetry can often help to determine part of the medial axis.

The examples demonstrated in this chapter help to illustrate our characterizations of fractal trees and their closed ϵ-neighbourhoods. We start with a detailed example of a specific tree, the self-contacting tree $T(r_{sc}, 45°)$. Then we discuss other self-contacting trees. Four self-contacting trees are particularly interesting because they are related to the 'golden ratio'. We also discuss various self-avoiding trees. This includes a discussion of self-avoiding trees with the angles 90° and 135°.

Note. In this chapter and the next chapter, we often leave the symbols r and θ in symbolic form, even when we are considering specific values.

6.1 Detailed Example: $T(r_{sc}(45°), 45°)$

We begin with a detailed example of a specific tree, the self-contacting tree with branching angle 45° (see Figure 6.1). This tree is a nice example to start with be-

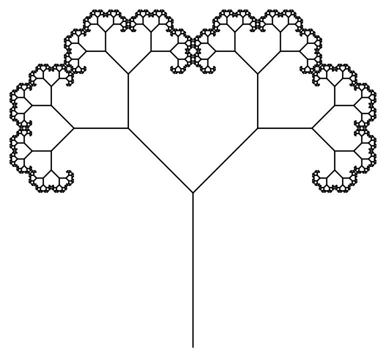

Figure 6.1: A finite approximation of $T(r_{sc}(45°), 45°)$.

cause the geometric calculations are more straightforward than with other angles. In some of the following calculations we do not simplify the equations but rather show them in forms that would be easier to generalize for other angles. This example is worked through in detail to demonstrate the methods that are used for other trees.

First, we need to determine r_{sc}. With turning number $N = 2$, we set $x_{c1} = 0$ to find r_{sc} (see equation[3.3.1] in Chapter 3):

$$r_{sc} \sin 45° - \frac{r_{sc}^3 \sin 45°}{1 - r_{sc}^2} - \frac{r_{sc}^4}{1 - r_{sc}^2} = 0.$$

Thus

$$r_{sc}^3 + \sqrt{2} r_{sc}^2 - \frac{1}{\sqrt{2}} = 0$$

and $r_{sc} \approx 0.59347$.

6.1.1 Level 0 Holes and Critical Values

We have already established that the self-contacting hole classes are located by the main pair $(\mathbf{A}_0, RL^3(LR)^\infty)$ and canopy pairs of the subtree S_{RL^3}, so pairs of the form $(\mathbf{A}\mathbf{C}_L, \mathbf{A}\mathbf{C}_R)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$.

We claim that the self-contacting hole classes are the only hole classes for the tree. There is no portion of the subtree below the line $y = 1$, so there are no holes below this line.

So consider local minima above the trunk. Are there any points that are not on the y-axis that could be hole locators?

To answer this, we first investigate whether there are any points on the y-axis, disjoint from the tree and between the lowest and highest top tip points of S_{RL^3}, and such that the closest point on the tree is at the same height (same y-value). If so, then by the scaling nature of the tree, there would have to be a point $P = (x, y_{\max})$ that is in some canopy interval of the tree such that the closest point on the tree to P has the same x-value (because the tree has a vertical trunk and the subtree S_{RL^3} has a horizontal trunk), but this is never true. To see this, it suffices to look at the degree 0 canopy interval, since every other canopy interval is similar. For any point on this interval, the closest point on the tree is a tip point of either the subtree S_{RLL} or S_{LRR}, and this tip point is always at an x-value that is closer to one of the ends of the canopy interval. Any open ball around P that has a point (x, y) of the tree on the boundary of the ball always contains other points of the tree within the ball.

Now consider S_{RL^2} and S_{RL^4}. The line segments through the tops of these subtrees each form an angle of $45°$ with the vertical line through the top of S_{RL^3}. Let $P = (x, y)$ be a top tip point of S_{RL^2}. Let $P' = (0, y)$. Let L be the line through P' that is perpendicular to the top line segment of S_{RL^2}. Let P'' be the intersection of L and the top line segment of S_{RL^2}. P'' may or may not be a top tip point. If it is, then it is closer to P' than P is. If not, then there are tip points between P'' and P on L, and each of them is closer to P' than P is, so P cannot be a hole locator. Likewise, there can be no hole locators on the top of S_{RL^4}.

Thus there are no other hole classes. Let M denote the main hole class. Let $P_{c1} = (x_{c1}, y_{c1})$ be the point with address $RL^3(LR)^\infty$ (the lowest tip point on the y-axis). Other holes are of the canopy type. Let C_0 denote the hole class identified by $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$. All other canopy holes are similar to C_0. To study this hole more carefully, consider the 'double tree' in Figure 6.3. The first canopy hole is similar to the hole at the center of the double tree, with a contraction factor of r_{sc}^4 (since the subtree S_{RL^3} is similar to the tree with contraction factor r_{sc}^4). We use the double tree to determine the critical ϵ-values for the canopy holes.

Since any canopy hole class is the image of the hole class C_0, we only need to determine the critical ϵ-values for M and C_0. In general, there are 2^j level 0 canopy holes that are similar to C_0 with contraction factor r_{sc}^{2j}, via the address maps from addresses in $\mathcal{AL}_{2j}$. We denote these holes C_j. The persistence interval for each hole begins with 0, since each hole class is a self-contacting hole class. The exact collapse values of the persistence intervals can also be determined. All hole classes here are symmetric about the y-axis, so to determine the collapse values we determine the smallest ϵ so that the interval between the two locator points of the class is within ϵ of the tree.

First we calculate the collapse value $\overline{\epsilon_M}$ of the largest hole. To do this, we determine which point on the y-axis inside the hole is furthest from the boundary of the hole. Consider Figure 6.2, which is a close-up view of the main hole, along with a square formed from the branches $b(R)$ and $b(L)$, and two other new line segments. The top right line segment intersects the tree at the top of the subtree S_{RL^4}. Let $P_c = (x_c, y_c)$ be the left corner point of the subtree S_{RL^4}, so the point with address $RL^4(LR)^\infty$. Let L_c denote the line of the square that goes through P_c.

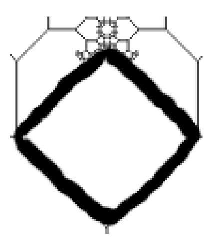

Figure 6.2: Main Hole of $T(r_{sc}, 45°)$

For the square shown in Figure 6.2, the center of the square is at $(0, r_{sc}/2)$. The top of the subtree S_{RL^4} is more than $r_{sc}/2$ away from the centre, so a point on the y-axis that is equidistant to the branch $b(R)$ and the subtree S_{RL} is above this centre point. There will be a unique value y' with $0 < y' < y_{c1}$ such that the point $(0, y')$ is equidistant from the branch $b(R)$ and the tip points of the tree that are on L_c. Some basic geometry tells us that for any point $(0, y)$ with $0 < y \leq y_c - x_c$, the tip point on

L_c that is closest to $(0, y)$ is in fact P_c. Now we will solve for y' using the point P_c, and provided we obtain $y' \leq y_c - x_c$, we know that P_c is indeed the closest tip point on L_c. To solve for y', we need to equate the distance to the branch with endpoint R and the distance to P_c. Thus

$$(y' \sin 45°)^2 = (x_c - 0)^2 + (y_c - y')^2$$

and hence the value in the desired region is

$$y' = \frac{y_c - \sqrt{y_c^2 - \cos^2 45°(x_c^2 + y_c^2)}}{\cos^2 45°}$$

To find the exact value of y', we first need the exact value of the coordinates x_c and y_c. For this tree we have

$$\begin{aligned} x_c &= r_{sc} \sin 45° - \frac{r_{sc}^3 \sin 45°}{1 - r_{sc}^2} - r_{sc}^4 \approx 0.06743, \\ y_c &= r_{sc} \cos 45° + r_{sc}^2 + r_{sc}^3 \cos 45° - \frac{r_{sc}^5}{1 - r_{sc}^2}[\cos 45° + r_{sc}] \approx 0.77185 \end{aligned} \tag{6.1.1}$$

and $y' \approx 0.45631 < y_c - x_c$.

The above calculations tell us that the collapse value of the hole class M equals the distance from $(0, y')$ to the branch $b(R)$. More precisely:

$$\overline{\epsilon_M} = y' \sin 45° \approx 0.32266 \tag{6.1.2}$$

We can use a similar method to determine the value of $\overline{\epsilon_{C_0}}$. To study this hole class, we first consider the double tree that is formed by taking the tree itself together with its reflection along the line $y = y_{\max}$, as in Figure 6.3.

Consider the hole that occurs around the point $(0, y_{\max})$ in the double tree. We will refer to it as the center hole, and denote it by D_0. The hole C_0 of the actual tree is similar to D_0 with contraction factor r_{sc}^4, since the relevant subtree is S_{RL^3}, which is similar to the tree with contraction factor r_{sc}^4. We will determine when the center hole in the double tree collapses, and this will in turn give us the value of $\overline{\epsilon_{C_0}}$.

It is clear that the point $(0, y_{\max})$ is the furthest away from the double tree out of all points in D_0. Let $\overline{\epsilon_{D_0}}$ be the collapse value of D_0. We wish to find a point on the double tree that has minimal distance to $(0, y_{\max})$, and then this distance will equal $\overline{\epsilon_{D_0}}$. Let P_1 denote the point $(0, y_{\max})$.

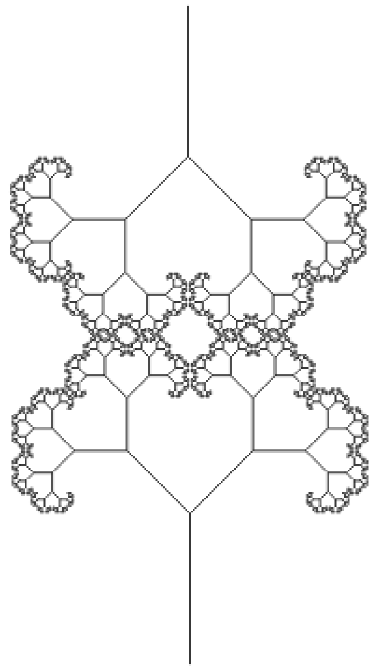

Figure 6.3: Double Tree of $T(r_{sc}, 45°)$

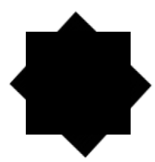

Figure 6.4: Canopy Holes Boundary Approximation for $T(r_{sc}, 45°)$

The border of D_0 can be approximated by a polygon of the form shown in Figure 6.4. Consider the subtree S_{RL}. The linear extension of the branch $b(RL)$ crosses the y-axis at P_1. The canopy endpoints of the degree 0 canopy interval of the subtree S_{RLL} are equidistant from P_1, and we claim that they have minimal distance to P_1. They are closer than any other top tip points of the subtree S_{RLL}. There are no points between the two canopy points that are closer, because they would be in the triangular region shown in Figure 6.4, and any circle that reaches a point there would have one of the canopy points inside.

Let the right endpoint of the degree 0 canopy interval of S_{RLL} (*i.e.* the point with address $RLL\mathbf{C}_R$) be denoted by $P_D = (x_D, y_D)$. For this tree we have

$$x_D = r_{sc}\sin 45^\circ - \frac{r_{sc}^3 \sin 45^\circ}{1 - r_{sc}^2} - \frac{r_{sc}^6}{1 - r_{sc}^2} \approx 0.12404$$

and

$$y_D = \frac{r_{sc}\cos 45^\circ}{1 - r_{sc}^2} + r_{sc}^2 + r_{sc}^4 \approx 1.12407$$

We now have that $\overline{\epsilon_{D_0}}$ equals the distance between P_1 and P_D, so

$$\overline{\epsilon_{D_0}} = \sqrt{x_D^2 + (y_D - y_{\max})^2} \approx 0.1412$$

and hence

$$\overline{\epsilon_{C_0}} = r_{sc}^4 \overline{\epsilon_{D_0}} \approx 0.0175. \tag{6.1.3}$$

Now we are able to completely determine the persistence intervals, the hole partition and the hole sequence for this tree.

6.1.2 Hole Partition and Hole Sequence

Proposition 6.1.2.1 *The level 0 critical ϵ-values for the self-contacting tree with $\theta = 45^\circ$ are:*

$$\begin{aligned} \underline{\epsilon_M} &= 0, \quad \overline{\epsilon_M} \approx 0.3227 \\ \underline{\epsilon_{C_0}} &= 0, \quad \overline{\epsilon_{C_0}} \approx 0.0175 \\ \underline{\epsilon_{C_j}} &= 0, \quad \overline{\epsilon_{C_j}} = r_{sc}^{2j}\overline{\epsilon_{C_0}} \quad j \geq 0 \end{aligned}$$

Proof. The contact values are all 0 because the hole classes are self-contacting hole classes. We discussed the first and second collapse values above, and gave geometric arguments for their values. For the value of $\overline{\epsilon_{C_j}}$, we can use the fact that a hole class C_j is obtained from the hole class C_0 via an address map that is of level $2j$. □

To order these critical values in decreasing order, we need to relate $\overline{\epsilon_{C_0}}$ to $\overline{\epsilon_M}$. For these values we have

$$r_{sc}^6 \overline{\epsilon_M} < \overline{\epsilon_{C_0}} < r_{sc}^5 \overline{\epsilon_M} \tag{6.1.4}$$

Now we can count the number of holes for all possible ϵ-ranges. In Table 6.1, the symbol " is used to indicate that the same classes of holes are present as in the

Angle Description	**Self-Contacting Addresses**	r_{sc}
Equivalence Class	Description of Holes Present	$N([\epsilon])$
$[\overline{\epsilon_M}, \infty)$		0
$[r_{sc}\overline{\epsilon_M}, \overline{\epsilon_M})$	4 M	1
$[r_{sc}^2\overline{\epsilon_M}, r_{sc}\overline{\epsilon_M})$	", 2 M^1	3
$[r_{sc}^3\overline{\epsilon_M}, r_{sc}^2\overline{\epsilon_M})$	" , 4 M^2	7
$[r_{sc}^4\overline{\epsilon_M}, r_{sc}^3\overline{\epsilon_M})$	", 8 M^3	15
$[r_{sc}^5\overline{\epsilon_M}, r_{sc}^4\overline{\epsilon_M})$	", 16 M^4	31
$[\overline{\epsilon_{C_0}}, r_{sc}^5\overline{\epsilon_M})$	",32 M^5	63
$[r_{sc}^6\overline{\epsilon_M}, \overline{\epsilon_{C_0}})$	", 1 C_0	64
$[r_{sc}\overline{\epsilon_{C_0}}, r_{sc}^6\overline{\epsilon_M})$	", 64 M^6	128
$[r_{sc}^7\overline{\epsilon_M}, r_{sc}\overline{\epsilon_{C_0}})$	", 2 C_0^1	130
$[r_{sc}^2\overline{\epsilon_{C_0}}, r_{sc}^7\overline{\epsilon_M})$	", 128 M^7	258
$[r_{sc}^8\overline{\epsilon_M}, r_{sc}^2\overline{\epsilon_{C_0}})$	",2 C_1	260

Table 6.1: Summary Of the first twelve persistence intervals and numbers of holes in the hole partition for $T(r_{sc}, 45°)$

previous line, along with whatever new class is given. In the description of the holes, we have counted the holes by level and type. The superscript denotes the level of the hole.

The first five non-trivial equivalence classes are given in Table 6.1. Any other equivalence class is of the form $[r_{sc}^j\overline{\epsilon_{C_0}}, r_{sc}^{5+j}\overline{\epsilon_M})$, where $j \geq 0$, or of the form $[r_{sc}^{6+j}\overline{\epsilon_M}, r_{sc}^j\overline{\epsilon_{C_0}})$, where $j \geq 0$. We will look at these two forms of classes, and we consider the cases when j is odd and when j is even.

First we will look at equivalence classes of the form $[r_{sc}^{6+2k}\overline{\epsilon_M}, r_{sc}^{2k}\overline{\epsilon_{C_0}})$, where $k \geq 0$.

- Holes of class M are present at levels 0 through $6+2k-1$, thus $N_M = 2^{6+2k} - 1$.

- Holes of class C_0 are present at levels 0 through $2k$ and $N_{C_0} = 2^{2k+1} - 1$.

- Holes of class C_1 are present at levels 0 through $2(k-1)$ and $N_{C_1} = 2(2^{2k-1} - 1)$.

- In general, for integers l such that $0 \leq l \leq k$, holes of type C_l are present at levels 0 through $2(k-l)$. We have $N_{C_l} = 2^l(2^{2k-2l+1} - 1) = 2^{2k-l+1} - 2^l$.

- There are no other classes of holes.

The total number of holes for this equivalence class is then

$$\begin{aligned} N([r_{sc}^{6+2k}\overline{\epsilon_M}, r_{sc}^{2k}\overline{\epsilon_{C_0}}]) &= 2^{6+2k} - 1 + \sum_{l=0}^{k}(2^{2k-l+1} - 2^l) \\ &= 2^{6+2k} - 1 + 2^{k+1}(2^{k+1} - 1) - (2^{k+1} - 1) \\ &= 2^{6+2k} + 2^{k+1}(2^{k+1} - 2) \end{aligned}$$

Next we will look at equivalence classes of the form $[r_{sc}^{6+2k+1}\overline{\epsilon_M}, r_{sc}^{2k+1}\overline{\epsilon_{C_0}}]$, where $k \geq 0$.

- Holes of class M are present at levels 0 through $6 + 2k$, thus $N_M = 2^{7+2k} - 1$.
- In general, for integers l such that $0 \leq l \leq k$, holes of class C_l are present at levels 0 through $2(k - l) + 1$. We have $N_{C_l} = 2^l(2^{2k-2l+2} - 1) = 2^{2k-l+2} - 2^l$.
- There are no other classes of holes.

The total number of holes for this equivalence class is then

$$\begin{aligned} N([r_{sc}^{6+2k+1}\overline{\epsilon_M}, r_{sc}^{2k+1}\overline{\epsilon_{C_0}}]) &= 2^{7+2k} - 1 + \sum_{l=0}^{k}(2^{2k-l+2} - 2^l) \\ &= 2^{7+2k} + 2^{k+1}(2^{k+2} - 3) \end{aligned}$$

Now we will look at equivalence classes of the form $[r_{sc}^{2k}\overline{\epsilon_{C_0}}, r_{sc}^{5+2k}\overline{\epsilon_M}]$, where $k \geq 1$.

- Holes of class M are present at levels 0 through $5 + 2k$, thus $N_M = 2^{6+2k} - 1$.
- In general, for integers l such that $0 \leq l \leq k - 1$, holes of class C_l are present at levels 0 through $2(k - l) - 1$. We have $N_{C_l} = 2^l(2^{2k-2l} - 1) = 2^{2k-l} - 2^l$.
- There are no other classes of holes.

The total number of holes for this equivalence class is then

$$\begin{aligned} N([r_{sc}^{2k}\overline{\epsilon_{C_0}}, r_{sc}^{5+2k}\overline{\epsilon_M}]) &= 2^{6+2k} - 1 + \sum_{l=0}^{k-1}(2^{2k-l} - 2^l) \\ &= 2^{6+2k} + 2^k(2^k - 2) \end{aligned}$$

Finally we will look at equivalence classes of the form $[r_{sc}^{2k+1}\overline{\epsilon_{C_0}}, r_{sc}^{5+2k+1}\overline{\epsilon_M}]$, where $k \geq 0$.

- Holes of class M are present at levels 0 through $5+2k+1$, thus $N_M = 2^{7+2k}-1$.
- In general, for integers l such that $0 \le l \le k$, holes of class C_l are present at levels 0 through $2(k-l)$. We have $N_{C_l} = 2^l(2^{2k-2l+1}-1) = 2^{2k-l+1}-2^l$.
- There are no other classes of holes.

The total number of holes for this equivalence class is then

$$\begin{aligned}
N([r_{sc}^{2k+1}\overline{\epsilon_{C_0}}, r_{sc}^{5+2k+1}\overline{\epsilon_M}]) &= 2^{7+2k}-1+\sum_{l=0}^{k}(2^{2k-l+1}-2^l) \\
&= 2^{7+2k}-1+2^{k+1}(2^{k+1}-1)-(2^{k+1}-1) \\
&= 2^{7+2k}+2^{k+1}(2^{k+1}-2)
\end{aligned}$$

We conclude:

Theorem 6.1.2.2 *For the self-contacting tree with branching angle* $45°$*, the equivalence classes with respect to hole congruence are completely determined, and thus the hole sequence is also completely determined. We have*

$$N_0 = 0, \quad N_1 = 1, \quad N_2 = 3, \quad N_3 = 7, \quad N_4 = 15, \quad N_5 = 31, \quad N_6 = 63 \tag{6.1.5}$$

and for $k \ge 0$*,*

$$\begin{aligned}
N_{7+4k} &= N([r_{sc}^{6+2k}\overline{\epsilon_M}, r_{sc}^{2k}\overline{\epsilon_{C_0}}]) = 2^{6+2k}+2^{k+1}(2^{k+1}-2) \\
N_{8+4k} &= N([r_{sc}^{2k+1}\overline{\epsilon_{C_0}}, r_{sc}^{5+2k+1}\overline{\epsilon_M}]) = 2^{7+2k}+2^{k+1}(2^{k+1}-2) \\
N_{9+4k} &= N([r_{sc}^{6+2k+1}\overline{\epsilon_M}, r_{sc}^{2k+1}\overline{\epsilon_{C_0}}]) = 2^{7+2k}+2^{k+1}(2^{k+2}-3) \\
N_{10+4k} &= N([r_{sc}^{2(k+1)}\overline{\epsilon_{C_0}}, r_{sc}^{5+2(k+1)}\overline{\epsilon_M}]) = 2^{6+2(k+1)}+2^{k+1}(2^{k+1}-2)
\end{aligned} \tag{6.1.6}$$

Remarks.

1. The hole sequence is order-isomorphic to the natural numbers.
2. The hole sequence is monotonically increasing.
3. For any sequence of ϵ-values where $\epsilon_n = r_{sc}\epsilon_{n-1}$, we have the following growth rate of holes:
$$\lim_{n\to\infty}\frac{\log(N([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r_{sc}}$$
4. The tree has infinite complexity because it is self-contacting and not space-filling.

6.2 The Golden Ratio and Self-contacting Trees

This section presents an interesting connection between self-contacting symmetric binary fractal trees and the famed 'golden ratio'.

There are four self-contacting trees that are quite special because of their unique symmetries and because each has a self-contacting ratio that is equal to the reciprocal of the golden ratio ϕ. These golden trees occur at the angles $60°, 108°, 120°$ and $144°$. We shall see that each of the golden trees "lines up" in some sense. Each of the trees lines up in a different way, so we will discuss the features for each tree separately.

6.2.1 Introduction to the Golden Ratio and the Golden Fractal Trees

The golden ratio, also known as the divine proportion, golden mean or golden section, is a number that is often encountered when taking ratios of distances in simple geometric figures. It appears in the pentagram, decagon and dodecagon. It is generally denoted ϕ, or sometimes τ. We use ϕ. See [10] and [57] for more information about the golden ratio.

Given a rectangle having sides in the ratio $1 : \phi$, ϕ is defined such that partitioning the original rectangle into a square and a new rectangle results in a new rectangle having sides with a ratio $1 : \phi$. See Figure 6.5. Thus ϕ satisfies

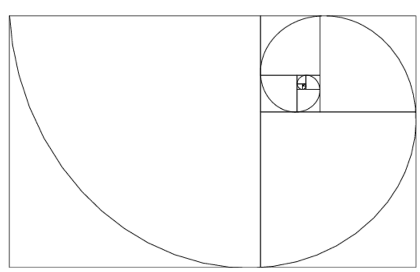

Figure 6.5: The Golden Rectangle

$$\phi^2 - \phi - 1 = 0$$

and hence

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

We also have

$$1 - \left(\frac{1}{\phi}\right) - \left(\frac{1}{\phi^2}\right) = 0. \tag{6.2.1}$$

ϕ can be considered to be the most 'irrational' number because it has a continued fraction representation

$$\phi = [1, 1, 1, \cdots] = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots}}}$$

ϕ is related to the Fibonacci numbers F_n:

$$\phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

There are many other interesting aspects of the golden ratio ϕ. Many people, including the ancient Greeks and Egyptians, found ϕ to be the most aesthetically pleasing ratio. It was often used in building monuments, such as the Parthenon.

While investigating various images of symmetric binary fractal trees, we found two particular self-contacting trees that possess remarkably nice geometrical properties. These two branching angles are 60° and 120°. It can easily be shown that the self-contacting scaling ratio for both these two angles is $1/\phi$. This led us to wonder if there were any other self-contacting trees with scaling ratio equal to $1/\phi$. One can use the results from [31] to prove that there are indeed two other such branching angles. From Figure 3.6, we see that one of the angles is between 90° and 135°, and the other is between 135° and 180°. We determined that the exact angles are 108° and 144°. In the next four subsections we will look at each 'golden tree' separately. We will highlight their nice geometrical properties, and discuss various aspects of their closed ϵ-neighbourhoods.

Because the scaling ratio $r_{sc} = 1/\phi$, this immensely simplifies many of the geometrical calculations. In particular, we have

$$1 - r_{sc}^2 = r_{sc} \tag{6.2.2}$$

We often encounter $1 - r_{sc}^2$ in the denominator, so this equation is helpful.

6.2.2 Golden 60

When the branching angle θ equals $60°$, from equation (3.3.1) the self-contacting ratio r_{sc} must satisfy

$$r_{sc}\sin 60° - \frac{r_{sc}^3 \sin 60°}{1-r_{sc}^2} - \frac{r_{sc}^4 \sin 120°}{1-r_{sc}^2}$$

thus

$$(1+r_{sc})(1-r_{sc}-r_{sc}^2)=0$$

and we recognize that the solution between 0 and 1 is

$$r_{sc}=\frac{-1+\sqrt{5}}{2}=\frac{1}{\phi}$$

Figure 6.6 displays an image of $T(1/\phi, 60°)$.

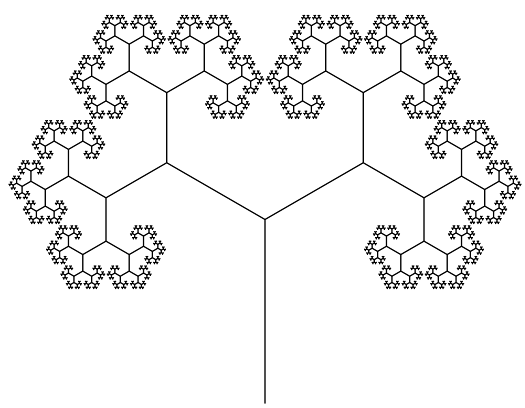

Figure 6.6: $T(1/\phi, 60°)$

Special Geometrical Properties:

1. This tree is unique in its angle range because the line through the maximal height tip points of the subtree S_{RLL} is coincident with the line through the maximal height tip points of the subtree S_{LRRR}.

2. There is a Koch-like curve that is a subset of this tree. A line segment of length a is replaced by two of length a/ϕ^2 on the ends, and two of length a/ϕ^3 in the middle, instead of all having length $a/3$ with the usual Koch curve. The construction of this subset is as follows. Start with the line segment through the top corner points (so between the point with address $(RL)^\infty$ and the point

with address $(LR)^\infty$. The generator for the fractal is the curve formed from four line segments: from the right corner point to the left corner point of the subtree S_{RL} (with address $RL(LR)^\infty$, from this point to the self-contact point with address $RL^3(RL)^\infty$, and the mirror images of these two line segments. We can think of this curve as being a 'golden' Koch curve.

3. The branches $b(RLLL)$ and $b(LL)$ have the same linear extension, so there are infinitely many branches with the same linear extension. Any branch of level 2 or higher shares its linear extension with infinitely many other branches.

We will now show that this tree has holes of the main type and the secondary contact.

Proposition 6.2.2.1 *The tree $T(r_{sc}, 60°)$ has no canopy holes.*

Proof. First we will consider subtrees above the trunk, beginning with the subtree S_{RL^2}. The degree 0 canopy interval of this subtree is given by the pair $RLL\mathbf{C}_L$ and $RLL\mathbf{C}_R$. Let $P_a = (x_a, y_a)$ be the point with address $RLL\mathbf{C}_R$, and let $P_b = (x_b, y_b)$ be the point with address $RLL\mathbf{C}_L$. Using some basic geometry and the fact that $1 - r_{sc}^2 = r_{sc}$, we have

$$\begin{aligned}
x_a &= r_{sc}\sin\theta - \frac{r_{sc}^3 \sin\theta}{1 - r_{sc}^2} - \frac{r_{sc}^6 \sin(2\theta)}{1 - r_{sc}^2} \\
&= r_{sc}\frac{\sqrt{3}}{2} - r_{sc}^2\frac{\sqrt{3}}{2} - r_{sc}^5\frac{\sqrt{3}}{2} \\
&= r_{sc}\frac{\sqrt{3}}{2}(1 - r_{sc} - r_{sc}^4) \\
&= r_{sc}\frac{\sqrt{3}}{2}(r_{sc}^2 - r_{sc}^4) \\
&= r_{sc}^3\frac{\sqrt{3}}{2}(1 - r_{sc}^2) \\
&= r_{sc}^4\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{3}}{2\phi^4} \\
&\approx 0.1263514 \qquad (6.2.3)
\end{aligned}$$

Similarly, one can show that:

$$\begin{aligned} y_a &\approx 1.982779 \\ x_b &= r_{sc}^5 \frac{\sqrt{3}}{2} \approx 0.0780895 \\ y_b &\approx 1.899187 \end{aligned} \tag{6.2.4}$$

There can only be a hole located by this pair if the point $P_c = (0, y_a)$ is more distant than x_a from the point P_b. The distance d between P_c and P_b is given by

$$d^2 = x_b^2 + (y_a - y_b)^2 \tag{6.2.5}$$

Thus $d \approx 0.11439 < x_a$. Hence P_c is closer to P_b than to P_a, and thus the pair of points are not hole locators.

Now we will show that this implies that no other canopy pair of this subtree are hole locators either.

Consider a pair of the form $RLL\mathbf{A}'\mathbf{C}_R$ and $RLL\mathbf{A}'\mathbf{C}_L$, where $\mathbf{A}' \in \mathcal{AL}_{2k}$ for some $k \geq 1$. Let $P'_a = (x'_a, y'_a)$ be the point with address $RLL\mathbf{A}'\mathbf{C}_R$ and let $P'_b = (x'_b, y'_b)$ be the point with address $RLL\mathbf{A}'\mathbf{C}_L$. The canopy interval specified by this pair is similar to the degree 0 canopy interval specified by P_a and P_b, with contraction factor r_{sc}^{2k}, and also has the same slope. Thus $y'_a > y'_b$ and $x'_a \geq x'_b$. This implies that $y'_a - y'_b = r_{sc}^{2k}(y_a - y_b)$. It is not necessarily true that $x'_b = r_{sc}^{2k} x_b$, that only occurs if $\mathbf{A}' = (LR)^k$. We do have $x'_b \geq r_{sc}^{2k} x_b$. If $x'_b = C + r_{sc}^{2k} x_b$ for some $C > 0$, then $x'_a = C + r_{sc}^{2k} x_a$ (since $x'_a - x'_b = r_{sc}^{2k}(x_a - x_b)$). Let P'_c be the point $(0, y'_a)$, and let d' be the distance between P'_c and P'_b. Then

$$\begin{aligned} (d')^2 &= (x'_b)^2 + (y'_a - y'_b)^2 \\ &= (C + r_{sc}^{2k} x_b)^2 + (y'_a - y'_b)^2 \\ &= C^2 + 2Cr_{sc}^{2k} x_b + r_{sc}^{4k} x_b^2 + r_{sc}^{4k}(y_a - y_b)^2 \\ &= C^2 + 2Cr_{sc}^{2k} x_b + r_{sc}^{4k} d^2 \\ &\leq C^2 + 2Cr_{sc}^{2k} x_b + r_{sc}^{4k} x_a^2 \\ &\leq C^2 + 2Cr_{sc}^{2k} x_a + r_{sc}^{4k} x_a^2 \\ &= (C + r_{sc}^{2k} x_a)^2 \\ &= (x'_a)^2 \end{aligned} \tag{6.2.6}$$

Hence the pair cannot locate a hole.

The other subtree we need to check is S_{RL^3}. This subtree is such that the top has negative slope. For $\theta = 60°$, it turns out that the top of the subtree S_{RL^3} meets the y-axis with an angle of $30°$. The top of the subtree S_{RLL} also meets the y-axis with an angle of $30°$. So the degree 0 canopy interval of S_{RL^3} is similar to degree 0 canopy interval of S_{RLL}. Moreover, the degree 0 canopy interval of S_{RL^3} has the same magnitude of slope as degree 0 canopy interval of S_{RLL}, hence the endpoints of this interval cannot be hole locators. Thus there are no canopy pairs of S_{RL^3} that are hole locators.

Finally we consider canopy pairs below the line $y = 1$. The only possible subtrees that have local minima are S_{R^5} and S_{R^4}. From Figure 6.6, it should be easy to see that the canopy pairs of S_{R^5} are not hole locators. For S_{R^4}, the top of the subtree is $30°$ away from being vertical, so it can't contain hole locator pairs because the subtrees S_{RLL} and S_{RLLL} do not. $\square$

The main level 0 hole class is identified by the pair $(\mathbf{A}_0, RL^3(RL)^\infty)$ and has a contact value of 0. The only other possible hole locator pair to consider is the secondary contact pair $(\mathbf{A}_0, R^5(LR)^\infty)$. We claim that this pair is indeed a hole locator. Let $P_1 = (x_1, y_1)$ be the secondary contact point (the point with address $R^5(LR)^\infty$). Let P_2 be the point $(x_1/2, y_1)$. If P_2 is more than $x_1/2$ away from the branch $b(R)$, then there is a hole for $\epsilon = x_1/2$. Let d_1 be the distance between P_2 and $b(R)$. One can show that

$$x_1 \approx 0.53523$$

$$x_1/2 \approx 0.26762 \tag{6.2.7}$$

$$y_1 \approx 1 - 0.16312$$

$$d_1 \approx 0.27507 \tag{6.2.8}$$

Since $d_1 > x_1/2$, there is indeed a hole at $\epsilon = x_1/2$.

Let M denote the main type hole class located by $(\mathbf{A}_0, RL^3(LR)^\infty)$, and let S denote the hole class located by $(\mathbf{A}_0, R^5(LR)^\infty)$. Now we will show that M has persistence equal to 0. At $\epsilon = r_{sc}^k \underline{\epsilon_S}$, for any $k \geq 1$, the region of M is split into a level 0 main hole and a level k secondary contact hole. Let M_k denote the remaining main mixed hole that has a contact value of $r_{sc}^k \underline{\epsilon_S}$. Then for $k \geq 2$, the collapse value is $r_{sc}^{k-1} \underline{\epsilon_S}$. For M_1, the collapse value is when the entire region is covered by the closed ϵ-neighbourhood. Thus the only critical values that we need to determine are $\overline{\epsilon_S}$ and $\overline{\epsilon_{M_1}}$. The nice geometry of this tree becomes evident here, because we can show that the collapse values of these two hole classes are actually the same.

Recall that $P_{c1} = (x_{c1}, y_{c1})$ denotes the self-contact point with address $RL^3(RL)^\infty$.

First we will determine $\overline{\epsilon_{M_1}}$, then we will show that this must equal $\overline{\epsilon_S}$. The collapse value of the M_1 hole class corresponds to the smallest ϵ-value such that the region of the y-axis given by $\mathbf{y}_{[0,y_{c1}]}$ is covered by the ϵ-neighbourhood. So consider points $(0, y)$ in this region. If y is sufficiently small, then the the closest point on the subtree S_R is on the branch $b(R)$. As y increases, there is a minimal value where the closest point on the subtree S_R is no longer on the branch $b(R)$. Similar to the tree $T(r_{sc}, 45°)$, this point is the left corner point of the subtree whose top is on the boundary of the original main hole M, which is the subtree S_{RLLL}.

Let $P_3 = (x_3, y_3)$ denote this corner point (with address $RLLL(LR)^\infty$). Similar to the tree $T(r_{sc}, 45°)$, we find the unique value of y such that the point $(0, y)$ is equidistant from the branch $b(R)$ and the point P_3, then find this distance. We do not give the calculations, because they are similar to previous calculations that we have presented. We find that the collapse value is

$$\overline{\epsilon_{M_1}} \approx 0.26769 \tag{6.2.9}$$

This value is only slightly higher than the contact value for the S class (which is equal to $x_1/2$ given above in 6.2.8), and we might wonder if they aren't actually the same. However, some geometry shows that $\overline{\epsilon_{M_1}}$ is actually the collapse value of the S hole class (so this class has small persistence compared to the M hole class).

We have the point P_1 with address $R^5(LR)^\infty$ and the point P_3 with address $RL^3(LR)^\infty$. Let $P_4 = (0, y_4)$ be the point on the y-axis that is equidistant from P_3 and $b(R)$. Let $P_5 = (x_5, y_5)$ be the point that is obtained by reflecting P_4 across the branch $b(R)$. We also have that P_1 can be obtained by reflecting P_3 across $b(R)$. So P_5 must be equidistant from P_1 and $b(R)$ also. The reflection of the y-axis across $b(R)$

gives the linear extension of the branch $b(L)$, denote this line λ. The angle between $\mathbf{y}$ and $b(R)$ is $60°$, hence the angle between $b(R)$ and λ is also $60°$. Finally, the angle between λ and the trunk is also $60°$. The distance between P_5 and the trunk, *i.e.* x_5, equals the distance between P_4 and $b(R)$, *i.e.*, $\overline{\epsilon_M}$. So P_5 is equidistant from the trunk, $b(R)$ and P_1. Thus $\overline{\epsilon_S} = \overline{\epsilon_{M_1}}$.

Now we can give the hole partition and hole sequence. We will not go into the same details as with the tree $T(r_{sc}, 45°)$. We have

$$\begin{aligned} \underline{\epsilon_{M_1}} &= r_{sc}\underline{\epsilon_S}, & \overline{\epsilon_{M_1}} &\approx 0.26769 \\ \underline{\epsilon_S} &\approx 0.26762, & \overline{\epsilon_S} &= \overline{\epsilon_{M_1}} \end{aligned} \tag{6.2.10}$$

Thus

$$\begin{aligned} N_0 &= N([\overline{\epsilon_{M_1}}, \infty]) = 0 \\ N_1 &= N([\underline{\epsilon_S}, \overline{\epsilon_{M_1}})) = 3 \text{ (1 main, 2 secondary)} \\ N_2 &= N([r_{sc}\overline{\epsilon_{M_1}}, \underline{\epsilon_S})) = 1 \text{ (1 main)} \\ N_3 &= N([r_{sc}\underline{\epsilon_S}, r_{sc}\overline{\epsilon_{M_1}})) = 7 \text{ (3 main, 4 secondary)} \\ N_4 &= N([r_{sc}^2\overline{\epsilon_{M_1}}, r_{sc}\underline{\epsilon_S})) = 3 \text{ (3 main)} \end{aligned}$$

and in general:

$$\begin{aligned} N_{2j+1}\ (j \geq 0) &= N([r_{sc}^j\underline{\epsilon_S}, r_{sc}^j\overline{\epsilon_{M_1}})) = 2^{j+2} - 1 \\ N_{2j}\ (j \geq 1) &= N([r_{sc}^j\overline{\epsilon_{M_1}}, r_{sc}^{j-1}\underline{\epsilon_S})) = 2^j - 1 \end{aligned}$$

Remarks.

1. The hole sequence is order-isomorphic to the natural numbers.

2. The hole sequence is not monotonically increasing.

3. For any sequence of ϵ-values where $\epsilon_n = r_{sc}\epsilon_{n-1}$, we have the following growth rate:

$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r_{sc}}$$

6.2.3 Golden 108

When the branching angle θ equals $108°$, equation (3.3.8) tells us that the self-contacting ratio r_{sc} must satisfy

$$r_{sc} = \frac{-\cos 108° - \sqrt{2 - \cos^2 108°}}{4\cos^2 108° - 2}$$

Using the relationship given by

$$\cos 108° = \frac{-1}{2\phi},$$

it is straightforward to prove that $r_{sc}(108°) = 1/\phi$. An image of the tree is displayed in Figure 6.7.

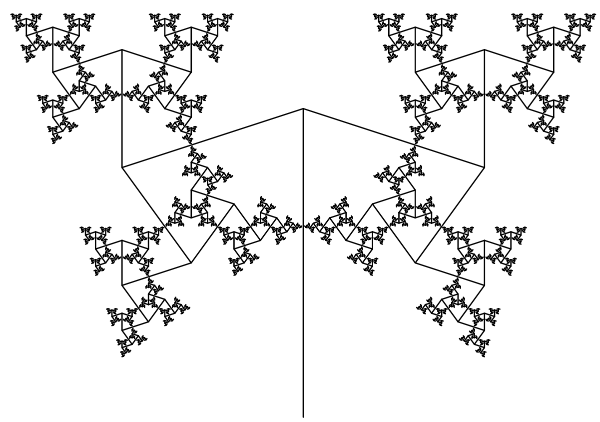

Figure 6.7: $T(1/\phi, 108°)$

Special Geometrical Properties:

1. The tip points with addresses $RRR(LR)^\infty$ and $RRR(RL)^\infty$ belong to the trunk and the branch $b(R)$, respectively, and they are equidistant from the point $(0, 1)$. This distance is equal to r_{sc}^2. Thus the line segment through the top of the subtree S_{RRR} forms an isosceles triangle with portions of the trunk and the branch $b(R)$.

2. The branches $b(R)$ and $b(LRR)$ have the same linear extension. Hence any branch of the form $b((LR)^k R)$ shares this same linear extension.

3. The line segment obtained by extending the branch $B(RR)$ to the y-axis meets the y-axis at the origin. Moreover, it has length 1, and together with the trunk and the branch $b(R)$, forms another isosceles triangle with sides of length $1, 1, 1/\phi$.

4. The linear extension of the branch $b(RRR)$ meets the y-axis at $(0, 1)$, bisects the angle between the trunk and $b(R)$, and meets the point with address $LR(RL)^\infty$ (the left endpoint of the degree 0 canopy interval of the tree).

5. The lines through the tops of the four subtrees S_{RLL}, S_{RRR}, S_{LRR} and S_{LLL}, along with the degree 0 canopy interval of the tree, form a regular pentagon with sides of length $2\sin(108°)/\phi^2$.

6. The top lines of the subtrees S_{RR} and S_{RRR} both meet the trunk with an acute angle of $54°$.

This tree has main and secondary contact hole types. The main level 0 hole locator pair is $(\mathbf{A}_0, RRR(LR)^\infty)$. The secondary contact hole locator pair is $(\mathbf{A}_0, RL(LR)^\infty)$.

Why are there no canopy holes? First consider the subtree S_{RLL}. The linear extension of the branch $b(RLL)$ meets the y-axis at the top of the trunk. Now consider the points P_a with address $RLL\mathbf{C}_R$ and $P_b = (x_b, y_b)$ with address $RLL\mathbf{C}_L$. Any point on the linear extension of $b(RLL)$ is equidistant from P_a and P_b, and above this linear extension is closer to P_a. In particular, the point $(0, y_b)$ is closer to P_a than P_b, which means that P_a and P_b cannot locate a hole. The same is true for any other canopy pair of this subtree. We could use a similar argument to show that the same is true for canopy pairs of either S_{RRR} or S_{RR}.

Consider the main hole class identified by the pair $(\mathbf{A}_0, RRR(LR)^\infty)$. Let M denote this hole class. The secondary contact hole class is identified by the pair $(\mathbf{A}_0, RL(LR)^\infty)$. Let S denote this hole class. Just as with the tree $T(r_{sc}, 60°)$, because we have the secondary contact holes, the main hole M has persistence 0. Let M_k denote the main mixed hole class that has contact value $r_{sc}^k \underline{\epsilon_S}$ for $k \geq 1$. Then the collapse value of M_k is the contact value of M_{k-1}, for $k \geq 2$. The collapse value of M_1 is when the entire region is covered. To estimate $\overline{\epsilon_{M_1}}$, we determine when a point on the linear extension of $b(RRR)$ is equidistant from the trunk and the left canopy point of the subtree S_{RRR}, so with address $RRR\mathbf{C}_R$. Let $P_1 = (x_1, y_1)$ denote such a point on the linear extension. Any point on the linear extension is equidistant from the trunk and $b(R)$. The linear extension is given by $y = -\tan(54°)x + 1$, since it goes through $(0, 1)$ and meets the trunk with an angle of 36°. When $\epsilon = x_1$, the closed ϵ-neighbourhood of the trunk and the branch $b(R)$ will intersect in the point $(x_1, -\tan(54°)x_1)$, and everything in the region of the hole class will be within ϵ to the tree. If $P_2 = (x_2, y_2)$ is the point with address $RRRLR(RL)^\infty$, then we find x_1 by equating the distances:

$$(x_2 - x_1)^2 + [y_2 - (-\tan(54°)x + 1]^2 = x_1^2$$

which gives $x_1 \approx 0.119006$, and this is the value we will take for $\epsilon_{M_1}^{gc}$.

Now we consider the collapse value of S. Let $P_3 = (x_3, y_3)$ be the point with address $RL(LR)^\infty$. To show how the golden ratio simplifies such calculations, we present the details for this point:

$$\begin{aligned} x_3 &= r_{sc}\sin\theta - \frac{r_{sc}^3\sin\theta}{1 - r_{sc}^2} \\ &= r_{sc}\sin\theta - r_{sc}^2\sin\theta \\ &= r_{sc}^3\sin\theta \approx 0.224514 \end{aligned} \tag{6.2.11}$$

and

$$\begin{aligned} y_3 &= \frac{r_{sc}\cos\theta + r_{sc}^2}{1 - r_{sc}^2} \\ &= \frac{r_{sc}\cos\theta + r_{sc}^2}{r_{sc}} \\ &= \cos\theta + r_{sc} \\ &= \frac{r_{sc}}{2} \approx 0.309017 \end{aligned} \tag{6.2.12}$$

The branch $b(R)$ has negative slope, so for any point on the y-axis above the trunk, the closest point on $b(R)$ is just $(0, 1)$. To estimate $\overline{\epsilon_S}$, we will determine when a point on the y-axis is equidistant from $(0, 1)$ and P_3. When ϵ is equal to the distance between such a point and $(0, 1)$, then the entire region $\mathbf{y}_{[1,y_3]}$ is within ϵ of the tree and there can be no hole left in the hole class. Let $(0, 1+\epsilon)$ be a point that is ϵ away from P_3. Then

$$x_3^2 + [y_3 - (1+\epsilon)^2] = \epsilon^2$$

which gives

$$\epsilon = \frac{x_3^2 + y_3^2}{2y_3} \approx 0.236607$$

So we use $\epsilon_S^{gc} \approx 0.236607$, because it is sufficient to determine the relationship between $\overline{\epsilon_{M_1}}$ and the other critical values.

To summarize, we have:

$$\begin{array}{c} \underline{\epsilon_{M_1}} = r_{sc}^2 \underline{\epsilon_S}, \quad \overline{\epsilon_{M_1}} < \epsilon_{M_1}^{gc} 0.119006 \\ \underline{\epsilon_S} = 0.224514, \quad 0.224514 < \overline{\epsilon_S} < \epsilon_S^{gc} \approx 0.236068 \end{array} \tag{6.2.13}$$

This implies

$$r_{sc}\overline{\epsilon_{M_1}} < r_{sc}^2 \underline{\epsilon_S} < r_{sc}^2 \overline{\epsilon_S} < \overline{\epsilon_{M_1}} \tag{6.2.14}$$

Thus

$$\begin{aligned} N_0 &= N([\overline{\epsilon_S}, \infty]) = 0 \\ N_1 &= N([\underline{\epsilon_S}, \overline{\epsilon_S})) = 1 \text{ (level 0 secondary)} \\ N_2 &= N([r_{sc}\overline{\epsilon_S}, \underline{\epsilon_S})) = 0 \\ N_3 &= N([r_{sc}\underline{\epsilon_S}, r_{sc}\overline{\epsilon_S})) = 2 \text{ (level 1 secondary)} \\ N_4 &= N([\overline{\epsilon_{M_1}}, r_{sc}\underline{\epsilon_S})) = 0 \\ N_5 &= N([r_{sc}^2\overline{\epsilon_S}, \overline{\epsilon_{M_1}})) = 2 \text{ (level 0 main)} \\ N_6 &= N([r_{sc}^2\underline{\epsilon_S}, r_{sc}^2\overline{\epsilon_S})) = 6 \text{ (4 secondary, 2 main)} \end{aligned} \tag{6.2.15}$$

and in general:

$$\begin{aligned} N_{5+3k}\ (k \geq 0) &= N([r_{sc}^{k+2}\overline{\epsilon_S}, r_{sc}^{k}\overline{\epsilon_{M_1}})) = 2^{k+2} - 2 \\ N_{6+3k}\ (k \geq 0) &= N([r_{sc}^{k+2}\underline{\epsilon_S}, r_{sc}^{k+2}\overline{\epsilon_S})) = 2^{k+3} - 2 \\ N_{7+3k}\ (k \geq 0) &= N([r_{sc}^{k+1}\overline{\epsilon_{M_1}}, r_{sc}^{k+2}\overline{\epsilon_S})) = 2^{k+2} - 2 \end{aligned} \tag{6.2.16}$$

Remarks.

1. The hole sequence is order-isomorphic to the natural numbers.

2. The interesting thing to note here is that in the closed ϵ-neighbourhood for $\epsilon = r_{sc}^2 \epsilon_{\underline{S}}$, there are holes in level 2 and level 0, but no holes of level 1. This provides a counter-example to an early conjecture that it is not possible to have a discontinuity in the levels of holes possible for any specific ϵ.

3. For any sequence of ϵ-values where $\epsilon_n = r_{sc}\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r_{sc}}$$

6.2.4 Golden 120

When the branching angle θ equals 120°, from equation (3.3.6), we have that the self-contacting ratio r_{sc} must satisfy

$$r_{sc} \sin 120° + \frac{r_{sc}^2 \sin 240°}{1 - r_{sc}^2} + \frac{r_{sc}^3 \sin 360}{1 - r_{sc}^2} = 0$$

Hence

$$1 - r_{sc} - r_{sc}^2 = 0$$

and $r_{sc}(120°) = 1/\phi$. An image of the tree is displayed in Figure 6.8

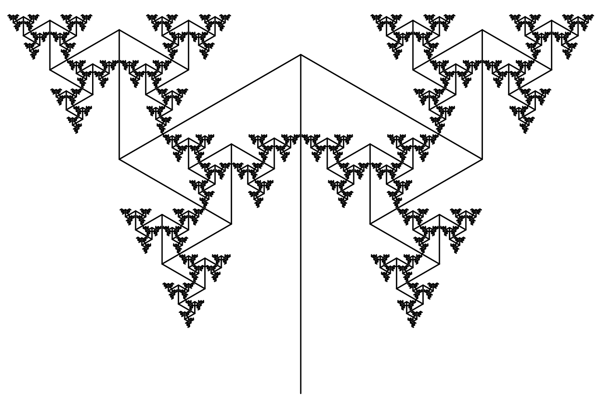

Figure 6.8: $T(1/\phi, 120°)$

Special Geometrical Properties:

1. The line through the tip points of maximal height of the subtree S_{RRR} is horizontal, and it forms a special 30-60-90 triangle with the trunk and the branch $b(R)$.

2. The line segments through the tip points of maximal height of the subtrees S_{RRR} and S_{RLL} form 2 sides of an equilateral triangle, with the other side being a line segment that is on the same line as the top of the subtree S_{LL}.

3. The linear extension of the branch $b(L)$ meets the point with address $RL(LR)^\infty$ and is perpendicular to the top of the subtree S_{RLL}.

4. The line segment starting at the origin, through the point with address $R(RL)^\infty$ (the lowest tip point on the subtree S_R) and up to where it intersects with the linear extension of the branch $b(R)$, forms one side of an equilateral triangle. The other sides are the trunk and part of the linear extension of $b(R)$, so each side has length 1.

The main hole is located by the pair $(\mathbf{A}_0, RRR(LR)^\infty)$. There are no secondary contact holes, because the point $P_s = (x_s, y_s)$ with address $RL(LR)^\infty$ is such that $y_s - 1 > x_s$, so at $\epsilon = x_c$, the interval of the y-axis between the top of the trunk and $(0, y_c)$ is covered by the ϵ-neighbourhood. This tree has mixed holes (located by a pair that correspond to a vertex and canopy point). The pair $(RR, RR\mathbf{C}_L)$ are the hole locator pair for the largest level 0 mixed hole class. Any pair of the form $(RR(RL)^k, RR(RL)^k\mathbf{C}_L)$ are also hole locators. There are other mixed level 0 hole classes, that are located by $\mathbf{A}_0$ and level 1 mixed hole locators. We will discuss this after first discussing the level 0 mixed hole classes.

Let M denote the main hole class. The maximal hole of this class is bounded by part of the trunk, part of $b(R)$, and part of the subtree S_{RRR}. The interesting thing about this hole class is that it has 0 persistence, as with the main holes for the previous two golden trees. To see this, we have to look at the mixed holes first.

Let V_k, for $k \geq 0$, denote the hole class located by the pair $(RR(RL)^k, RR(RL)^k\mathbf{C}_L)$. By the scaling nature of the tree, we have

$$\underline{\epsilon_{V_k}} = r_{sc}^{2k}\underline{\epsilon_{V_0}}, \quad \text{and} \quad \overline{\epsilon_{V_k}} = r_{sc}^{2k}\overline{\epsilon_{V_0}} \tag{6.2.17}$$

For any $\delta > 0$, there is $k \geq 0$ such that

$$\underline{\epsilon_{V_k}} < \overline{\epsilon_{V_k}} < \delta \tag{6.2.18}$$

Now consider the subtree S_R. The critical values of the corresponding mixed hole classes are equal to the critical values of the level 0 mixed hole classes all scaled by a factor of r_{sc}. This means that the main hole class M must have persistence equal to 0, because for any $\delta > 0$, there are critical values less than δ that result in the hole class splitting. So the original main level 0 hole class splits into level 1 mixed holes and level 0 mixed holes (located by $\mathbf{A}_0$ and some vertex of the subtree S_{RRR}). Let MV_k denote the hole class that is the remaining part of the M hole class and has a contact value of $r_{sc}\underline{\epsilon_{V_k}}$ (so the contact value of the level 1 mixed hole class V_k). We refer to this class as the main mixed class. For $k \geq 1$, the collapse value of MV_k is $r_{sc}\underline{\epsilon_{V_{k-1}}}$, because that is when the hole class splits into the level 1 hole class V_{k-1} and the level 0 hole class MV_{k-1}. For $k = 0$, we find an upper bound. Our estimate is based on finding when there is a point within the hole class MV_0 that is equidistant from the trunk, the branch $b(R)$, and the tip point $RRRLR\mathbf{C}_R$ (the right degree 0 canopy point of the subtree S_{RRRLR}). Then the distance from this point to the trunk will be an upper bound, because every other point in the region is closer to the tree, so would be covered by the corresponding closed ϵ-neighbourhood. Without showing the calculations, this gives

$$\epsilon^{gc}_{MV_0} \approx 0.086615 \tag{6.2.19}$$

The only critical values left to determine or estimate are the critical values for the hole classes V_k. Consider the hole class V_0, located by RR and $RR\mathbf{C}_L$. Let $P_1 = (x_1, y_1)$ be with address RR and $P_2 = (x_2, y_2)$ be with address $RR\mathbf{C}_L$. Then

$$x_1 = x_2 = \frac{\sqrt{3}}{2} r_{sc}^3 \approx 0.2044409, \quad y_1 = \frac{1}{2} \tag{6.2.20}$$

The value of $\underline{\epsilon_{V_0}}$ is half the value of x_1. This gives

$$\underline{\epsilon_{V_0}} = \frac{\sqrt{3}}{4} r_{sc}^3 \approx 0.1022204 \tag{6.2.21}$$

The distance between the point $(0,1)$ and the tip point with address $LR(RL)^\infty$ can be shown to be equal to r_{sc}^3, hence $y_1 - y_2 = r_{sc}^5$ (since they are the corresponding points on a level 2 subtree). To find an upper bound for $\overline{\epsilon_{V_0}}$, we determine when a point of the form $(\epsilon, \frac{y_1+y_2}{2})$ is ϵ away from P_1. Then all points in the region of the hole class V_0 would be within ϵ of the tree. To solve for such an ϵ, we have

$$(x_1 - \epsilon)^2 + \left(y_1 - \left[\frac{y_1 + y_2}{2}\right]\right)^2 = \epsilon^2 \tag{6.2.22}$$

Thus

$$\epsilon = \frac{x_1}{2} + \frac{r_{sc}^7}{4\sqrt{3}} \tag{6.2.23}$$

So the upper bound for $\overline{\epsilon_{V_0}}$ we have is

$$\epsilon_{V_0}^{gc} = \underline{\epsilon_{V_0}} + \frac{r_{sc}^7}{4\sqrt{3}} \approx 0.1071922 \tag{6.2.24}$$

In general, for $k \geq 0$, we have

$$\underline{\epsilon_{V_k}} = \frac{\sqrt{3}}{4} r_{sc}^{3+2k}, \quad \epsilon_{V_k}^{gc} = \frac{\sqrt{3}}{4} r_{sc}^{3+2k} + \frac{r_{sc}^{7+2k}}{4\sqrt{3}} \tag{6.2.25}$$

The critical values satisfy

$$r_{sc}\overline{\epsilon_{V_0}} < \overline{\epsilon_{MV_0}} < \underline{\epsilon_{V_0}} \tag{6.2.26}$$

So for the hole partition and hole sequence we have:

$$\begin{aligned}
N_0 &= N(\overline{\epsilon_{V_0}}, \infty]) = 0 \\
N_1 &= N(\underline{\epsilon_{V_0}}, \overline{\epsilon_{V_0}}) = 2 \text{ (level 0 mixed)} \\
N_2 &= N(\overline{\epsilon_{MV_0}}, \underline{\epsilon_{V_0}}) = 0 \\
N_3 &= N(r_{sc}\overline{\epsilon_{V_0}}, \overline{\epsilon_{MV_0}}) = 2 \text{ (level 0 main mixed)} \\
N_4 &= N(r_{sc}\underline{\epsilon_{V_0}}, r_{sc}\overline{\epsilon_{V_0}}) = 4 \text{ (2 level 0 main mixed and 2 level 1 mixed)}
\end{aligned}$$

In general, for $k \geq 1$:

- For intervals of the form $[r_{sc}^k \overline{\epsilon_{MV_0}}, r_{sc}^k \underline{\epsilon_{V_0}})$, there are main mixed holes of levels 0 through $k-1$, and
$$N([r_{sc}^k \overline{\epsilon_{MV_0}}, r_{sc}^k \underline{\epsilon_{V_0}})) = 2^{k+1} - 2 \tag{6.2.27}$$

- For intervals of the form $[r_{sc}^{k+1} \overline{\epsilon_{V_0}}, r_{sc}^k \overline{\epsilon_{MV_0}})$, there are main mixed holes of levels 0 through k, and
$$N([r_{sc}^{k+1} \overline{\epsilon_{V_0}}, r_{sc}^k \overline{\epsilon_{MV_0}})) = 2^{k+2} - 2 \tag{6.2.28}$$

- For intervals of the form $[r_{sc}^{2k}\underline{\epsilon_{V_0}}, r_{sc}^{2k}\overline{\epsilon_{V_0}})$, there are main mixed holes of levels 0 through $2k-1$ and mixed holes in even levels 0 through $2l$. We have

$$N([r_{sc}^{2k}\underline{\epsilon_{V_0}}, r_{sc}^{2k}\overline{\epsilon_{V_0}})) = 2^{2k+1} - 2 + \frac{2}{3}(4^{k+1} - 1) \qquad (6.2.29)$$

- For intervals of the form $[r_{sc}^{2k+1}\underline{\epsilon_{V_0}}, r_{sc}^{2k+1}\overline{\epsilon_{V_0}})$, there are main mixed holes of levels 0 through $2k$, and mixed holes in odd levels 1 through $2k+1$. We have

$$N([r_{sc}^{2k+1}\underline{\epsilon_{V_0}}, r_{sc}^{2k+1}\overline{\epsilon_{V_0}})) = 2^{2k+2} - 2 + \frac{4}{3}(4^{k+1} - 1) \qquad (6.2.30)$$

Remarks.

1. The hole sequence is order-isomorphic to the natural numbers.

2. For any sequence of ϵ-values where $\epsilon_n = r_{sc}\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r_{sc}}$$

6.2.5 Golden 144

When the branching angle θ equals 144°, from equation (3.3.13) we have the self-contacting ratio r_{sc} is such that

$$r_{sc} = \frac{-1}{2\cos 144°}$$

and using the relationship

$$\cos 144° = \frac{-\phi}{2}$$

we have $r_{sc}(144) = 1/\phi$. Figure 6.9 displays an image of the tree.

Special Geometrical Properties:

1. The branches $b(LL)$ and $b(RRR)$ are collinear, thus all branches of the form $(RL)^kLL$ or $(RL)^kRRR$, are also on the same line.

2. The line through the origin and the point with address R goes through all points with addresses of the form $R(LR)^k$. Let l_1 be the line segment from $(0,1)$ to the tip point P with address $R(LR)^\infty$ (the left-most tip point of the tree). Then l_1

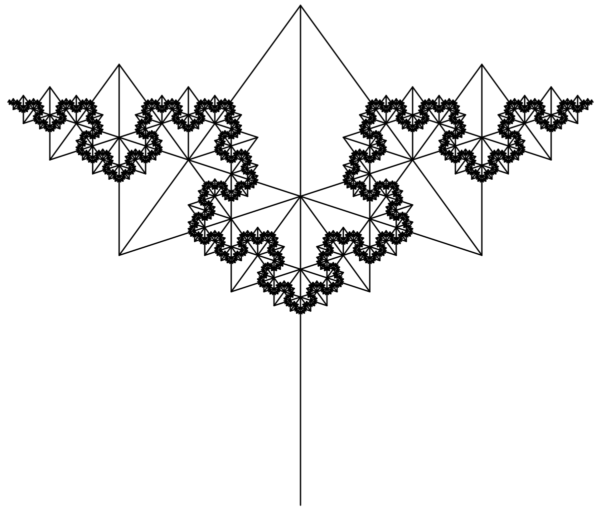

Figure 6.9: $T(1/\phi, 144°)$

has length 1. Let $l-2$ be the line segment from $(0,1)$ to the point P. Then l_2 has length r. Moreover, the trunk together with l_1 and l_2 form an isosceles triangle, with angles $72°, 72°, 36°$.

3. Consider all branches of the form R^k. They form a spiral of isosceles triangles.

4. The point with address RR splits the trunk into line segments of length $1/\phi^2$ above and $1/\phi$ below.

5. The point with address $RR((LR)^\infty$ splits the trunk into line segments of length $1/\phi$ above and $1/\phi^2$ below.

6. The line segment l_3 from the point with address RR through the point with address $RRRL$ up to the branch $b(R)$, together with a portion of the trunk and a portion of $b(R)$ form an isosceles triangle with angles $36°, 36°, 108°$. In addition, l_3 bisects the acute angle between the trunk and the branch $b(RR)$.

The self-contacting hole classes of this tree are the main hole class M located by $\mathbf{A}_0$ and RR; and the vertex classes V_k located by $RR(LR)^k$ and $RR(LR)^{k+1}$ for $k \geq 0$. Let $P_1 = (x_1, y_1)$ be the point with address $RRRL$. Then

$$x_1 \approx 0.085757, \quad y_1 \approx 1 - 0.263932 \tag{6.2.31}$$

There are non-self-contacting hole classes that correspond to the self-contacting hole classes splitting. The main class splits when $\epsilon = x_1/2$. The vertex class V_k splits when ϵ is half the value of the x-coordinate of the point with address $RR(LR)^{k+1}RL$. We shall show that there are no other hole locators.

Let $P_2 = (x_2, y_2)$ be the point with address $RRRLLR$. Then $x_2 = x_1$. As mentioned in the special properties of this tree, the line l_1 bisects the acute angle between the trunk and the branch $b(RR)$. This means that the distance from the point $(x_2/2, y_2)$ to the branch $b(RR)$ equals $x_2/2$ (the distance to the trunk). Hence there is no new hole class located by this point. Similarly, for any other vertex not of the form $RR(LR)^{k+1}RL$ there can be no new level 0 hole class located.

The region of the tree bounded by the two branches $b(RR(LR)^kL)$ and $b(RR(LR)^{k+1})$, along with the portion of the trunk between the starting point of $b(RR(LR)^kL)$ and the endpoint of $b(RR(LR)^{k+1})$, is similar to the region of the tree bounded by the branch $b(R)$, the branch $b(RR)$ and the portion of the trunk between $(0, 1)$ and the endpoint of $b(RR)$, with a contraction factor of $r_{sc}^{2(k+1)}$. This fact reduces the set of critical values that we need to determine.

Let MS denote the hole class with contact value $x_1/2$ that is above the point P_1 (so the upper part of the hole class M that remains after it splits). We refer to it as the 'main split' class. Let VS denote the hole class with contact value $x_1/2$ that is below the point P_1 (so the lower part of the hole class M after it splits). We refer to this as the 'vertex split' class.

The contact values we have so far are

$$\underline{\epsilon_M} = \underline{\epsilon_{V_k}} = 0, \quad \underline{\epsilon_{MS}} = \underline{\epsilon_{VS}} = x_1/2 \tag{6.2.32}$$

The collapse values we have so far are

$$\overline{\epsilon_M} = x_1/2, \quad \overline{\epsilon_{V_k}} = r_{sc}^{2(k+1)} x_1/2 \tag{6.2.33}$$

To find an upper bound for the hole class MS, we determine the ϵ-value such that there is a point in the region that is ϵ away from the trunk, the branch $b(R)$ and the point P_1. This implies that the point $(\epsilon, \epsilon(-\tan(54°) + \csc(36°))$ is at a distance of ϵ from P_1. Solving for ϵ gives

$$\epsilon_{MS}^{gc} \approx 0.0655123 \tag{6.2.34}$$

To find an upper bound for the hole class VS, we find the ϵ-value such that the point $(\epsilon, (y_1+y_2)/2)$ is ϵ away from the point P_1. Solving for this value gives

$$\epsilon_{VS}^{gc} \approx 0.0474052 \tag{6.2.35}$$

Now we have

$$r_{sc}\overline{\epsilon_{MS}} < \underline{\epsilon_{VS}} < \overline{\epsilon_{VS}} < \overline{\epsilon_{MS}} \tag{6.2.36}$$

The hole partition and sequence is given by

$$\begin{aligned}
N_0 &= N([\overline{\epsilon_{MS}}, \infty]) = 0 \\
N_1 &= N([\overline{\epsilon_{VS}}, \overline{\epsilon_{MS}})) = 2 \text{ (level 0 main split)} \\
N_2 &= N([\underline{\epsilon_{VS}}, \overline{\epsilon_{VS}})) = 4 \text{ (2 level 0 main split, 2 level 0 vertex split)} \\
N_3 &= N([r_{sc}\overline{\epsilon_{MS}}, \underline{\epsilon_{VS}})) = 2 \text{ (level 0 main)} \\
N_4 &= N([r_{sc}\overline{\epsilon_{VS}}, r_{sc}\overline{\epsilon_{MS}})) = 6 \text{ (2 level 0 main split, 4 level 1 main split)} \\
N_5 &= N([r_{sc}\underline{\epsilon_{VS}}, r_{sc}\overline{\epsilon_{VS}})) = 10 \text{ (same as previous line, plus 4 level 1 vertex split)}
\end{aligned} \tag{6.2.37}$$

In general, for integers $k \geq 0$:

- For intervals of the form $[r_{sc}^{2k+1}\overline{\epsilon_{VS}}, r_{sc}^{2k+1}\overline{\epsilon_{MS}})$, there are main split holes at levels 0 through $2k$, with the total number of holes given by

$$\begin{aligned}
N([r_{sc}^{2k}\overline{\epsilon_{VS}}, r_{sc}^{2k}\overline{\epsilon_{MS}})) &= \sum_{i=1}^{k+1} 2^{2i} - 2 \\
&= \frac{4}{3}(4^{k+1}-3) - 2(k+1)
\end{aligned} \tag{6.2.38}$$

- For intervals of the form $[r_{sc}^{2k}\overline{\epsilon_{VS}}, r_{sc}^{2k}\overline{\epsilon_{MS}})$, there are main split holes at levels 0 through $2k+1$, with the total number of holes given by

$$\begin{aligned}
N([r_{sc}^{2k+1}\overline{\epsilon_{VS}}, r_{sc}^{2k+1}\overline{\epsilon_{MS}})) &= \sum_{i=1}^{k+1} 2^{2i+1} - 2 \\
&= \frac{8}{3}(4^{k+1}-3) - 2(k+1)
\end{aligned} \tag{6.2.39}$$

- For intervals of the form $[r_{sc}^{2k}\underline{\epsilon_{VS}}, r_{sc}^{2k}\overline{\epsilon_{VS}})$, there are main and vertex split holes at levels 0 through $2k$, with

$$N([r_{sc}^{2k}\underline{\epsilon_{VS}}, r_{sc}^{2k}\overline{\epsilon_{VS}}) = \frac{4}{3}(4^{k+1}-3) - 2(k+1) + \frac{2}{3}(4^{k+1}-1) \tag{6.2.40}$$

- For intervals of the form $[r_{sc}^{2k+1}\underline{\epsilon_{VS}}, r_{sc}^{2k+1}\overline{\epsilon_{VS}})$, there are main and vertex split holes at levels 0 through $2k+1$, with

$$N([r_{sc}^{2k+1}\underline{\epsilon_{VS}}, r_{sc}^{2k+1}\overline{\epsilon_{VS}}) = \frac{8}{3}(4^{k+1}-3) - 2(k+1) + \frac{4}{3}(4^{k+1}-1) \tag{6.2.41}$$

- For intervals of the form $[r_{sc}^{2k+1}\overline{\epsilon_{MS}}, r_{sc}^{2k}\underline{\epsilon_{VS}})$, there are main holes in levels 0 through $2k$, and

$$N([r_{sc}^{2k+1}\overline{\epsilon_{MS}}, r_{sc}^{2k}\underline{\epsilon_{VS}})) = \frac{4}{3}(4^{k+1}-3) - 2(k+1) \tag{6.2.42}$$

- For intervals of the form $[r_{sc}^{2k+2}\overline{\epsilon_{MS}}, r_{sc}^{2k+1}\underline{\epsilon_{VS}})$, there are main holes in levels 0 through $2k+1$, and

$$N([r_{sc}^{2k+2}\overline{\epsilon_{MS}}, r_{sc}^{2k+1}\underline{\epsilon_{VS}})) = \frac{8}{3}(4^{k+1}-3) - 2(k+1) \tag{6.2.43}$$

Remarks.

1. The hole sequence is order-isomorphic to the natural numbers.

2. The main hole class has persistence greater than zero, though it does split. This is different from the first three golden trees where the main hole class has persistence equal to 0.

3. For any sequence of ϵ-values where $\epsilon_n = r_{sc}\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r_{sc}}$$

6.3 Other Self-Contacting Trees

6.3.1 $T(r_{sc}(67.5°), 67.5°)$

As remarked in earlier chapters, the trees with this branching angle are interesting because there are infinitely many tip points on the subtree S_{RR} that have minimal distance to the trunk. These are the points at addresses of the form $RRRR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$. For sufficiently large scaling ratios, a closed ϵ-neighbourhood of the tree $T(r_{sc}, 67.5°)$ can have infinitely many holes. These arise from the canopy intervals

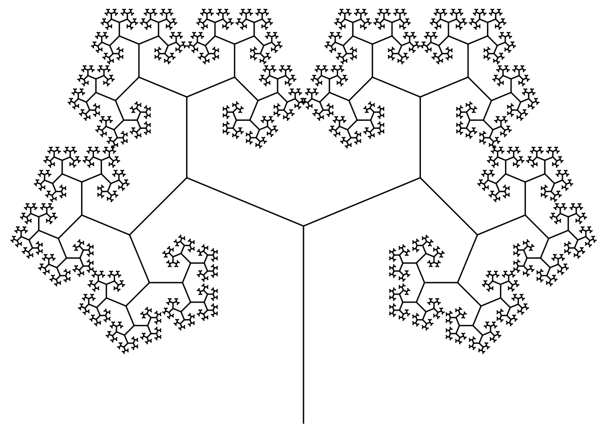

Figure 6.10: $T(r_{sc}, 67.5°)$

of the subtree S_{RRRR}. Figure 6.10 displays an image of the self-contacting tree with branching angle 67.5°.

There is a unique self-contacting tip point on the right side of the tree, namely the point $P_{c1} = (x_{c1}, y_{c1})$ with address $RLL^2(RL)^\infty$. Setting x_{c1} to zero to find r_{sc} gives $1 - 2r_{sc}^2 + 2r_{sc}^3 \cos\theta = 0$, and $r_{sc} \approx 0.6343$.

This tree has one type of hole at $\epsilon = 0$ since there is a unique self-contacting tip point. What are the other types of holes? As noted, there are an infinite number of tip points that have minimal horizontal distance to the trunk, and so these should yield new types of holes.

Consider the subtree S_{RLL} and its canopy points. For the degree 0 canopy interval of this subtree, let $P_a = (x_a, y_a)$ be the right endpoint with address $RLL\mathbf{C}_R$ and let $P_b = (x_b, y_b)$ be the left endpoint with address $RLL\mathbf{C}_L$. Then P_a is further from $\mathbf{y}$ than P_b, because the top of the subtree S_{RLL} has positive slope. We need to check if there is a level 0 hole specified by these two points when ϵ is equal to x_a. We could show that x_a is indeed less than the distance between the point $(0, y_a)$ and P_b, which would prove that there is a hole. Instead we will use an alternate method.

Consider the linear extension of the branch $b(RLL)$. Every point on this line is equidistant from P_a and P_b. Denote this line L_{ab}. If L_{ab} crosses $\mathbf{y}$ at a point that is lower than P_a, then this proves that the distance from $(0, y_a)$ to P_b is indeed greater than x_a. Moreover, this would also give us a collapse value for this type of hole, because it would be equal to the distance of the y-intercept of L_{ab} to P_a (which necessarily equals the distance from the y-intercept to P_b).

Without showing the tedious calculations, this method proves that there is indeed a hole specified by the points P_a and P_b, which we will denote by type C_0. We have

$$\underline{\epsilon^{C_0}} \approx 0.114467$$

and

$$\overline{\epsilon^{C_0}} \approx 0.114598$$

Although there is a hole, the persistence is very small:

$$P(C_0) \approx 0.000131$$

There are infinitely many other types of level 0 holes that are related to the C_0 hole. The canopy points of any degree 0 canopy interval of a subtree $S_{RLL\mathbf{A}}$, where $\mathbf{A} = (LR)^k$, locate a new hole class, and the corresponding critical values are equal to the critical values of C_0 scaled by r_{sc}^{2k}.

There are no other holes due to gaps in the subtree S_{RLL}, nor are there any holes due to the subtree S_{RLLL}. To prove this claim, we would follow the same approach as we used to show that there was a hole specified by P_a and P_b. We leave out the calculations, because a quick look at the image of the finite approximation of the tree should easily convince the reader that the claim is indeed true.

Finally we have the holes that involve the trunk. These are the canopy holes that arise from the canopy pairs of the subtree S_{RRRR}. Let CT_k denote the hole class located by a canopy pair corresponding to a degree k canopy pair of S_{RRRR} Each of these level 0 holes has a contact value equal to half the distance between the trunk and the points at addresses of the form $RRRR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_\infty$. Let x_t be the minimal x-distance of these points. Thus

$$x_t = r_{sc}\sin\theta + r_{sc}^2\sin(2\theta) + r_{sc}^3\sin(3\theta) + \frac{1}{1 - r_{sc}^2}[r_{sc}^4\sin(4\theta) + r_{sc}^5\sin(5\theta)]$$

The critical values of these canopy holes are quite complicated because of the branch $b(R)$. The highest tip point on the subtree S_{RRRR} that has minimal distance to the trunk is at $RRRR(RL)^\infty$. Denote this point $P_c = (x_c, y_c) = (x_t, y_c)$. Now consider the point $(x_t/2, y_c)$. It can be shown that this point is closer to the branch $b(R)$

than the point P_c. In fact, for $\epsilon = x_t/2$, the ϵ-neighbourhood of the branch $b(R)$ will cover all points on the line $x = x_t/2$ that have y-values between 1 and approximately 0.8543, which is lower than the point at P_c, which has $y_c \approx 0.8812$. This implies that the branch $b(R)$ will cover some of the possible trunk holes. This makes the actual calculations quite difficult. There are still infinitely many trunk hole types, since the branch $b(R)$ doesn't affect the hole specified by the degree 0 canopy pair, or any pair below. So for this tree we don't have the hole partition and hole sequence completely determined.

Remark. Although we have not completely determined the hole partition and hole sequence for this tree, we can make the observation that the hole sequence is not order-isomorphic to the natural numbers. One would need to use a double index to label the elements of the hole sequence, similar to the hole sequence for the tree $T(0.5, 90°)$ that is discussed below in Subsection 6.4.1.

6.3.2 Self-Contacting Trees with only Main Holes

If a self-contacting tree has only the main type of holes, then the hole partition and hole sequence are straightforward. For an example, consider the tree $T(r_{sc}, 112.5°)$, as displayed in Figure 6.11. One can show that this tree has only the main type of holes. We don't need to find the collapse value of the main hole to determine the hole sequence and partition, for we necessarily have, for $k \geq 1$,

$$\begin{aligned} N_0 &= N([\overline{\epsilon_M}, \infty]) = 0 \\ N_k &= N([r_{sc}^k \overline{\epsilon_M}, r_{sc}^{k-1} \overline{\epsilon_M})) = 2^{k+1} - 2 \end{aligned} \tag{6.3.1}$$

Any self-contacting tree with angle greater than 90° and not equal to 135° that only has the main type of holes would have the same hole partition and sequence. For self-contacting trees with angles less than 90° that have only the main type of holes, the partition is the same, but now $N_k = 2^k - 1$.

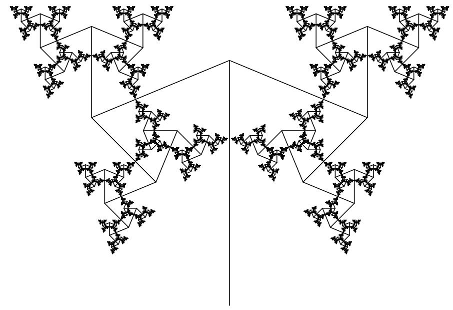

Figure 6.11: $T(r_{sc}, 112.5°)$

6.4 Self-Avoiding Trees

In this section, we discuss various examples of self-avoiding trees. For these trees, the complexity is a more interesting feature than for the self-contacting trees, since they all have infinite complexity if they are not space-filling. We begin with a discussion on self-avoiding trees with the critical angle 90°, then continue with self-avoiding trees with the other critical angle 135°, along with other examples.

6.4.1 Trees with the Critical Angle 90°

Before looking at any specific examples, we will first make some observations about general trees with branching angle 90°. As mentioned throughout this thesis, Mandelbrot and Frame identify this angle as being topologically critical. We shall see that it is critical in new ways based on the analysis of closed ϵ-neighbourhoods. An image of a tree with branching angle 90° is displayed in Figure 6.12.

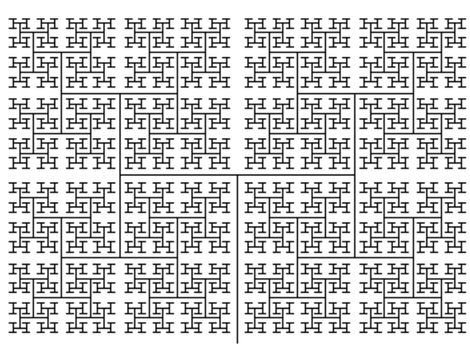

Figure 6.12: $T(0.7, 90°)$

As discussed in Chapter 3, Section 3.3, the self-contacting scaling ratio for $\theta = 90°$ is $1/\sqrt{2}$, and the corresponding self-contacting tree is space-filling. It is also contractible, so it is a simple tree.

The possible hole locators are the canopy pairs of the subtree S_{RLL} or the subtree S_{RRR}. There are no main holes for the following reason. Let $P_1 = (x_1, y_1)$ denote the point with address $RL^2(LR)^\infty$ (one of the contact addresses for 90°). Then P_1 is the lowest tip point of the subtree S_{RLL}, and it cannot be a hole locator above the trunk either. By the scaling nature of the tree, and because the subtree S_{RLLL} is vertical, this point is at a distance of rx_1 to branch $b(R)$, so $y_1 = rx_1 + 1$. Then the point $P' = (0, y_1)$ is closer to the top of the trunk, since $rx_1 < x_1$, so it couldn't be the top of a hole.

We have

$$\begin{aligned} x_1 &= r - r^3 - r^5 - \cdots \\ &= r - \frac{r^3}{1-r^2} \\ &= \frac{r(1-2r^2)}{1-r^2} \end{aligned} \tag{6.4.1}$$

and

$$\begin{aligned} y_1 &= 1 + r^2 - r^4 - r^6 \cdots \\ &= 1 + rx_1 \end{aligned} \tag{6.4.2}$$

The height of a tree with branching angle 90° is given by

$$\begin{aligned} h &= 1 + r^2 + r^4 \cdots \\ &= \frac{1}{1-r^2} = 1 + \frac{r^2}{1-r^2} \end{aligned} \tag{6.4.3}$$

The length of the degree 0 canopy interval is $2x_1$.

First we will consider holes above the trunk. Let $d = h - 1$. The region $\mathbf{y}_{[1,h]}$ is such that every point is at a distance to the tree that is less than or equal to d. So if

$d \leq x_1$, there can be no level 0 holes above the trunk. For what values of r is $d \leq x_1$?

$$\begin{aligned} d \leq x_1 \quad &\Rightarrow \quad \frac{r^2}{1-r^2} \leq \frac{r(1-2r^2)}{1-r^2} \\ &\Rightarrow \quad 2r^2 + r - 1 \leq 0 \\ &\Rightarrow \quad r \leq \frac{1}{2} \end{aligned} \tag{6.4.4}$$

So there are no holes whatsoever for any tree with scaling ratio less than or equal to 1/2.

Suppose $r > 1/2$. The contact value of any pair $(RLL\mathbf{AC}_R, RLL\mathbf{AC}_L)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$, is equal to x_1 (since canopy points of the subtree S_{RLL} have the same x-coordinate). Consider a degree k canopy interval of S_{RLL}, corresponding to the address $RLL\mathbf{A}$ (where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 0$). The length of this interval is $2r^{3+2k}x_1$. Let $P_2 = (x_1, y_2)$ denote the point with address $RLL\mathbf{AC}_R$ and let $P_3 = (x_1, y_3)$ denote the point with address $RLL\mathbf{AC}_L$. If P_2 and P_3 do locate a hole, how can we find a collapse value? Let $P_4 = (0, y_4)$ be the point on the y-axis whose y-coordinate is halfway between the y-coordinates of the canopy points (so at the same height as the branch $b(RLL\mathbf{A})$, and $y_4 = (y_2 + y_3)/2$). The point P_4 is clearly equidistant to the two canopy points P_2 and P_3, and there are no other tip points on the tree that are closer to P_4. When ϵ is equal to the distance between P_4 and P_2, then there can be no hole left of the hole class, because the region $\mathbf{y}_{[y_3,y_2]}$ is within ϵ to the tree. Let d_C be this distance ('C' for canopy). The difference in the y-values in the points P_2 and P_4 is equal to half the width of the degree k canopy interval of the subtree, which equals $r^{3+2k}x_1$. Then

$$\begin{aligned} d_C^2 &= x_1^2 + (y_2 - y_4)^2 \\ &= x_1^2 + (r^{3+2k}x_1)^2 \\ &= x_1^2(1 + r^{6+4k}) \end{aligned} \tag{6.4.5}$$

Hence

$$d_C = x_1\sqrt{1 + r^{6+4k}} \tag{6.4.6}$$

Now let d_B be the distance from the point P_2 to the branch $b(R)$. If $\epsilon = d_B$, then there can be no hole because the region $\mathbf{y}_{[y_3,y_2]}$ is within ϵ to the tree (since P_2 is the furthest from the branch). We have

$$d_B = y_2 - 1 \tag{6.4.7}$$

There will be collapse of the hole class precisely when ϵ is the minimum of d_C and d_B.

Now consider holes below the line $y = 1$. The relevant pairs are the canopy pairs of the subtree S_{RRR}. Each canopy point on this subtree has the same x-coordinate, which is equal to x_1. So if a pair indeed locates a hole, then the contact value is $x_1/2$. As with holes above the trunk, there is a maximum value of the scaling ratio such that the tip point with address $RRR(LR)^\infty$ (the highest tip point of the subtree S_{RRR} that also has minimal distance to the trunk) is within a distance of $x_1/2$ to the branch $b(R)$. Note that this distance is equal to d as discussed above. To find such a scaling ratio, we need to satisfy

$$\begin{aligned} d \le \frac{x_1}{2} \quad &\Rightarrow \quad \frac{r^2}{1-r^2} \le \frac{r(1-2r^2)}{2(1-r^2)} \\ &\Rightarrow \quad 2r^2 + 2r - 1 \le 0 \\ &\Rightarrow \quad r \le \frac{\sqrt{3}-1}{2} \approx 0.3660254 \end{aligned} \tag{6.4.8}$$

Proposition 6.4.1.1 *Let* $r_0 = \dfrac{\sqrt{3}-1}{2}$. *For any* $r \le r_0$, *the tree* $T(r, 90°)$ *is simple.*

Proof. If $r \le r_0$, then there are no level 0 holes below the line $y = 1$. If $r \le r_0$, then $r < 1/2$, so there are no level 0 holes above the trunk either. Hence there are no holes of any level, and the tree is simple. □

Suppose $r > r_0$. The contact value of any pair $(RRR\mathbf{AC}_R, RRR\mathbf{AC}_L)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \ge 0$, is equal to $x_1/2$. Consider a degree k canopy interval of S_{RRR}, corresponding to the address $RLL\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \ge 0$. The length of this interval is $2r^{3+2k}x_1$. Let $P_5 = (x_1, y_5)$ denote the point with address $RRR\mathbf{AC}_R$ and let $P_6 = (x_1, y_6)$ denote the point with address $RRR\mathbf{AC}_L$. If P_5 and P_6 do locate a hole, how can we find a collapse value? Let $y_7 = (y_5 + y_6)/2$. We will determine the ϵ-value for which the point (ϵ, y_7) is at a distance from P_5 (and also from P_6 since P_7 is halfway between them). Such an ϵ would satisfy

$$(x_1 - \epsilon)^2 + (y_5 - y_7)^2 = \epsilon^2 \tag{6.4.9}$$

Solving for ϵ gives

$$\epsilon = \frac{x_1}{2}(1 + r^{6+4k}) \tag{6.4.10}$$

Let $d'_C = \frac{x_1}{2}(1 + r^{6+4k})$ and let d'_B be the distance from the branch $b(R)$ to the lower canopy point P_6. Then the collapse value of the hole located by P_5 and P_6 is the minimum of d'_C and d'_B.

Specific Example: $T(0.5, 90°)$

Now we consider a specific tree with branching angle 90°. As discussed above, this tree has no holes above the trunk, so we just need to look at the canopy pairs of the subtree S_{RRR}.

When $r = 0.5$, we have

$$x_1 = \frac{1}{3}, \quad \frac{x_1}{2} = \frac{1}{6} \tag{6.4.11}$$

The tip point with address $RRR(RL)^\infty$ has maximal height for tip points of the subtree S_{RRR}, and it is at a distance of $rx_1 = 1/6$ from the branch $b(R)$. Thus every canopy interval will locate a hole because they are all sufficiently far away from the branch $b(R)$ when $\epsilon = x_1/2 = 1/6$. This means that there are infinitely many level 0 holes at $\epsilon = \frac{1}{6}$.

Given a degree k interval, the collapse value is the minimum of d'_C and d'_B. We have

$$d'_C = \frac{x_1}{2}(1 + r^{6+4k}) = \frac{1}{6} + \frac{1}{6(2^{6+4k})} \tag{6.4.12}$$

The value of d'_B must be greater than or equal to the distance from $b(R)$ to the nearest tip point of S_{RRR} plus the width of the gap of the specific canopy interval $(2x_2r^{3+2k})$. So

$$d'_B \geq rx_1 + 2x_1r^{3+2k} = \frac{1}{6} + \frac{1}{6(2^{2+2k})} \tag{6.4.13}$$

For any $k \geq 0$, we have

$$\frac{1}{6(2^{6+4k})} \leq \frac{1}{6(2^{2+2k})} \tag{6.4.14}$$

so $d'_C \leq d'_B$ for any k. This means we can always use d'_C as the collapse value. Let C_k denote the hole classes corresponding to degree k canopy intervals. Then for any $k \geq 0$,

$$\underline{\epsilon_{C_k}} = \frac{1}{6}, \quad \overline{\epsilon_{C_k}} = \frac{1}{6} + \frac{1}{6(2^{6+4k})} \tag{6.4.15}$$

The collapse values decrease as k increases, and

$$\lim_{k\to\infty} \overline{\epsilon_{C_k}} = \frac{1}{6} \tag{6.4.16}$$

We also have

$$r\overline{\epsilon_{C_0}} \approx 0.0846354 < \frac{1}{6} \tag{6.4.17}$$

so there can be holes in at most one level at a time.

For the hole sequence we will use a double index. The first index denotes the level and the second denotes the highest degree of the canopy interval possible, plus 1. We have

$$\begin{aligned} N_{00} &= N([\overline{\epsilon_{C_0}}, \infty]) = 0 \\ N_{01} &= N([\overline{\epsilon_{C_1}}, \overline{\epsilon_{C_0}})) = 2 \\ N_{02} &= N([\overline{\epsilon_{C_2}}, \overline{\epsilon_{C_1}})) = 6 \end{aligned}$$

and in general, for level 0 and $k \geq 1$:

$$N_{0k} = N([\overline{\epsilon_{C_k}}, \overline{\epsilon_{C_{k-1}}})) = 2^{k+1} - 2 \tag{6.4.18}$$

The value 1/6 is not hole congruent to any other real number, because of Equation 6.4.16. Thus

$$N_{0\infty} = N(\{\underline{\epsilon_{C_0}}\}) = \infty \tag{6.4.19}$$

We use this notation to signify that there are infinitely many critical values before $\underline{\epsilon_{C_0}}$, and that it is the limit of these values as in Equation 6.4.16. For general levels

$j \geq 1$ and $k \geq 1$:

$$\begin{aligned} N_{j0} &= N([r^j \overline{\epsilon_{C_0}}, r^{j-1} \underline{\epsilon_{C_0}})) = 0 \\ N_{jk} &= N([r^j \overline{\epsilon_{C_k}}, r^j \overline{\epsilon_{C_{k-1}}})) = 2^j(2^{k+1} - 2) \\ N_{j\infty} &= N(\{r^j \underline{\epsilon_{C_0}}\}) = \infty \end{aligned} \tag{6.4.20}$$

Remarks.

1. The hole sequence is not order-isomorphic to the natural numbers, we need to use a double index to order the sequence.

2. There are non-zero values of ϵ for which there are infinitely many holes in the corresponding closed ϵ-neighbourhood, and these values of ϵ form singleton sets in the hole partition.

3. The tree has complexity equal to 1.

4. The hole classes have small persistence. For the level 0 hole classes, the persistence of the class C_k is
$$P([C_k]) = \frac{1}{6(2^{6+4k})} \tag{6.4.21}$$
For example, for $k = 5$, $P([C_5]) \approx 2.4835 \times 10^{-9}$. This small persistence is related to the space-filling nature of the angle. In fact, the persistence of such hole classes decreases as the scaling ratio gets closer to the self-contacting scaling ratio $1/\sqrt{2}$.

6.4.2 Trees with Critical Angle $135°$

As with the critical angle $90°$ discussed in the previous subsection, here we will make a few observations before presenting a specific example. This angle is the only other angle identified by Mandelbrot and Frame as being topologically critical [31]. When $r = r_{sc} = 1\sqrt{2}$, the tree is space-filling and contractible. Figure 6.13 displays an image of a tree with branching angle $135°$.

Trees with branching angle $135°$ all have $y_{\max} = 1$. If a tree $T(r, 135)$ is non-simple, then there can only be holes below the line $y = 1$. As discussed in Chapter 5, the possible hole locators for this angle are the contact point at address RR and the

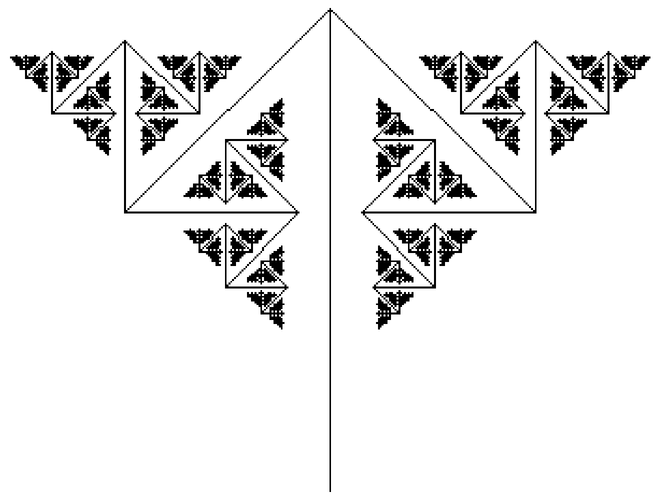

Figure 6.13: $T(0.6, 135°)$

other vertex points of the subtree S_{RR}, so with addresses of the form $RR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 1$.

Let $P_c = (x_c, y_c)$ denote the point with address RR. Then

$$x_c = \frac{r}{\sqrt{2}} - r^2, \quad y_c = 1 - \frac{r}{\sqrt{2}} \tag{6.4.22}$$

If $k \geq 1$, then the x-coordinate of a vertex point $RR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$, is equal to $x_c(1 + r^2 \cdots r^{2k})$.

Specific Example: $T(0.5, 135°)$

The contact point $P_c = (x_c, y_c)$ with address RR has coordinates

$$\begin{aligned} x_c &= \frac{r}{\sqrt{2}} - r^2 \approx 0.10355 \\ y_c &= 1 - \frac{r}{\sqrt{2}} \approx 1 - 0.3535534 \end{aligned} \tag{6.4.23}$$

One can show that the only vertex points that locate holes are the two points with addresses $RRRL$ and $RRLR$. We do not provide complete details, but give a quick explanation for this. Consider the vertex point $P_v = (x_v, y_v)$ with address $RRRLRL$.

This point is at a distance of $x_v/2 = x_c(1+r^2+r^4)/2$ from the trunk. Let $P_2 = (x_2, y_2)$ be the vertex with address $RRRL$. To show that P_v does not locate a hole, it suffices to show that the distance from the point $(x_v/2, y_v)$ to P_2 is less than $x_v/2$. Similarly for any other vertex point with addresses $RR\mathbf{A}$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ and $k \geq 2$, the point does not locate a hole.

Let M denote the main hole class located by P_c. Then

$$\underline{\epsilon_M} = \frac{x_c}{2}$$

At $\epsilon = \frac{x_c}{2}(1+r^2)$, the hole class M splits into three new classes. The uppermost hole class is located by P_2 and the top of the trunk, let MS denote this hole class. Let V denote the other two hole classes, located by vertex points and the point P_c. (We use the same symbol to denote these 2 classes because they have the same critical values.) Then

$$\underline{\epsilon_V} = \underline{\epsilon_{MV}} = \frac{x_c}{2}(1+r^2)$$

We find upper bounds for the collapse values of the classes as follows. For MV, we determine when a point is equidistant from the trunk, the branch $b(R)$ and the point P_1 with address $RR(RL)^\infty$. When ϵ is equal to this distance, then the hole class MV is covered. For V, we determine when a point is equidistant from the trunk, P_c and P_2. We have

$$\begin{aligned} \underline{\epsilon_M} \approx \quad & 0.05178, & \overline{\epsilon_M} = \underline{\epsilon_{MV}} = \underline{\epsilon_V} \approx 0.06472 \\ \epsilon_V^{gc} \approx 0.06821, \quad & \epsilon_{MV}^{gc} = 0.07755 & \end{aligned} \tag{6.4.24}$$

Since $r\epsilon_{MV}^{gc} < \underline{\epsilon_M}$, there can only be holes at one level for any given ϵ. The hole partition and hole sequence is given by

$$\begin{aligned} N_{4k+1} &= N([r^k\overline{\epsilon_V}, r^k\overline{\epsilon_{MV}})) = 2^{k+1} \\ N_{4k+2} &= N([r^k\underline{\epsilon_V}, r^k\overline{\epsilon_V})) = 2^{k+1} \cdot 3 \\ N_{4k+3} &= N([r^k\underline{\epsilon_M}, r^k\underline{\epsilon_V})) = 2^{k+1} \\ N_{4k+4} &= N([r^{k+1}\overline{\epsilon_{MV}}, r^k\underline{\epsilon_M})) = 0 \end{aligned}$$

6.4.3 Other Examples of Self-Avoiding Trees

The Tree $T(0.55, 35°)$

This tree has only the main type of holes, and one can show that

$$r^4\overline{\epsilon_M} < \underline{\epsilon_M} < r^3\overline{\epsilon_M}.$$

Thus the tree has complexity equal to 4. The hole sequence and partition is given by

$$\begin{aligned} N_0 &= N([\overline{\epsilon_M}, \infty]) = 0 \\ N_1 &= N([r\overline{\epsilon_M}, \overline{\epsilon_M})) = 1 \\ N_2 &= N([r^2\overline{\epsilon_M}, r\overline{\epsilon_M})) = 3 \\ N_3 &= N([r^3\overline{\epsilon_M}, r^2\overline{\epsilon_M})) = 7 \\ N_4 &= N([\underline{\epsilon_M}, r^3\overline{\epsilon_M})) = 15 \\ N_5 &= N([r^4\overline{\epsilon_M}, \underline{\epsilon_M})) = 14 \end{aligned} \tag{6.4.25}$$

In general, for $k \geq 0$,

$$\begin{aligned} N_{4+2k} &= N([r^k\underline{\epsilon_M}, r^{3+k}\overline{\epsilon_M})) = 2^k(15) \\ N_{5+2k} &= N([r^{4+k}\overline{\epsilon_M}, r^k\underline{\epsilon_M})) = 2^k(14) \end{aligned} \tag{6.4.26}$$

Observation. If a tree with branching angle in the first range has only main holes and

$$r^j\overline{\epsilon_M} < \underline{\epsilon_M} < r^{j-1}\overline{\epsilon_M}$$

so that the complexity equals j, we can easily determine the hole partition and sequence. For $1 \leq k \leq j$,

$$N_k = N([r^k\overline{\epsilon_M}, r^{k-1}\overline{\epsilon_M})) = 2^k - 1$$

and for $k \geq 0$,

$$\begin{aligned} N_{j+2k} &= N([r^k\underline{\epsilon_M}, r^{j+k-1}\overline{\epsilon_M})) = 2^k(2^j - 1) \\ N_{j+2k+1} &= N([r^{j+k}\overline{\epsilon_M}, r^k\underline{\epsilon_M})) = 2^k(2^j - 2) \end{aligned} \tag{6.4.27}$$

Similarly, if a tree with branching angle greater than 90° has only main holes and has complexity equal to j, then for $1 \leq k \leq j$,

$$N_k = N([r^k \overline{\epsilon_M}, r^{k-1}\overline{\epsilon_M}]) = 2^{k+1} - 2$$

and for $k \geq 0$,

$$\begin{aligned} N_{j+2k} &= N([r^k \underline{\epsilon_M}, r^{j+k-1}\overline{\epsilon_M}]) = 2^{k+1}(2^j - 1) \\ N_{j+2k+1} &= N([r^{j+k}\overline{\epsilon_M}, r^k \underline{\epsilon_M}]) = 2^{k+1}(2^j - 2) \end{aligned} \tag{6.4.28}$$

Remarks.

1. This tree has complexity equal to 4 (as mentioned above).

2. For any sequence of ϵ-values where $\epsilon_n = r\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

The Tree $T(0.4, 50°)$

Figure 6.14 displays an image of $T(0.4, 50°)$.

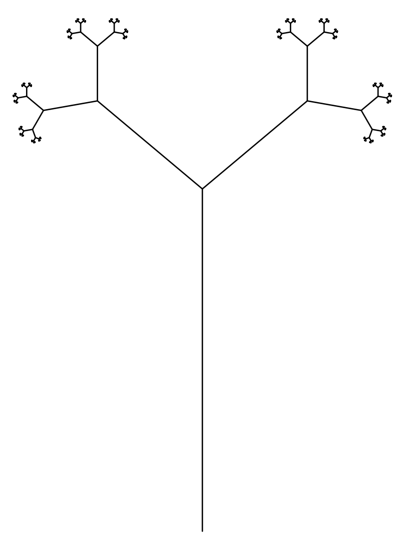

Figure 6.14: $T(0.4, 50°)$

This tree can be shown to have only the main type of holes, and has complexity equal to 1. Let M denote the main class of holes. $P_c = (x_c, y_c)$ denotes the contact point with address $RL^3(RL)^\infty$. Then

$$\begin{aligned} x_c &= r\sin\theta - \frac{r^3\sin\theta}{1-r^2} - \frac{r^4\sin\theta}{1-r^2} \approx 0.21804 \\ y_c &= 1 + r\cos\theta + r^2 + \frac{r^3\cos\theta}{1-r^2} + \frac{r^4\cos(2\theta)}{1-r^2} \approx 1 + .46080 \end{aligned} \tag{6.4.29}$$

We have $\underline{\epsilon_M} = x_c$. To estimate the collapse value, we determine when a point on the y-axis is equidistant from the branch $b(R)$ and the point P_c. This gives $\epsilon_M^{gc} \approx 0.76540$, which suffices to show that $r\overline{\epsilon_M} < \underline{\epsilon_M}$. The hole sequence and partition is given by

$$\begin{aligned} N_0 &= N([\overline{\epsilon_M}, \infty]) = 0 \\ N_1 &= N([\underline{\epsilon_M}, \overline{\epsilon_M})) = 1 \\ N_2 &= N([r\overline{\epsilon_M}, \underline{\epsilon_M})) = 0 \\ N_3 &= N([r\underline{\epsilon_M}, r\overline{\epsilon_M})) = 2 \\ N_4 &= N([r^2\overline{\epsilon_M}, r\underline{\epsilon_M})) = 0 \end{aligned} \tag{6.4.30}$$

and in general, for $j \geq 1$,

$$\begin{aligned} N_{2j} &= N([r^j\overline{\epsilon_M}, r^{j-1}\underline{\epsilon_M})) = 0 \\ N_{2j+1} &= N([r^j\underline{\epsilon_M}, r^j\overline{\epsilon_M})) = 2^j \end{aligned} \tag{6.4.31}$$

Remarks.

1. This tree has complexity equal to 1 (as mentioned above).

2. The hole sequence is not monotonically increasing, and every number in the sequence is either 0 or a power of 2.

3. For any sequence of ϵ-values where $\epsilon_n = r\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

4. Any tree with angle less than 90° that has only the main type of holes and has complexity equal to 1 has the same hole partition and hole sequence.

The Tree $T(0.5, 45°)$

The point $P_c = (x_c, y_c)$ with address $RL^3(LR)^\infty$ is the lowest tip point of the subtree S_R that has minimal distance to the y-axis. The point $(0, y_c)$ is at a distance of x_c to the tree. For $\epsilon = x_c$, there are infinitely many level 0 holes. The main hole class M is located by P_c and the top of the trunk, while the other classes are located by canopy pairs of the subtree S_{RL^3}. Let C_k denote the hole classes identified by a canopy pair of the subtree S_{RL^3} of degree k. The persistence of C_k decreases as k increases. The contact value for every level 0 hole class is $x_c \approx 0.15237$. By determining when a point on the y-axis is equidistant from P_c and the branch $b(R)$ we have the upper bound $\overline{\epsilon_M} \approx 0.29217$. Thus $r\overline{\epsilon_M} < \underline{\epsilon_M}$ and the tree has complexity equal to 1. We can determine the exact collapse value of the class C_0 as with canopy holes for the tree $T(0.5, 90°)$, this gives $\overline{\epsilon_{C_0}} \approx 0.15308$, so the persistence of C_0 is approximately 0.00071. The partition and sequence is similar to $T(0.5, 90°)$, except now we also have the main holes as well as the canopy holes. For the hole sequence we will use a double index. The first index denotes the level and the second denotes the highest degree of the canopy interval possible, plus 1. We have

$$
\begin{aligned}
N_{00} &= N([\overline{\epsilon_M}, \infty]) = 0 \\
N_{01} &= N([\overline{\epsilon_{C_0}}, \overline{\epsilon_M})) = 1 \\
N_{02} &= N([\overline{\epsilon_{C_1}}, \overline{\epsilon_{C_0}})) = 3 \\
N_{03} &= N([\overline{\epsilon_{C_2}}, \overline{\epsilon_{C_1}})) = 7
\end{aligned}
$$

and in general, for level 0 and $k \geq 2$:

$$N_{0k} = N([\overline{\epsilon_{C_{k-1}}}, \overline{\epsilon_{C_{k-2}}})) = 2^k - 1 \tag{6.4.32}$$

The value $\underline{\epsilon_M}$ is not hole congruent to any other real number, so we denote the equivalence class

$$N_{0\infty} = N(\{\underline{\epsilon_M}\}) = \infty \tag{6.4.33}$$

For general levels $j \geq 1$ and $k \geq 2$:

$$\begin{aligned} N_{j0} &= N([r^j\overline{\epsilon_M}, r^{j-1}\underline{\epsilon_M})) = 0 \\ N_{j1} &= N([r^j\overline{\epsilon_{C_0}}, r^j\overline{\epsilon_M})) = 2^j \\ N_{jk} &= N([r^j\overline{\epsilon_{C_{k-1}}}, r^j\overline{\epsilon_{C_{k-2}}})) = 2^j(2^k - 1) \\ N_{j\infty} &= N(\{r^j\underline{\epsilon_M}\}) = \infty \end{aligned} \tag{6.4.34}$$

The Tree $T(0.55, 75°)$

Figure 6.15 displays an image of $T(0.55, 75°)$.

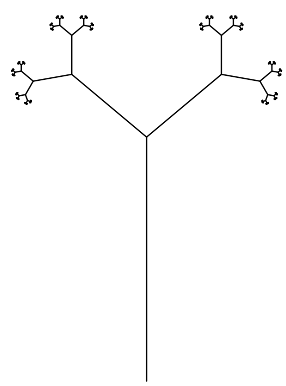

Figure 6.15: $T(0.55, 75°)$

It can be shown that this tree has only the main and secondary contact types of holes. Let M denote the main hole class and let S denote the secondary contact hole class. The hole class M is located by P_c with address $RL^3(RL)^\infty$ and the top of the trunk, while S is located by the point $P_s = (x_s, y_x)$ with address $RRRR(LR)^\infty$. We find

$$\begin{aligned} \underline{\epsilon_M} &\approx 0.23526, \quad \epsilon_M^{gc} \approx 0.26396 \\ \underline{\epsilon_S} &\approx 0.20011, \quad \epsilon_S^{gc} = 0.20603 \end{aligned} \tag{6.4.35}$$

Thus the hole partition and sequence, for $k \geq 0$, is given by

$$\begin{aligned}
N_0 &= N([\overline{\epsilon_M}, \infty]) = 0 \\
N_{4k+1} &= N([r^k, \underline{\epsilon_M}, r^k \overline{\epsilon_M})) = 2^k \\
N_{4k+2} &= N([r^k \overline{\epsilon_S}, r^k \underline{\epsilon_M})) = 0 \\
N_{4k+3} &= N([r^k \underline{\epsilon_S}, r^k \overline{\epsilon_S})) = 2^{k+1} \\
N_{4k+4} &= N([r^{k+1} \overline{\epsilon_M}, r^k \underline{\epsilon_S})) = 0
\end{aligned} \tag{6.4.36}$$

Remarks.

1. This tree has complexity equal to 1 (as mentioned above).

2. The hole sequence is not monotonically increasing, and every number in the sequence is either 0 or a power of 2.

3. The first two items on this list are the same as for the tree $T(0.4, 50°)$ discussed above, but the sequence is different and the types are different (here we have main and secondary contact while $T(0.4, 50°)$ has only main holes).

4. For any sequence of ϵ-values where $\epsilon_n = r\epsilon_{n-1}$, we have the following growth rate:
$$\lim_{n \to \infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

The Tree $T(0.6, 105°)$

It can be shown that this tree has only the main and secondary types of holes. Let M denote the main hole class located by the point $P_c = (x_c, y_c)$ with address $RRR(LR)^\infty$. Let S denote the secondary contact hole class located by the point $P_s = (x_s, y_s)$ with address $RL(LR)^\infty$. This tree is interesting because although there are only two types of holes as with the previous example $T(0.55, 75°)$, here the hole partition and hole sequence is more complicated. One can show that the critical values satisfy the following relationships:

$$\begin{aligned}
&r^4\overline{\epsilon_M} < \underline{\epsilon_M} < r^3\overline{\epsilon_M} \\
&r\overline{\epsilon_S} < \underline{\epsilon_S} \\
&r\overline{\epsilon_M} < r^2\overline{\epsilon_S} < \overline{\epsilon_M} < r\underline{\epsilon_S} \\
&r^5\overline{\epsilon_S} < \underline{\epsilon_M} < r^4\underline{\epsilon_S}
\end{aligned} \tag{6.4.37}$$

So the hole partition and sequence is given by:

$$\begin{aligned}
N_0 &= N(\overline{\epsilon}, \infty])) = 0 \\
N_1 &= N([\underline{\epsilon_S}, \overline{\epsilon_S})) = 1 \text{ (1 sec.con. level 0)} \\
N_2 &= N([r\overline{\epsilon_S}, \underline{\epsilon_S})) = 0 \\
N_3 &= N([r\underline{\epsilon_S}, r\overline{\epsilon_S})) = 2 \text{ (2 sec.con. level 1)} \\
N_4 &= N([\overline{\epsilon_M}, r\underline{\epsilon_S})) = 0 \\
N_5 &= N([r^2\overline{\epsilon_S}, \overline{\epsilon_M})) = 2 \text{ (2 main level 0)} \\
N_6 &= N([r^2\underline{\epsilon_S}, r^2\overline{\epsilon_S})) = 6 \text{ (2 main level 0, 4 sec.con. level 2)} \\
N_7 &= N([r\overline{\epsilon_M}, r^2\underline{\epsilon_S})) = 2 \text{ (2 main level 0)} \\
N_8 &= N([r^3\overline{\epsilon_S}, r\overline{\epsilon_M})) = 6 \text{ (main levels 0 and 1)} \\
N_9 &= N([r^3\underline{\epsilon_S}, r^3\overline{\epsilon_S})) = 14 \text{ (6 main, 8 sec.con.)} \\
N_{10} &= N([r^2\overline{\epsilon_M}, r^3\underline{\epsilon_S})) = 6 \\
N_{11} &= N(r^4\overline{\epsilon_S}, r^2\overline{\epsilon_M})) = 14 \\
N_{12} &= N([r^4\underline{\epsilon_S}, r^4\overline{\epsilon_S})) = 30 \\
N_{13} &= N([r^3\overline{\epsilon_M}, r^4\underline{\epsilon_S})) = 14
\end{aligned} \tag{6.4.38}$$

In general, for $k \geq 0$,

$$\begin{aligned}
N_{13+4k} &= N([r^{3+k}\overline{\epsilon_M}, r^{4+k}\underline{\epsilon_S})) = 2^k(14) \\
N_{14+4k} &= N([r^k\underline{\epsilon_M}, r^{3+k}\overline{\epsilon_M})) = 2^k(30) \\
N_{15+4k} &= N([r^{k+5}\overline{\epsilon_S}, r^k\underline{\epsilon_M})) = 2^k(28) \\
N_{16+4k} &= N([r^{k+5}\underline{\epsilon_S}, r^{k+5}\overline{\epsilon_S})) = 2^k(60)
\end{aligned} \tag{6.4.39}$$

Remark. The hole sequence clearly shows that the growth rate of holes agrees with the scaling of the tree. For any sequence of ϵ-values where $\epsilon_n = r\epsilon_{n-1}$, we have the following growth rate:

$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

6.5 Brief Chapter Summary

In this chapter, we have presented a sample of analysis of specific trees and their closed ϵ-neighbourhoods. The examples clearly illustrated that the study of the closed ϵ-neighbourhoods reveals a great deal of information about a tree. Now that we have seen these examples, we are better prepared to discuss the theory further.

Chapter 7

Discussion, Conclusions and Future Work

In this final chapter of the thesis, we begin with a discussion of our theory in light of the specific examples presented in the previous chapter. We form conclusions about the theory, highlighting the main accomplishments along with questions that arise. We give an overview of future work that could stem from the thesis.

7.1 Discussion

In the previous chapters, we developed new concepts and theory to study symmetric binary fractal trees, based on an analysis of their closed ϵ-neighbourhoods as ϵ ranges through the non-negative real numbers. The specific examples presented in the previous chapter have shown how rich this extra structure can be. We now revisit the theory. In particular, we discuss the persistence intervals of hole classes and critical ϵ-values for a tree; complexity and critical scaling ratios as a function of branching angle; hole locations and critical angles based on hole locations; and the hole sequences of trees.

7.1.1 Persistence Intervals of Hole Classes and Critical Values of ϵ for a Specific Tree

In studying the closed ϵ-neighbourhoods of a tree as ϵ ranges over the non-negative real numbers, we are interested in various aspects of the holes (if holes exist at all), not just the number of holes. One important feature is the persistence interval (the interval of ϵ-values for which there is exactly one hole that has non-empty intersection with the maximal hole of the hole class) and the persistence (length of the persistence interval) of a hole class. One of the main results of our theory is that every hole class can be obtained from some level 0 hole class via a suitable address map, and the persistence is equal to the persistence of the level 0 hole class scaled by some power of the scaling ratio r. Once we have determined the persistence intervals of the level 0 holes, we then know all the persistence intervals for the tree, and thus all critical ϵ-values.

As seen in the examples, there are trees that have certain hole classes with relatively small persistence (compared to other hole classes associated with the tree). So relatively small holes can make a big difference in our characterizations of the trees. For example, consider the three self-contacting trees $T(r_{sc}, 108°)$, $T(r_{sc}, 112.5°)$ and $T(r_{sc}, 120°)$ discussed in the previous chapter. A quick glance at their images indicates that they look 'similar' (see Figures 6.7, 6.11 and 6.8). The tree $T(r_{sc}, 112.5)$ has only holes of the main type, while the other two have other types of holes as well. In the case of $T(r_{sc}, 108°)$, there are also secondary contact holes, while $T(r_{sc}, 120°)$ has mixed canopy holes as well as the main type. The persistence of the secondary contact holes in the case of $T(r_{sc}, 108°)$ is relatively small compared with the main holes, as is the persistence of the mixed canopy holes of $T(r_{sc}, 120°)$. So these small persistence holes make a big difference in the hole sequences of the trees. If we didn't want to make such a distinction between the trees, perhaps we could restrict our attention to level 0 hole classes that have some minimum persistence, so a persistence cutoff value. We have not studied this idea in detail yet, but it is definitely worth investigating. Depending on the persistence cutoff value, a tree with a complicated hole sequence based on our original theory could have a more straightforward hole sequence. However, from a theoretical point of view, it is interesting that our theory does distinguish between trees such as $T(r_{sc}, 108°)$, $T(r_{sc}, 112.5°)$ and $T(r_{sc}, 120°)$, though the distinction may be too fine for the sake of applications.

Another issue is that a persistence cutoff value would eliminate hole classes that have 0 persistence. We have seen trees whose main holes have 0 persistence (see Section 6.2). To include these types of holes, perhaps we could consider a new definition of hole class and persistence interval. For example, one could define the persistence interval of a hole to be the range of ϵ-values for which there is at least one hole that has non-empty intersection with the original hole, as opposed to our requirement that there is exactly one. This would change the nature of persistence intervals, because they would no longer be independent of the hole chosen (because of splitting holes).

The set of critical ϵ-values would remain the same, but the hole classes would be different. Another definition of persistence could make a distinction between level 0 hole classes that split into hole classes that are all level 0 (*eg.* $T(r_{sc}, 144°)$) and level 0 hole classes that split into more than one hole class, but only one hole class is still level 0 (*eg.* $T(r_{sc}, 120°)$). We are currently looking into other definitions of persistence, but have no major results yet. Future work includes a general definition for persistence of holes in closed ϵ-neighbourhoods of any set in $\mathbb{R}^2$, not just symmetric binary fractal trees.

The special angles θ_N for $N \geq 2$ are such that the corresponding self-contacting trees have infinitely many canopy holes located by the canopy pairs of the subtree $S_{RL^{N+1}}$. In the case of $T(r_{sc}, 45°)$, the canopy holes are not negligible compared to the main hole, at least for lower degrees. For the smaller special angles, the persistence of the degree 0 canopy hole decreases as N increases, because the level of the subtree goes up. Following the ideas used to determine the collapse value of the degree 0 canopy hole for $T(r_{sc}, 45°)$, we obtain the following upper bounds for the collapse values of the degree 0 canopy holes of the next 4 self-contacting trees with special angles:

Label	Angle	ϵ^{gc} for degree 0 canopy hole
θ_3	30°	0.0040
θ_4	22.5°	0.0013
θ_5	18°	0.00048
θ_6	15°	0.00020

The size of the canopy holes becomes quite insignificant as N increases.

Other questions naturally arise about persistence intervals.

Question. For a specific angle and hole locator pair, how does the persistence of a hole locator pair vary as a function of r?

An early conjecture was that the persistence of any hole class increases as r increases towards r_{sc}, with the maximum value at $r = r_{sc}$.

For example, consider the angle 45°. A rough proof that persistence of the main hole class located by the pair $(\mathbf{A}_0, RL^3(RL)^\infty)$ increases as r increases to r_{sc} is as follows.

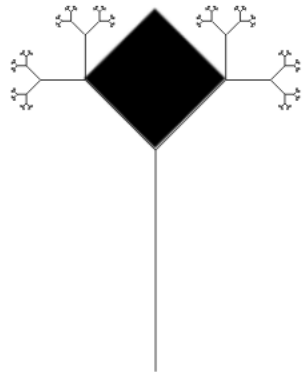

Figure 7.1: Square in main hole area of tree with angle 45°

For a given r, consider the square formed from the branches $b(R)$, $b(L)$ and two other line segments to form a square (see Figure 7.1). The interior of this square is disjoint from the tree, and is a subset of the region of the tree that is necessary for the formation of the main hole. Moreover, for any ϵ in the persistence interval of the main hole, any hole bounded by the closed ϵ-neighbourhood of the boundary of the square is a subset of the main hole at the same ϵ value. Thus the smallest ϵ such that the corresponding closed ϵ-neighbourhood for the boundary of the square is contractible is less than or equal to the collapse value of the main hole. The collapse ϵ for the boundary of the square is $r/2$, since the center of the centre of the square is at a distance of $r/2$ from the boundary of the square, and any other point inside the square is closer to the boundary of the square. Thus the collapse value for the boundary of the square is an increasing function of r. As r increases, the contact value of the main hole decreases, while the collapse value increases, so the persistence is also increasing (since it is equal to the difference between the collapse value and the contact value).

However, it is not true in general that for a fixed angle and hole locator pair, persistence increases as r increases. Two counter-examples are the angles 90° and 135°, which both yield space-filling trees at r_{sc}. Are these two angles just exceptions? We conjecture that there are angles distinct from 90° and 135° with hole locator pairs such that the persistence is not strictly increasing as r approaches r_{sc}. We have not found a specific counter-example, nor have we been able to disprove the conjecture.

For a fixed angle θ, one could use the change in persistence of hole classes as a function of r to indicate the space-filling or space-minimizing nature of trees with that branching angle. This is quite complicated to study, as persistence depends on collapse values of hole classes, and we do not have a general method to determine collapse values for all hole classes.

To obtain more general methods of determining the collapse values, we could try to use the ideas of finding the medial axis as described in [8]. The methods of [8] are suitable for polygonal regions. The shapes we are dealing with are not polygonal, but we can take polygonal approximations.

Question. For a fixed tree that is non-simple and has more than one level 0 hole class, how do the persistence intervals of the different level 0 hole classes compare?

Question. How does persistence of a hole class of a certain locator pair vary over the angle range of the pair? What angle values correspond to local maxima or minima for persistence, and do they relate to critical angles based on hole location?

The last question leads to the next subsection dealing with hole location and critical angles based on hole location.

7.1.2 Hole Location and Critical Angles Based on Hole Location

In general, there are infinitely many pairs of addresses that are hole locators. There are infinitely many hole locator pairs just for the self-contacting hole classes. We now discuss the critical angles of the various types of hole locator pairs. The angle range of a pair $(\mathbf{A}_1, \mathbf{A}_2)$ is the set of all angles θ such that $(\mathbf{A}_1, \mathbf{A}_2) \in \mathcal{HL}(\theta)$ (the hole location set of the angle θ). The critical angles for a given locator pair are the lower and upper bounds for the angle range of the pair. We cannot give a complete list of critical angles, because the list of hole locator pairs is infinite. Note that there may be more than two critical angles associated with any one address, because it is possible for an address to be part of more than one locator pair.

Contact Addresses and Main Holes

The main hole classes are located by the contact addresses. The hole locator pair of a main hole class is of the form $(\mathbf{A}_0, \mathbf{A}_c)$, where $\mathbf{A}_c$ is the contact address for the specific tree. Given any tree whose branching angle is not equal to 90° or 135°, the corresponding self-contacting tree has holes located by the contact address. The contact addresses are summarized in Table 3.1.

Pairs of the form $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$

Proposition 7.1.2.1 *The angle range of the pair* $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$ *is* $(\theta_N, \theta_{N-1}]$ *if* $N > 2$, *and it is* (θ_N, θ_{N-1}) *if* $N = 2$. *Thus the critical angles for the pair* $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$, *for* $N \geq 2$, *are* θ_N *and* θ_{N-1}.

Proof. In the first angle range, for $N > 2$, the address $RL^{N+1}(RL)^\infty$ locates a hole for every angle θ such that $\theta_N < \theta \leq \theta_{N-1}$ because it locates the main hole in the corresponding self-contacting tree. For $N = 2$ the address $RL^3(RL)^\infty$ locates a hole for every angle θ such that $\theta_2 = 45° < \theta < 90°$.

For any $N \geq 2$ and any angle θ_N, the point with address $RL^{N+1}(RL)^\infty$ is the right corner point of the subtree $S_{RL^{N+1}}$, which has a horizontal trunk, and so it cannot be a hole locator for θ_N. If $\theta < \theta_N$, then the point with address $RL^{N+1}(RL)^\infty$ is not even a local minimum, so cannot be a hole locator. Thus the critical angles associated with the pair $(\mathbf{A}_0, RL^{N+1}(LR)^\infty)$ are θ_N and θ_{N-1}, for $N \geq 2$. For any $N > 2$ and $\theta > N_{N-1}$, the point with address $RL^{N+1}(RL)^\infty$ is not a local minimum and so cannot be a hole locator point. The pair $(\mathbf{A}_0, RL^3(RL)^\infty)$ is not a hole locator pair for $\theta = 90°$, as discussed in the previous chapter.

Therefore, the angle range of the pair $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$ is $(\theta_N, \theta_{N-1}]$ if $N > 2$, and it is (θ_N, θ_{N-1}) if $N = 2$; and the critical angles for the pair $(\mathbf{A}_0, RL^{N+1}(RL)^\infty)$, for $N \geq 2$, are θ_N and $\theta - N - 1$. □

The pair $(\mathbf{A}_0, R^3(LR)^\infty)$

This pair locates the main hole for trees with θ such that $90° < \theta < 135°$, but it can also locate secondary contact holes for $\theta < 90°$. Here we just consider the angle range of the pair in terms of holes of the main type, and we will discuss the angle range with respect to the secondary contact holes after the discussion on main holes.

Proposition 7.1.2.2 *The angle range of the pair* $(\mathbf{A}_0, R^3(LR)^\infty)$ *as a hole locator of holes of the main type is* $(90°, 135°)$*, and the critical angles associated with the pair are* $90°$ *and* $135°$*.*

Proof. Clearly for any $\theta \in (90°, 135°)$, the pair $(\mathbf{A}_0, R^3(LR)^\infty)$ is in $\mathcal{HL}(\theta)$ because it locates the main self-contacting hole class. For $\theta = 90°$ or $135°$, we have already discussed why $(\mathbf{A}_0, R^3(LR)^\infty)$ is not a hole locator pair in the previous chapter. For any θ such that $\theta < 90°$, the point with address $R^3(LR)^\infty$ may be a hole locator point, but it would locate a secondary contact hole, and we are only considering the main holes here. Therefore, the angle range of the pair $(\mathbf{A}_0, R^3(LR)^\infty)$ as a hole locator of holes of the main type is $(90°, 135°)$, and the critical angles are $90°$ and $135°$. $\square$

The pair $(\mathbf{A}_0, RR)$

The contact address for the third angle range including $135°$ range is RR. This address is also the contact address for certain trees with angles in the second angle range. Recall from Section 3.3 of Chapter 3 that for any θ in the second angle range and scaling ratio such that $r \leq -\sin(3\theta)\csc(2\theta)$, the point with address RR has minimal x-value for any point on the subtree S_{RR}. So it is also possible for a tree with angle $90° < \theta < 135°$ to have $(\mathbf{A}_0, RR)$ as a hole locator pair.

Proposition 7.1.2.3 *The angle range of the pair* $(\mathbf{A}_0, RR)$ *is* $(\theta_l, 180°)$*, where* $\theta_l \approx 123.1884°$*.*

Proof. For any $\theta \geq 135°$, we have already established that $(\mathbf{A}_0, RR)$ is in $\mathcal{HL}(\theta)$. So now consider θ in the second angle range. As discussed above and in Subsection 3.3.3 in Chapter 3, the point with address RR has minimal distance to the trunk for any r such that $r \leq -\sin(3\theta)\csc(2\theta)$. This is one of two conditions that must be met in order that the pair $(\mathbf{A}_0, RR)$ is in $\mathcal{HL}(\theta)$. The second condition is that the point with address RR is a hole locator, and this condition will put a lower bound on the value of r (otherwise the closed ϵ-neighbourhood of the branch $b(R)$ would cover the possible hole region). The critical angle θ_l is found by determining when the upper bound from the first condition is equal to the lower bound from the second condition.

For θ in the second angle range, let $P_c = (x_c, y_c)$ denote the point with address $R^3(LR)^\infty$, and let $P_1 = (x_1, y_1)$ denote the point with address RR.

Let $m(\theta)$ be defined as

$$m(\theta) = -\frac{\sin(3\theta)}{\sin(2\theta)} = -\sin(3\theta)\csc(2\theta) \tag{7.1.1}$$

This function is increasing for $90° < \theta < 135°$, see Figure 7.2. For $90° < \theta \leq 120°$, the value of $m(\theta)$ is always non-positive, so the inequality cannot be satisfied. For $120° < \theta < 135°$, $m(\theta)$ is always positive, so there are always values of r for which P_1 is closer to the trunk than P_c is. However, this doesn't guarantee that there is a hole. Assuming that P_1 is closer, when does it locate a hole as part of the pair $(\mathbf{A}_0, RR)$? The pair locates a hole if the point $P_2 = (x_1/2, y_1)$ is at a distance of more than $x_1/2$ from the branch $b(R)$. The line $lin(R)$ is given by

$$y_{lin(R)}(x) = \cot(\theta)x + 1 \tag{7.1.2}$$

Let d_1 denote the distance from P_2 to $b(R)$. Then

$$d_1 = [y_{lin(R)}(x_1/2) - y_1]\sin(180° - \theta) \tag{7.1.3}$$

Solving the inequality $d_1 > x_c/2$ gives

$$r > \frac{\cos\theta + 1}{2(\cos^2\theta - \cos\theta - \cos(2\theta))} \tag{7.1.4}$$

Let $M(\theta)$ be defined as follows:

$$M(\theta) = \frac{\cos\theta + 1}{2(\cos^2\theta - \cos\theta - \cos(2\theta))} \tag{7.1.5}$$

Then $M(\theta)$ is decreasing for $120° < \theta < 135°$. Figure 7.2 displays $M(\theta)$ (see Equation 7.1.5) in the line curve and $m(\theta)$ (see Equation 7.1.1) in the dotted curve. Using Maple to find the value θ_l that corresponds to the intersection point of the curves $m(\theta)$ and $M(\theta)$ gives $\theta_l \approx 123.1884°$. For any $\theta_l < \theta < 135°$, there exists r in the interval $(M(\theta), m(\theta))$, which means that the pair $(\mathbf{A}_0, RR)$ is in $\mathcal{HL}(\theta)$.

Therefore, the angle range of the pair $(\mathbf{A}_0, RR)$ is $(\theta_l, 180°)$, where $\theta_l \approx 123.1884°$.

□

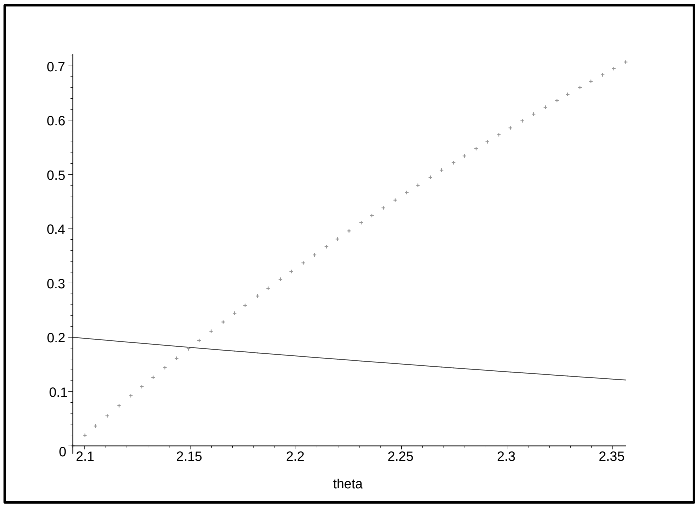

Figure 7.2: Plots of $m(\theta)$ (Eq. 7.1.1, dotted line) and $M(\theta)$ (Eq. 7.1.5, line curve) as functions of θ (in radians)

Secondary Contact Addresses and Holes

In general, the secondary contact addresses are relevant for angles between 45° and 135°. In the first angle range, the secondary contact address is $R^{N_2}(LR)^\infty$, where N_2 is the secondary turning number (the smallest integer such that $N_2\theta \geq 270°$). In the second angle range, the secondary contact address is $RL(LR)^\infty$.

The pair $(\mathbf{A}_0, R^6(LR)^\infty)$

Proposition 7.1.2.4 *The pair* $(\mathbf{A}_0, R^6(LR)^\infty)$ *is not a hole locator pair for any branching angle.*

Proof. The address $R^6(LR)^\infty$ is the secondary contact address for angles between 45° and 54°. The self-contacting tree with branching angle 54° does not have any holes below the line $y = 1$. To prove this, let $P_c = (x_c, y_c)$ be the lowest top tip point of the subtree S_{R^5} (which would correspond to the address $R^6(LR)^\infty$). It is straightforward to determine the values x_c and y_c, and to show that the point $(x_c/2, y_c)$ is closer to the branch $b(R)$ than to the point P_c. This implies that P_c is not a hole locator point for $T(r_{sc}, 54°)$. This also implies that the same is true for any tree with $45° < \theta \leq 54°$ and $r \leq r_{sc}$. For any angle greater than 54°, the point with address

$R^6(LR)^\infty$ isn't even a local minimum, and so is not a hole locator point. □

The pair $(\mathbf{A}_0, R^5(LR)^\infty)$

Recall in the previous chapter that the address $R^5(LR)^\infty$ located a secondary contact class of holes for the angle 60°. Thus the angle range of the pair $(\mathbf{A}_0, R^5(LR)^\infty)$ must include 60°.

Proposition 7.1.2.5 *The angle range of the pair* $(\mathbf{A}_0, R^5(LR)^\infty)$ *is the interval* $(\theta_{s1}, \theta_{s2})$ *where* $\theta_{s1} \approx 57.0057°$ *and* $\theta_{s2} \approx 63.8359°$.

Proof. The pair $(\mathbf{A}_0, R^5(LR)^\infty)$ locates a secondary contact hole class provided the following condition is met. Let $P_s = (x_s, y_s)$ denote the point with address $R^5(LR)^\infty$, and let $P_1 = (x_s/2, y_s)$. This address is the secondary contact address for angles between 45° and 67.5°. There is a secondary contact type of hole at $\epsilon = x_s/2$ if the point P_1 is more than ϵ away from the branch $b(R)$. For a given angle in this range, if P_s does not locate a hole for the self-contacting tree, then it doesn't locate a hole for any self-avoiding tree with the same angle. So we can restrict our attention to the self-contacting trees. We use Maple to determine the angle range where P_1 is more than $x_s/2$ away from $b(R)$ for the self-contacting trees. The coordinates of the point P_s are given by:

$$x_s = r\sin(\theta) + r^2\sin(2\theta) + r^3\sin(3\theta) + \frac{r^4\sin(4\theta) + r^5\sin(5\theta)}{1-r^2} \quad (7.1.6)$$

$$y_s = r\cos(\theta) + r^2\cos(2\theta) + r^3\cos(3\theta) + \frac{r^4\cos(4\theta) + r^5\cos(5\theta)}{1-r^2} \quad (7.1.7)$$

Points on the branch $b(R)$ satisfy

$$y = \cot(\theta)x \quad (7.1.8)$$

Let $P_2 = (x_2, y_2)$ be the point on the branch $b(R)$ with the same x-coordinate as P_1. Then $x_2 = x_s/2$ and $y_2 = \cot(\theta)x_s/2$. The distance d between P_1 and the branch is given by

$$d = (y_2 - y_s)\sin\theta \quad (7.1.9)$$

By setting $x_s/2$ equal to d, we obtain the lower limit angle to be approximately 57.0057° and the upper limit angle to be 63.8359°. For any angle between these two values, there is a secondary contact type of hole located by the pair $(\mathbf{A}_0, R^5(LR)^\infty)$. $\square$

The pair $(\mathbf{A}_0, R^4(LR)^\infty)$

The only other secondary contact address for the first angle range is $R^4(LR)^\infty$.

Proposition 7.1.2.6 *The angle range of the pair* $(\mathbf{A}_0, R^4(LR)^\infty)$ *is the interval* $(67.5°, 90°)$.

Proof. Let $P_s = (x_s, y_s)$ denote the point with address $R^4(LR)^\infty$. At $67.5°$, P_s is the left corner point of the subtree S_{R^4}. This subtree has a horizontal trunk, and so although the point may be a local minimum if r is large enough, it doesn't locate a hole. Any vertical neighbourhood of the point contains other tip points of the subtree S_{R^4} that are at the same distance to the trunk and are higher than P_s. For angles strictly between $67.5°$ and $90°$, the top of the subtree S_{R^4} has positive slope. For any angle in this range, the corresponding self-contacting tree is such that the point halfway between the trunk and the secondary contact point is always closer to the secondary contact point than the branch $b(R)$. This is also true for self-avoiding trees with sufficiently large r. For $\theta = 90°$, this address does not locate a hole (as discussed in the previous chapter). $\square$

The pair $(\mathbf{A}_0, RL(LR)^\infty)$

The secondary contact address for the second angle range is $RL(LR)^\infty$. Recall that $\mathbf{C}_R = RL(LR)^\infty$, because it corresponds to the right canopy point of the degree 0 canopy interval of the tree.

Proposition 7.1.2.7 *For trees with* $112.5° \leq \theta < 135°$*, the pair* $(\mathbf{A}_0, \mathbf{C}_R)$ *cannot be a hole locator.*

Proof. First consider all trees with angles between $90°$ and $135°$ and such that $y_{\max} > 1$ (otherwise the point with address $\mathbf{C}_R$ is not above the line $y = 1$).

Let $P_s = (x_s, y_s)$ denote the point with address $\mathbf{C}_R$. The point P_s is a hole locator point if and only if the point $P_1 = (0, y_s)$ is closer to P_s than the top of the trunk. This condition is met precisely when $x_s < y_s - 1$. Note that $y_s = y_{\max}$. If $x_s \geq y_s - 1$,

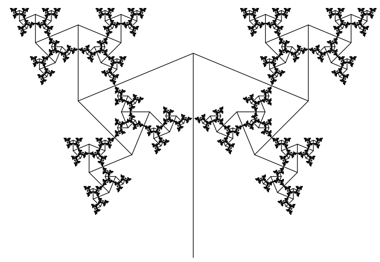

Figure 7.3: $T(r_{sc}, 112.5°)$

then the point P_1 is closer to the top of the trunk at $(0, 1)$ than it is to the point P_s, thus the point could not be a hole locator. If $x_s < y_s - 1$, then P_1 is at a distance of x_s to the tree, since any other portion of S_R that has smaller x-values is on the branch $b(R)$. This branch is negatively sloped, so the closest point on the branch $b(R)$ to P_1 is the point $(0, 1)$, which is further than x_s from P_1 (by assumption that $x_s < y_s - 1$). We have

$$\begin{aligned} x_s &= r \sin\theta \left(\frac{1 - 2r^2}{1 - r^2} \right) \\ y_s - 1 &= \frac{r\cos\theta + r^2}{1 - r^2} \end{aligned} \tag{7.1.10}$$

When is $x = y - 1$ for a self-contacting tree? We can use geometry to determine this. Let L denote the line segment between $(0, 1)$ and the point P_s, and let A denote the length of L. Then there is a right triangle whose hypotenuse is L, and the other two sides have length x_s and $y_s - 1$. Let β denote the angle between L and the y-axis. Then $x_s = A \sin\beta$ and $y_s - 1 = A\cos\beta$. The angle between L and $b(R)$ is equal to the angle between $b(R)$ and the trunk (by the symmetry of the subtree S_R). Let γ denote this angle. Then $\beta + 2\gamma = 180°$, and we also have $\beta + \gamma = \theta$. Hence $\theta = 90° + \beta/2$. There will be equality of x_s and $y_s - 1$ precisely when $\sin\beta = \cos\beta$, so when $\beta = 45°$. Thus $\theta = 112.5°$ (see Figure 6.11).

What about the self-avoiding trees with this angle? Recall that $y_{\max}$ is an increasing function of r. For values of $r < r_{sc}$, the angle β will increase past $45°$, and so $x_s > y_s - 1$. If $\theta > 112.5°$, then β will also increase past $45°$ and again we have $x_s > y_s - 1$. Thus any tree with angle greater than or equal to $112.5°$ cannot have $\mathbf{C}_R$ as a hole locator. $\square$

As a result of the previous proposition, we have the following corollary which applies to any hole locator pair, not just the pair $(\mathbf{A}_0, \mathbf{C}_R)$.

Corollary 7.1.2.8 *For trees with* $112.5° \leq \theta$*, there are no hole locators above the trunk.*

Proof. Let $P_s = (x_s, y_s)$ be the point with address $\mathbf{C}_R$. Any local minimum $P' = (x', y')$ of the subtree S_R that is above the line $y = 1$ is also below the line $y = y_{\max} = y_s$ and has an x-value greater than x_s. Thus

$$x' > x_s > y_s - 1 > y' - 1, \tag{7.1.11}$$

and hence the point P' cannot be a hole locator point, and there are no hole locators above the trunk. $\square$

Proposition 7.1.2.9 *The angle range of the pair* $(\mathbf{A}_0, \mathbf{C}_R)$ *is the interval* $(90°, 112.5°)$.

Proof. For $\theta = 90°$, the address does not locate any hole, as discussed in the previous chapter. For any θ such that $90° < \theta < 112.5°$, consider the self-contacting tree $T(r_{sc}, \theta)$. The point $P_s = (x_s, y_s)$ with address $\mathbf{C}_R$ is such that $x_s < y_s - 1$ (as discussed above). The pair locates a hole for such a tree, and the hole class has a contact value of x_s. For $\theta \geq 112.5$, we have already established that the pair cannot locate a hole.

Canopy Pairs and Canopy Holes

The holes located by canopy pairs (holes of the canopy type) are much more complicated than the main or secondary contact types. As seen with the self-contacting hole classes, the trees with special angles have infinitely many canopy holes. So any canopy pair of a subtree of the form $S_{RL^{N+1}}$, where $N \geq 2$, is a hole locator pair. In the previous chapter, we saw that canopy holes from other subtrees may arise. Trees with angle 90° may have holes located by the canopy pairs of S_{RL^2} or S_{R^3}. Trees with angle 67.5° may have holes due to the canopy pairs of the subtree S_{R^4}. Our analysis of critical angles corresponding to canopy pairs is based on looking at subtrees whose canopy pairs do locate a hole class for some angle, and focusing on just the degree 0 canopy pair of the subtree. For a specific subtree, the degree 0 canopy interval is the largest and generally corresponds with the largest canopy holes from the subtree (see $T(r_{sc}, 45°)$ as an example). The methods we use to determine critical angles corresponding to degree 0 canopy pairs can be used to find critical angles for other canopy pairs of a subtree, and quite often the angle range of a higher degree pair is just a subset of the angle range of the degree 0 pair. This is not always the case, because it is possible that for a given angle, there are no degree 0 canopy holes for any $r \leq r_{sc}$, but there are higher degree canopy holes because they are further away from the branch $b(R)$.

Unfortunately there is not one basic method to find critical angles for a canopy pair. However, we do have criteria to guarantee that a pair is not a hole locator pair for a given tree. There are two cases, one for canopy holes above the line $y = 1$, and one for below.

Theorem 7.1.2.10 *Let $T(r, \theta)$ be a tree and $\mathbf{A} = RA_2 \cdots \in \mathcal{A}_k$, for some $k \geq 0$, such that the following condition is met. The point $P_t = (x_t, y_t)$ with address $\mathbf{AC}_R$ and the point $P_b = (x_b, y_b)$ with address $\mathbf{AC}_L$ are such that $1 < y_b < y_t$, and one of the points is a local minimum, while the other is at least a one-sided local minimum. If the pair $(\mathbf{AC}_L, \mathbf{AC}_R)$ locates a hole, then the linear extension $lin(\mathbf{A})$ of the branch $b(\mathbf{A})$ crosses the y-axis strictly between y_b and y_t.*

Proof. Suppose $T(r, \theta)$ is a tree and $\mathbf{A} = RA_2 \cdots \in \mathcal{A}_k$, for some $k \geq 0$, such that the point $P_t = (x_t, y_t)$ with address $\mathbf{AC}_R$ is above the point $P_b = (x_b, y_b)$ with address $\mathbf{AC}_L$, and one of the points is a local minimum, the other is at least a one-sided local minimum. Suppose that the pair $(\mathbf{AC}_L, \mathbf{AC}_R)$ locates a hole. Let P_l be the point where $lin(\mathbf{A})$ crosses the y-axis. If $lin(\mathbf{A})$ crosses the y-axis at a value greater than or equal to y_t, then the top of the subtree $S_{\mathbf{A}}$ necessarily has positive slope, and the local minimum of the canopy pair is P_t. Then P_l is equidistant from P_t and P_b. The point $(0, y_t)$ is below P_l, so it is closer to P_b than to P_t. This means that the pair cannot locate a hole, which contradicts the assumption that they do. On the other hand, if $lin(\mathbf{A})$ crosses the y-axis at a value less than or equal to y_b, then the top of the subtree $S_{\mathbf{A}}$ necessarily has negative slope, and the local minimum of the canopy pair is P_b. Then the point $(0, y_b)$ is above P_l, so it is closer to P_t than to P_b, which again implies that the pair cannot locate a hole. $\square$

Note. The condition in the previous theorem is not a sufficient condition, because we would also have to consider the distance from the point $(0, y_t)$ to the rest of the tree, particularly the branch $b(R)$. The condition does give us one place to start when trying to find the critical angles.

We have a similar theorem for canopy pairs for holes below the line $y = 1$:

Theorem 7.1.2.11 *Let $T(r, \theta)$ be a tree and $\mathbf{A} = RA_2 \cdots \in \mathcal{A}_k$, for some $k \geq 0$, such that the following condition is met. The point $P_t = (x_t, y_t)$ with address $\mathbf{AC}_R$ and the point $P_b = (x_b, y_b)$ with address $\mathbf{AC}_L$ are such that $y_b < y_t < 1$, and one point is a local minimum, while the other is at least a one-sided local minimum. If the pair $(\mathbf{AC}_L, \mathbf{AC}_R)$ locates a hole, then*

- *If $x_b \geq x_t$, then the linear extension $lin(\mathbf{A})$ of the branch $b(\mathbf{A})$ crosses the line $x = x_t/2$ strictly between y_b and y_t.*
- *If $x_t \geq x_b$, then the linear extension $lin(\mathbf{A})$ of the branch $b(\mathbf{A})$ crosses the line $x = x_b/2$ strictly between y_b and y_t.*

Proof. If $x_b \leq x_t$, then the top of the subtree $S_{\mathbf{A}}$ has positive slope, and P_t is the local minimum. Let P_l be the intersection of $lin(\mathbf{A})$ and the line $x = x_t/2$, which is necessarily at a higher y-value than y_b (because the branch has negative slope). If P_l is at a y-value greater than or equal to y_t, then the point $(x_t/2, y_t)$ is closer to P_b than to P_t, and this means that there couldn't be a hole located by the pair. Thus P_l must have a y-value strictly between y_b and y_t. Similarly, if $x_t \leq x_b$, P_l must also have a y-value strictly between y_b and y_t. $\square$

Degree 0 Canopy Pairs of Subtrees $S_{RL^{N+1}}$

The subtrees that are relevant for level 0 canopy holes above the line $y = 1$ are $S_{RL^{N+1}}$, where $N \geq 2$. We use Theorem 7.1.2.10 as a starting place for finding critical values.

The pair $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$

Since we are familiar with the canopy holes due to the subtree S_{RL^3} (as detailed in the previous chapter in the example of $T(r_{sc}, 45°)$), we start with the degree 0 canopy pair $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$. This pair locates a hole for $T(r_{sc}, 45°)$ and for self-avoiding trees with angle $45°$ and sufficiently large r. When does this pair locate a hole for other angles close to $45°$?

Proposition 7.1.2.12 *The angle range of the pair* $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$ *is* $(30°, \theta_b)$, *where* $\theta_b \approx 58.1624°$.

Proof. Let $P_t = (x_t, y_t)$ denote the point with address $RL^3\mathbf{C}_R$, and let $P_b = (x_b, y_b)$ denote the point with address $RL^3\mathbf{C}_L$ ('t' for top and 'b' for bottom). If the pair does locate a hole, then the linear extension of the branch $b(RL^3)$ must cross the y-axis between the y-coordinates of the corresponding two canopy points, according to Theorem 7.1.2.10. Using Maple to determine the angle range where this is possible for self-contacting trees, we find the lower limit θ_a to be approximately $29.3979°$ and the upper limit θ_b to be approximately $58.1624°$. The linear extension of $b(RL^3)$ has slope $\cot(2\theta)$ and goes through the point with address RLL. From this information, we find the intercept of $lin(RL^3)$ to be:

$$b = \cot(2\theta)(r\sin\theta - r^3\sin\theta) + (r\cos\theta + r^2 + r^3\cos\theta) \tag{7.1.12}$$

The y-coordinates of P_t and P_b are given by

$$\begin{aligned} y_t &= r\cos\theta + r^2 + r^3\cos\theta + \frac{r^4\cos(2\theta)}{1-r^2} + r^5\cos\theta + \frac{r^7\cos(3\theta)}{1-r^2} && (7.1.13) \\ y_b &= r\cos\theta + r^2 + r^3\cos\theta + \frac{r^4\cos(2\theta)}{1-r^2} + r^5\cos(3\theta) + \frac{r^7\cos\theta}{1-r^2} && (7.1.14) \end{aligned}$$

The lower limit of the angle range was found by equating b and y_t when $r = r_{sc}$, while the upper limit was found by equating b and y_b when $r = r_{sc}$.

Are these the actual critical values associated with the pair?

First consider the upper limit $\theta_b \approx 58.1624°$. This angle is indeed the upper limit, because there is no other part of the tree interacting with the canopy holes for θ

between 45° and θ_b. For any angle between 45° and θ_b, the self-contacting tree is such that the point $(0, y_b)$ is at a distance of x_b from the tree, and it is more than x_b away from P_t.

Now consider the lower limit θ_a. For trees with angles between 30° and 45°, the top of the subtree S_{RL^3RLL} has negative slope, and at r_{sc}, the pair $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$ does locate a hole. At 30°, the subtree S_{RL^3RLL} has a horizontal trunk. The point P_t is the right corner point of this subtree, and so any open vertical neighbourhood of the point contains other top tip points of the subtree S_{RL^3RLL} that have the same y-coordinate. This means that the point P_t could not be a hole locator point, but there are other tip points of S_{RL^3RLL} that would be (canopy points of this subtree S_{RL^3RLL}). So there is a shift at 30°, and the pair $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$ is not a hole locator pair. For trees with angles between θ_a and 30°, the top of the subtree S_{RL^3RLL} now has positive slope, and the point P_t isn't even a local minimum, so the pair does not locate a hole. □

Remark. What about the other higher degree canopy intervals of the subtree S_{RL^3}? For a pair $(RL^3\mathbf{AC}_L, RL^3\mathbf{AC}_R)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 1$, we know that the pair locates a hole for $T(r_{sc}, 45°)$. To determine the angle range for the pair, one could use a similar method as for the degree 0 canopy interval. The difference between the y-coordinates of the endpoints of a degree k canopy interval is just r^{2k} times the difference between the y-coordinates of the degree 0 canopy interval. The values of the x-coordinates of the endpoints of the degree k canopy interval are greater than or equal to r^{2k} times those for the degree 0 one. This implies that the range of values for which the linear extension of the branch $b(RL^3\mathbf{A})$ crosses the y-axis between the y-coordinates of the canopy endpoints is a subset of the angle range for the degree 0 canopy pair, because if the degree k pair locates a hole for a given angle, then necessarily the degree 0 one must as well.

The pair $(RL^4\mathbf{C}_L, RL^4\mathbf{C}_R)$

Recall that the third special angle is $\theta_3 = 30°$. The self-contacting tree has infinitely many canopy holes, located by the canopy pairs of the subtree S_{RL^4}. As with the subtree S_{RL^3}, we just determine the angle range of the degree 0 pair, which must include 30°.

Proposition 7.1.2.13 *The angle range of the pair* $(RL^4\mathbf{C}_L, RL^4\mathbf{C}_R)$ *is the interval* $(22.5°, \theta_b)$*, where* θ_b *is the angle such that in the corresponding self-contacting tree,* $lin(RL^4)$ *crosses the* y*-axis at the same* y*-coordinate as in the point* P_b *with address* $RL^4\mathbf{C}_L$*.*

Proof. Following a similar argument as with the degree 0 canopy pair of the subtree S_{RL^3}, we determine the range of angles for which the linear extension of the branch $b(RL^4)$ crosses the y-axis between the y-coordinates of the degree 0 canopy pair $(RL^4\mathbf{C}_L, RL^4\mathbf{C}_R)$. This gives a lower limit of approximately 20.0874° and an upper limit of 40.0000°. The lower limit is below the fourth special angle $\theta_4 = 22.5°$ and this value is the real lower critical angle for the angle range (following a similar argument as for the pair $(RL^3\mathbf{C}_L, RL^3\mathbf{C}_R)$). □

The pair $(RL^5\mathbf{C}_L, RL^5\mathbf{C}_R)$

The fourth special angle is 22.5°. The self-contacting tree with this angle has infinitely many canopy holes, located by the pairs of the subtree S_{RL^5}. The angle range of the pair $(RL^5\mathbf{C}_L, RL^5\mathbf{C}_R)$ must include 22.5°.

Proposition 7.1.2.14 *The angle range of the pair* $(RL^5\mathbf{C}_L, RL^5\mathbf{C}_R)$ *is* $(18°, 30°)$*.*

Proof. First we use Theorem 7.1.2.10 to find the angle range for which the linear extension of the branch $b(RL^5)$ crosses the y-axis between the y-coordinates of the degree 0 canopy pair $(RL^5\mathbf{C}_L, RL^5\mathbf{C}_R)$. This gives a lower limit of approximately 15.2391° and an upper limit of approximately 30.2195°. This lower limit is below the fifth special angle 18°, and the upper limit is above the third special angle 30°. The real critical values are then 18° and 30°. If $\theta \leq 18°$, the point with address $RL^5\mathbf{C}_R$ is not a local minimum and can't be a hole locator, and for $\theta \geq 30°$, the point with address $RL^5\mathbf{C}_R$ is not a local minimum and can't be a hole locator. □

Pairs of the form $(RL^{N+1}\mathbf{C}_L, RL^{N+1}\mathbf{C}_R)$, where $N \geq 5$

Proposition 7.1.2.15 *For all degree 0 canopy pairs of the form $(RL^{N+1}\mathbf{C}_L, RL^{N+1}\mathbf{C}_R)$, where $N \geq 5$, the angle range is $(\theta_{N+1}, \theta_{N-1})$.*

Proof. As with the degree 0 canopy pair $(RL^5\mathbf{C}_L, RL^5\mathbf{C}_R)$, Theorem 7.1.2.10 gives a lower limit that is smaller than θ_{N+1} and an upper limit that is greater than θ_{N-1}. So we similarly have that the real angle range is $(\theta_{N+1}, \theta_{N-1})$. □

Note. The persistence of such hole classes is relatively small (as mentioned in the previous subsection).

The pair $(RL^2\mathbf{C}_L, RL^2\mathbf{C}_R)$

The first special angle is 90°, and the relevant subtree for canopy holes is RL^2.

Proposition 7.1.2.16 *The angle range of the pair $(RL^2\mathbf{C}_L, RL^2\mathbf{C}_R)$ is (θ_a, θ_b'), where $\theta_a \approx 65.5471°$ and $\theta_b' \approx 98.6548°$.*

Proof. Let P_t be the point with address $RL^2\mathbf{C}_R$ and let P_b be the point with address $RL^2\mathbf{C}_L$. If the pair $(RL^2\mathbf{C}_L, RL^2\mathbf{C}_R)$ locates a hole class for a given tree, then the linear extension of $b(RL^2)$ must cross the y-axis between y_t and y_b (following Theorem 7.1.2.10). First we find the range of angles for which this is true for the self-contacting trees. This gives a lower limit θ_a of approximately 65.5471° and an upper limit θ_b of approximately 101.3011°. The lower limit is the lower critical angle for the angle range. For any self-contacting tree with $\theta_a < \theta < 90°$, the pair does locate a hole, and for any $\theta \leq \theta_a$, the pair does not. The upper limit is not the upper critical angle, however, because we have not taken into consideration the branch $b(R)$. Consider the point $P_1 = (0, x_b)$. If P_1 is equidistant from P_t and the top of the trunk, then the pair does not locate a hole, because the region of the y-axis between y_b and y_t is within x_b of the tree. Using Maple to determine the angle for which this happens in self-contacting trees gives an angle θ_b' of approximately 98.6548°, which is lower than the other lower limit. Thus the upper critical angle is θ_b'. □

Degree 0 Canopy Pairs of Subtrees S_{R^j}, where $2 \leq j \leq 5$

The subtrees that are relevant for level 0 canopy holes below the line $y = 1$ are S_{R^j}, where $2 \leq j \leq 5$. We use Theorem 7.1.2.11 as a starting place for finding critical values.

The pair $(R^5\mathbf{C}_L, R^5\mathbf{C}_R)$

Proposition 7.1.2.17 *The pair $(R^5\mathbf{C}_L, R^5\mathbf{C}_R)$ is not a hole locator for any angle.*

Proof. The upper limit angle given by Theorem 7.1.2.10 is approximately equal to 54.6052°, and there is no degree 0 canopy hole from the subtree S_{R^5} for the corresponding self-contacting tree. This should not be surprising, since the upper limit is only slightly greater than 54°. The branch $b(R)$ always covers the area that a degree 0 canopy hole might have formed. At $\theta = 54°$, the subtree S_{R^5} is horizontal. Let $P_s = (x_s, y_s)$ denote the point with address $R^5(RL)^\infty = R^6(LR)^\infty$, *i.e.*, the secondary contact address. We have already mentioned that the point $(x_s/2, y_s)$ is at a distance to the branch $b(R)$ that is less than $x_s/2$. □

The pair $(R^4\mathbf{C}_L, R^4\mathbf{C}_R)$

Now consider the subtree S_{R^4}. In the previous chapter, we discussed the self-contacting tree with branching angle 67.5°. For this tree, the subtree S_{R^4} has a horizontal trunk. At ϵ equal to half the distance between the trunk and the top tip points of S_{R^4}, there are infinitely many canopy holes, including the class located by the pair $(R^4\mathbf{C}_L, R^4\mathbf{C}_R)$. Thus the angle range of the pair $(R^4\mathbf{C}_L, R^4\mathbf{C}_R)$ must include 67.5°.

Proposition 7.1.2.18 *The angle range of the pair $(R^4\mathbf{C}_L, R^4\mathbf{C}_R)$ is (θ_a, θ_b), where $\theta_a \approx 65.6389°$ and $\theta_b \approx 69.7134°$.*

Proof. Following Theorem 7.1.2.11 to determine the angle range of the pair $(R^4\mathbf{C}_L, R^4\mathbf{C}_R)$, we determine the lower limit to be approximately 65.6389° and the upper limit to be approximately 69.7134. These are the actual critical values for the pair, because the branch $b(R)$ is too far away to affect the degree 0 canopy holes. □

The pair $(R^3\mathbf{C}_L, R^3\mathbf{C}_R)$

Canopy pairs of the subtree S_{R^3} locate infinitely many canopy holes in the tree $T(0.5, 90°)$, as discussed in the previous chapter. Thus the angle range of the degree 0 canopy pair $(R^3\mathbf{C}_L, R^3\mathbf{C}_R)$ must include 90°.

Proposition 7.1.2.19 *The angle range of the pair* $(R^3\mathbf{C}_L, R^3\mathbf{C}_R)$ *is* (θ_a, θ_b)*, where* $\theta_a \approx 84.3571°$ *and* $\theta_b \approx 101.4330°$.

Proof. Again we use Theorem 7.1.2.11 to determine the angle range, we find the lower limit to be approximately 84.3571° and the upper limit to be approximately 101.4330°. These are the actual critical values because for trees with angles between these two values, the degree 0 canopy interval is far enough from the branch $b(R)$ that it doesn't interact with the degree 0 canopy holes. ☐

The pair $(R^2\mathbf{C}_L, R^2\mathbf{C}_R)$

Finally we have the subtree S_{RR}. The degree 0 canopy pair is $(RR\mathbf{C}_L, RR\mathbf{C}_R)$.

Proposition 7.1.2.20 *The angle range of the pair* $(R^2\mathbf{C}_L, R^2\mathbf{C}_R)$ *is* (θ_a, θ_b)*, where* $\theta_a \approx 115.4091°$ *and* $\theta_b' \approx 119.8517°$.

Proof. Let $P_t = (x_t, y_t)$ denote the point with address $RR\mathbf{C}_R$ and let $P_b = (x_b, y_b)$ denote the point with address $RR\mathbf{C}_L$. This pair locates a hole if there is an open vertical interval above the point $(x_b/2, y_b)$ that is more than $x_b/2$ away from the subtree S_R. To determine the lower critical angle, we find the angle for which the linear extension of the branch $b(RR)$ intersects the line $x = x_b/2$ at $y = y_b$ (following Theorem 7.1.2.11. This gives a value θ_a of approximately 115.4091°. To find the upper critical angle, we need to consider when the point P_c with address RR starts to change things (when $(x_b/2, y_b)$ becomes closer to P_c than P_t). At this angle, the locator pair becomes a mixed pair of the form $(RR, RR\mathbf{C}_L)$. The coordinates of P_c, P_b

and P_t are as follows:

$$\begin{aligned}
x_c &= r\sin\theta + r^2\sin(2\theta) \\
y_c &= 1 + r\cos\theta + r^2\cos(2\theta) \\
x_b &= x_c + r^3\sin\theta + \frac{r^4\sin(2\theta) + r^5\sin(3\theta)}{1-r^2} \\
y_b &= y_c + r^3\cos\theta + \frac{r^4\cos(2\theta) + r^5\cos(3\theta)}{1-r^2} \\
x_t &= x_c + r^3\sin(3\theta) + \frac{r^4\sin(2\theta) + r^5\sin\theta}{1-r^2} \\
y_t &= y_c + r^3\cos(3\theta) + \frac{r^4\cos(2\theta) + r^5\cos\theta}{1-r^2}
\end{aligned} \tag{7.1.15}$$

Let d_1 be the distance from $(x_b/2, y_b)$ to P_c and let d_2 be the distance from $(x/b/2, y_b)$ to P_t. Using Maple to determine the equality of d_1 and d_2 for the self-contacting scaling ratio gives a value θ'_b of approximately 119.8517°. For any $\theta \geq \theta'_b$, the point P_b may be a hole locator point, but as part of a pair with the point with address RR, not as part of a canopy pair. □

Holes of Mixed Type

The only mixed type of pairs that we will discuss are the pairs $(RR, RR\mathbf{C}_L)$ and $(RR\mathbf{C}_R, RR)$.

Proposition 7.1.2.21 *The angle range of the pair* $(RR, RR\mathbf{C}_L)$ *is* $[\theta'_b, 135°)$*, where* $\theta'_b \approx 119.8517$*. The angle range of the pair* $(RR\mathbf{C}_R, RR)$ *is* $(\theta_d, 135°)$*, where* $\theta_d \approx$ *122.9508°.*

Proof. We have already determined the lower critical angle for the pair $(RR, RR\mathbf{C}_L)$, because it is equal to the upper critical angle for the pair $(RR\mathbf{C}_R, RR\mathbf{C}_L)$ discussed in the previous item. The upper critical angle for the pair $(RR, RR\mathbf{C}_L)$ is 135°, because for every self-contacting tree with θ between the lower critical angle and 135°, the pair locates a hole. For $\theta \geq 135°$, the point with address $RR\mathbf{C}_L$ is no longer a local minimum. Similarly the upper critical angle for the pair $(RR\mathbf{C}_R, RR)$ is also 135°. We just need to determine the lower critical angle for the pair $(RR\mathbf{C}_R, RR)$.

Let $P_c = (x_c, y_c)$ denote the point with address RR and let $P_t = (x_t, y_t)$ denote the point with address $RR\mathbf{C}_R$ (as in Equations 7.1.15). The pair $(RR\mathbf{C}_R, RR)$ locates a hole if there is an open vertical interval above the point $(x_c/2, y_c)$ that is more than $x_c/2$ away from the subtree S_R. Let d_3 be the distance from $(x_c/2, y_c)$ to P_t. So the lower critical angle is the angle for which $d_3 = x_c/2$ in the self-contacting trees. Using Maple to solve this equation, we find the lower critical angle to be approximately $122.9508°$. □

Remark. As θ gets closer to $135°$, there are more and more mixed pairs that locate holes. These are pairs of the form $(RR\mathbf{A}, RR\mathbf{AC}_R)$ or $(RR\mathbf{A}, RR\mathbf{AC}_L)$, where $\mathbf{A} \in \mathcal{AL}_{2k}$ for some $k \geq 1$. They all have $135°$ as the upper critical angle, because that is where the canopy points stop being local minima. We do not provide the details for any other lower critical angles, but the method to determine them would be similar to how we determined the lower critical angle for the pair $(RR\mathbf{C}_R, RR)$.

Vertex Types

Recall that for the golden tree $T(r_{sc}, 144°)$, the pair $(\mathbf{A}_0, RRRL)$ is a hole locator of a hole of vertex type that causes the main hole to split. The pair is also a hole locator pair for the angle $135°$. The lower critical angle for the pair $(\mathbf{A}_0, RRRL)$ is not $135°$, however. For angles closer to $135°$, other vertex pairs may also be hole locators. Because there are infinitely many hole locator pairs of vertex types of holes, we cannot give a complete list of critical angles. We will just discuss the pair $(\mathbf{A}_0, RRRL)$.

The pair $(\mathbf{A}_0, RRRL)$

Proposition 7.1.2.22 *The angle range of the pair* $(\mathbf{A}_0, RRRL)$ *is* (θ_a, θ_b)*, where* $\theta_a \approx 123.7321°$ *and* $\theta_b \approx 151.2170°$*.*

Proof. Let $P_v = (x_v, y_v)$ denote the point with address $RRRL$. Then

$$\begin{aligned} x_v &= r\sin\theta + r^2\sin(2\theta) + r^3\sin(3\theta) + r^4\sin(2\theta) \\ y_v &= 1 + r\cos\theta + r^2\cos(2\theta) + r^3\cos(3\theta) + r^4\cos(2\theta) \end{aligned} \tag{7.1.16}$$

Let P_1 denote the point $(x_v/2, y_v)$. The point P_v is a hole locator if and only if there is an open vertical interval above and below P_1 that is more than $x_v/2$ away from the subtree S_R. This occurs if P_1 is more than $x_v/2$ away from the branch $b(RR)$ and the branch $b(R)$. We use the branch $b(RR)$ to determine the upper critical angle, and the branch $b(R)$ to determine the lower critical angle.

The equation of the linear extension of $b(RR)$ is given by

$$y_{lin(RR)}(x) = x\cot(3\theta) + (1 + r\cos\theta + r^2\cos(2\theta)) \tag{7.1.17}$$

Let d_1 denote the distance from P_1 to $b(RR)$. Then

$$d_1 = (y_v - y_{lin(RR)}(x_v/2))\sin(3\theta) \tag{7.1.18}$$

Equating d_1 and $x_v/2$ for the self-contacting scaling ratios, we obtain a value of $\theta \approx 151.2170°$. This angle is the upper critical angle for the pair $(\mathbf{A}_0, RRRL)$. For angles greater than or equal to this upper critical angle there are no hole locators between RR and $\mathbf{A}_0$, and by the scaling nature of the trees there are no hole locators between any consecutive vertex points with addresses $RR(LR)^k$ and $RR(LR)^{k+1}$, for $k \geq 1$. This implies that there can be no splitting of holes.

Now we determine the lower critical angle. Two conditions need to be met for $(\mathbf{A}_0, RRRL)$ to be a hole locator pair for $\theta < 135°$. Let $P_c = (x_c, y_c)$ denote the point with address $R^3(LR)^\infty$ (the contact address for angles in the second angle range). The first condition is that $x_v < x_c$ (so that there are no points higher than P_v that are closer to the trunk), and this places an upper bound on the possible scaling ratios. Secondly, we need that the point P_1 is more than $x_v/2$ away from the branch $b(R)$, and this forces the scaling ratios to have a lower bound. The lower critical angle for the pair $(\mathbf{A}_0, RRRL)$ is the angle for which the upper bound from the first condition is equal to the lower bound for the second condition. The equation of the linear extension of the branch $b(R)$ is given by:

$$y_{lin(R)}(x) = (\cot\theta)x + 1 \tag{7.1.19}$$

Let d_2 be the distance from P_1 to $b(R)$. Then

$$d_2 = ((y_{lin(R)}(x_v/2) - y_v)\sin\theta \tag{7.1.20}$$

Using Maple to find the unique angle for which $d_2 = x_v/2$ and $x_v = x_c$, we find the lower critical angle of the pair $(\mathbf{A}_0, RRRL)$ to be approximately $123.7321°$. $\square$

Remark. For any angle above the upper critical angle for this vertex pair, there is no splitting of holes.

7.1.3 Complexity and Critical Scaling Ratios Based On Complexity

For a given angle, the critical scaling ratios are defined to be values that mark a change in complexity. That is, $r' < r_{sc}$ is a critical scaling ratio if there exists a neighbourhood $U = (r_1, r_2)$ containing r' such that $C(r, \theta) = k_1$ for all $r \in (r_1, r)$, $C(r, \theta) = k_2$ for all $r \in (r, r_2)$ and $k_1 \neq k_2$. The self-contacting scaling ratio r_{sc} is also considered to be critical with respect to complexity, because only self-contacting trees have infinite complexity or are space-filling. In the case of angles different from 90° or 135°, any self-avoiding tree has finite complexity. For the two angles 90° and 135°, the self-contacting tree is space-filling, and so has complexity equal to 0. There are other scaling ratios that correspond to self-avoiding trees that are simple (and have 0 complexity), but any interval of the form (a, r_{sc}) contains scaling ratios whose corresponding self-avoiding trees have non-zero complexity.

For any angle, there are self-avoiding trees that are simple. The scaling ratio that separates the simple trees from the non-simple trees is certainly an important critical scaling ratio, and we refer to it as the simple scaling ratio for the angle (denoted by $r_0(\theta)$). Unfortunately we do not have a straightforward method to determine critical scaling ratios in general. Given a tree, to determine its complexity we need to know how many levels of holes can be present for any one ϵ-value. This depends on knowing what the hole locations are, and what the critical values for the hole classes are. As mentioned before, the determination of collapse values in general is not straightforward. In Subsection 6.4.1, we determined r_0 for $\theta = 90°$ to be $r_0 = (\sqrt{3}-1)/2 \approx 0.366$. We provide one example of the simple scaling ratio for the angle 45°, and a general result about angles greater than or equal to 135°.

Simple Scaling Ratios

The Angle 45°

Proposition 7.1.3.1 *For a tree $T(r, 45°)$, let $P_c = (x_c, y_c)$ denote the point with address $RL^3(RL)^\infty$. The tree is simple if and only if the distance from the point $(0, y_c)$ to the branch $b(R)$ is less than or equal to x_c. Moreover, the simple scaling ratio for 45°, $r_0(45°)$, is given by $r_0(45°) \approx 0.205147$.*

Proof. For trees with branching angle 45°, the possible hole locator pairs are the main pair $(\mathbf{A}_0, RL^3(LR)^\infty)$ and canopy pairs of the subtree S_{RL^3}. The minimum ϵ-value to get contact between the closed ϵ-neighbourhoods of S_{RL} and S_{LR} is equal to x_c (since P_c has minimal distance to the y-axis). The point P_c is the highest top tip point of the subtree S_{RL^3}. A tree is simple if and only if the region of the y-axis between $y = 1$ and $y = y_c$, so $\mathbf{y}_{(1,y_c)}$, is completely covered for $\epsilon \geq x_c$. If this region is covered for $\epsilon \geq x_c$, then there can be no main holes or holes due to canopy pairs of S_{RL^3} and hence the tree is simple. If the region is not covered, then there is at least one canopy pair of the subtree that does locate a hole, and so the tree is not simple. At $\epsilon = x_c$, the only way $\mathbf{y}_{(1,y_c)}$ is covered by the closed ϵ-neighbourhood is if the region is within ϵ of the branch $b(R)$. So a tree is simple if and only if the distance from the point $(0, y_c)$ to the branch $b(R)$ is less than or equal to x_c.

Now we can use this result to determine $r_0(45°)$, the critical scaling ratio that separates the simple trees from non-simple trees. The point P_c has coordinates given by

$$x_c = r\sin(45°) - \frac{r^3\sin(45°) + r^4\sin(90°)}{1 - r^2} \tag{7.1.21}$$

$$y_c = 1 + r\cos(45°) + r^2 + \frac{r^3\cos(45°)}{1 - r^2} \tag{7.1.22}$$

The distance from the point $(0, y_c)$ to $lin(R)$ is $y_c\cos(45°)$. Setting x_c equal to $y_c\cos(45°)$ gives

$$r_0(45°) \approx 0.205147 \tag{7.1.23}$$

□

Angles Greater Than Or Equal To 135°

There is a straightforward way to determine $r_0(\theta)$ if $\theta \geq 135°$. For such angles, the contact address is RR. Let $P_c = (x_c, y_c)$. The main type of holes (if they exist) have a contact value of $x_c/2$. First we will determine the largest scaling ratio as a function of θ such that the corresponding tree does not have main holes. Then we will show that this implies that there are no other types of holes either, and so this value is indeed $r_0(\theta)$.

Proposition 7.1.3.2 *Let $\theta \geq 135°$. The tree $T(r,\theta)$ does not have main holes if*

$$r \leq \frac{\cos\theta + 1}{2(\cos^2\theta - \cos\theta - \cos(2\theta))} \tag{7.1.24}$$

Proof. Let $\theta \geq 135°$. The main hole is located by the pair $(\mathbf{A}_0, RR)$. Let $P_c = (x_c, y_c)$ denote the point with address RR, and let $P_1 = (x_1, y_1)$ where $x_1 = x_c/2$ and $y_1 = y_c$. The only portion of the subtree S_R that has smaller x values than x_c is the branch $b(R)$. For a given tree, the main hole exists if P_1 is at a distance of more than $x_c/2$ from $b(R)$, because this means that there is an open vertical interval above the point P_1 that is more than $x_c/2$ away from the subtree S_R. If the distance from P_1 to $b(R)$ is less than or equal to $x_c/2$, then there can be no main hole. The coordinates of P_c are given by:

$$x_c = r\sin(\theta) + r^2\sin(2\theta) \tag{7.1.25}$$

$$y_c = 1 + r\cos\theta + r^2\cos(2\theta) \tag{7.1.26}$$

Let d_1 denote the distance from P_1 to $b(R)$. The equation of the line $lin(R)$ is given by

$$y_{lin(R)}(x) = (\cot\theta)x + 1 \tag{7.1.27}$$

Then

$$\begin{aligned} d_1 &= [y_{lin(R)}(x_c/2) - y_c]\sin(180° - \theta) \\ &= \left[\frac{(\cot\theta)(r\sin\theta + r^2\sin(2\theta))}{2} - (r\cos\theta + r^2\cos(2\theta))\right]\sin\theta \end{aligned} \tag{7.1.28}$$

Solving the inequality $d_1 \leq x_c/2$ gives

$$r \leq \frac{\cos\theta + 1}{2(\cos^2\theta - \cos\theta - \cos(2\theta))} \tag{7.1.29}$$

Thus the tree $T(r,\theta)$ does not have main holes if Equation 7.1.24 is satisfied. $\square$

Proposition 7.1.3.3 *Let $\theta \geq 135°$. If the tree $T(r, \theta)$ does not have main holes, then the tree has no other types of holes and hence it is simple.*

Proof. Let $\theta \geq 135°$. Let r be such that the tree $T(r, \theta)$ does not have main holes. Let P_c and P_1 denote the same points as in the previous proposition and proof, and let d_1 denote the distance from P_1 to $b(R)$ (so $d_1 \leq x_c/2$ to ensure that there is no main hole). Let $P_2 = (x_2, y_2)$ denote the point with address $RRRL$. Then

$$x_2 = x_1(1 + r^2), \quad y_2 = y_1 + r^2(y_1 - 1) \tag{7.1.30}$$

If there is no hole located by the pair $(\mathbf{A}_0, RR)$, then there are no hole locator points with y-coordinate between 1 and y_c, because the region is covered at $\epsilon = x_c/2$, and this is the smallest ϵ to get contact. Now consider the region between P_1 and P_2. P_2 is a local minimum, and it would locate a hole if the point $P_3 = (x_2/2, y_2)$ is more than $x_2/2$ away from the subtree S_R. The only portion of the subtree S_R that could be at a distance of $x_2/2$ or less to P_3 is the branch $b(RRL)$. This branch is parallel to $b(R)$. Let d_2 denote the distance from P_3 to the branch $b(RLL)$. Then by the scaling nature of the tree, we have

$$\begin{aligned} d_2 &= r^2 d_1 \\ &\leq r^2 x_c/2 \\ &\leq r^2 x_c/2(1 + r^2) = x_2/2 \end{aligned} \tag{7.1.31}$$

Thus the point P_3 is within $x_2/2$ of the branch $b(RRL)$, and there are no hole locator points between P_c and P_2. Similarly we could show that there are no hole locator points between any two points with addresses $RR(LR)^k$ and $RR(LR)^{k+1}$ for $k \geq 0$. This means that there are no hole locator points whatsoever and the tree is simple. $\square$

Theorem 7.1.3.4 *Let $\theta \geq 135°$. The simple scaling ratio $r_0(\theta)$ is given by*

$$r_0(\theta) = \frac{\cos\theta + 1}{2(\cos^2\theta - \cos\theta - \cos(2\theta))} \tag{7.1.32}$$

Proof. This theorem is a direct result of the previous two propositions. Given $\theta \geq 135°$, let r' be equal to the right hand side of Equation 7.1.32. Then for $r \leq r'$,

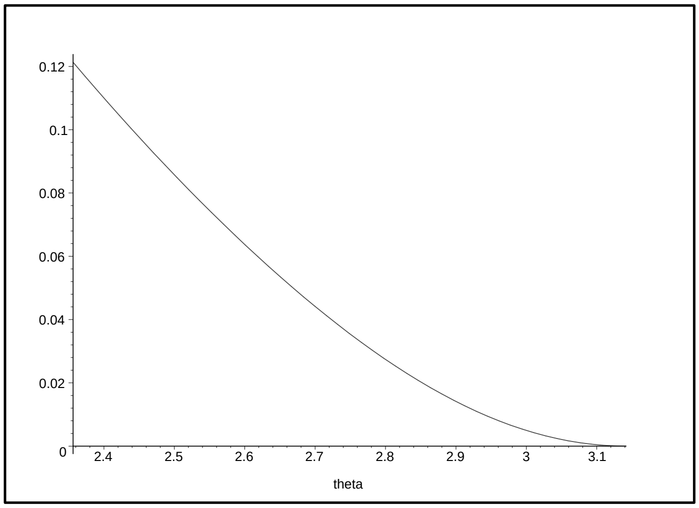

Figure 7.4: Plot of $r_0(\theta)$ as a function of θ (in radians)

the tree $T(r, \theta)$ is simple. If $r > r'$, then the tree $T(r, \theta)$ has at least the main holes, and is not simple. So $r' = r_0(\theta)$ by definition of simple scaling ratio. $\square$

The simple scaling ratio r_0 decreases as r increases from 135° to 180°. A plot of r_0 as a function of θ for these angles is given in Figure 7.4.

General Complexity of Self-Avoiding Trees

Now consider self-avoiding trees in general. Any self-avoiding tree has finite complexity, while any self-contacting tree has infinite complexity (if not space-filling) or it is simple (see Section 4.5). So in a sense, complexity is most interesting for the self-avoiding trees. For a given angle, is there a finite upper bound to the complexity that the self-avoiding trees can have? We discuss examples of three different angles. The first two angles are such that the self-avoiding trees do have a finite upper bound to complexity, while the third one does not.

The Angle $\theta = 90°$

For any non-simple tree with branching angle 90° the complexity is at most 2, and for the majority the complexity is at most 1.

Proposition 7.1.3.5 *For trees with branching angle* 90°*, the complexity of the tree is at most 2. The scaling ratio* r_1 *that cuts off the complexity 2 trees from complexity*

1 trees is $r_1 \approx 0.50866$*. Moreover, the proportion* ρ *of trees with complexity 2 to trees with complexity 1 is given by*

$$\rho = \frac{r_1 - 1/2}{1/\sqrt{2} - r_1} \approx 0.0436 \tag{7.1.33}$$

Proof. If $T(r, 90°)$ is simple, then the complexity is 0, which trivially satisfies the theorem. Let $T(r, 90°)$ be a non-simple tree. First we will show that there can't be holes above the trunk at more than one level for any given ϵ, and similarly for holes below. Then we will show that there can't be holes above and below at different levels.

Consider holes above the trunk (so assume $r > 0.5$). Let C_k denote the hole classes corresponding to the degree k canopy intervals of the subtree S_{RLL}. Then

$$\underline{\epsilon_{C_k}} = x_1, \quad \overline{\epsilon_{C_k}} \leq x_1\sqrt{1 + r^{6+4k}} \tag{7.1.34}$$

as discussed in the previous chapter (see Section 6.4.1). If $r\overline{\epsilon_{C_0}} < \underline{\epsilon_{C_0}} = x_1$, then there cannot be holes above the trunk in more than one level for any ϵ. Now $r\overline{\epsilon_{C_0}} = rx_1\sqrt{1 + r^6}$, so this inequality is satisfied if $r\sqrt{1 + r^6} < 1$. The largest value that $r\sqrt{1 + r^6}$ can take over the interval $(1/2, 1/\sqrt{2}]$ is at $1/\sqrt{2}$, and this value is $3/4$. Hence there can not be holes above the trunk at more than one level for any given ϵ.

Now consider holes below. Let C'_k denote the hole classes corresponding to the degree k canopy intervals of the subtree S_{RRR}. Then

$$\underline{\epsilon_{C'_k}} = \frac{x_1}{2}, \quad \overline{\epsilon_{C_k}} \leq \frac{x_1}{2}(1 + r^{6+4k}) \tag{7.1.35}$$

If $r\overline{\epsilon_{C'_0}} < \underline{\epsilon_{C'_0}} = x_1/2$, then there cannot be holes below in more than one level for any ϵ. Now $r\overline{\epsilon_{C'_0}} = rx_1(1 + r^6)/2$, so this inequality is satisfied if $r(1 + r^6) < 1$. The largest value that $r(1+r^6)$ can take over the interval $(1/2, 1/\sqrt{2}]$ is at $1/\sqrt{2}$, and this value is approximately 0.795. Hence there can not be holes below at more than one level for any given ϵ.

Finally we consider if it is possible to have a different level of holes above at the same time as holes below. Since the contact values of holes above are double the contact values of holes below for a fixed level, we consider level 1 holes above and level 0 holes below. If $\underline{\epsilon_{C'_0}} < r\underline{\epsilon_{C_0}} < \overline{\epsilon_{C'_0}}$ then there will indeed be holes at levels 0 and 1 for

$\epsilon = \underline{\epsilon_{C_0}}$. We show that this is not possible. Let $f(r) = r\underline{\epsilon_{C_0}} - \overline{\epsilon_{C_0'}}$ over the interval $(1/2, 1/\sqrt{2})$ (since r needs to be greater than $1/2$ to have any holes above the trunk). Whenever $f(r) > 0$, there can not be holes in more than one level. Then

$$\begin{aligned} f(r) &= rx_1 - \frac{x_1}{2}(1 + r^6) \\ &= \frac{x_1}{2}\left(2r - 1 - r^6\right) \end{aligned} \tag{7.1.36}$$

Now $f(r) > 0$ as long as $g(r) = 2r - 1 - r^6 > 0$. We have $g(1/2) = -1/64$ and $g(r)$ is increasing since $g'(r) = 2 - 6r^5$ (which is always positive for r in the interval $(1/2, 1/\sqrt{2})$). Let r_1 denote the root of $g(r)$ in $(1/2, 1/\sqrt{2})$, then $r_1 \approx 0.50866$. For $r > r_1$, the corresponding tree definitely has complexity at most 1, because $f(r) > 0$. What about $1/2 < r \leq r_1$? Now we need to show that there can be holes in at most 2 levels. In this case, the corresponding inequality is $\underline{\epsilon_{C_0'}} < r^2\underline{\epsilon_{C_0}} < \overline{\epsilon_{C_0'}}$ This inequality is not satisfied if $r^2\underline{\epsilon_{C_0}} < \underline{\epsilon_{C_0'}}$, *i.e.*, if

$$r^2 x_1 < \frac{x_1}{2}$$

We are only considering scaling ratios between $1/2$ and r_1, and any such scaling ratio r clearly satisfies $r^2 < 1/2$. Hence there cannot be holes in levels 0 and 2 for any given ϵ. Thus these trees have complexity at most 2.

The interval of scaling ratios that have complexity 2 has length $r_1 - 1/2$, and the interval of scaling ratios that has complexity 1 has length $1/\sqrt{2} - r_1$. The ratio ρ of these two interval lengths is given by

$$\rho = \frac{r_1 - 1/2}{1/\sqrt{2} - r_1} \approx 0.0436 \tag{7.1.37}$$

□

The Angle $\theta = 135°$

Proposition 7.1.3.6 *Trees with branching angle 135° have complexity at most 4.*

Proof. Let $T(r, 135°)$ be a non-simple tree. Let ϵ_0 be the minimum ϵ-value to get level 0 holes. Then by equation 6.4.22, we have

$$\epsilon_0 = \frac{x_c}{2} \tag{7.1.38}$$

Now consider the tip point $P_1 = (x_1, y_1)$ with address $RR(RL)^\infty$. This point is the highest tip point of S_{RR}. All other vertex points of the subtree S_{RR} are closer to the trunk than P_1. For any $\epsilon \geq x_1$, there can not be any level 0 holes. This is not the actual collapse value of a level 0 hole, but it is an upper bound that suffices to prove the proposition. By the scaling nature of the tree, we have

$$x_1 = x_c(1 + r^2 + r^4 \cdots) = \frac{x_c}{1 - r^2} \tag{7.1.39}$$

So, for any $k \geq 1$, there are no holes of level k or higher if $\epsilon \geq r^k x_1$. There cannot be holes at level k and level 0 for a specific ϵ if

$$r^k x_1 < \frac{x_c}{2}$$

That is, there cannot be holes at level k and level 0 for a specific ϵ if

$$\frac{r^k}{1 - r^2} x_c < \frac{x_c}{2}$$

Since we are assuming the tree is non-simple, we have $x_c > 0$ (for if $x_c = 0$ then the tree is self-contacting and space-filling). So the condition reduces to

$$\frac{r^k}{1 - r^2} < \frac{1}{2}$$

Let $f_k(r)$ be the function defined by

$$f_k(r) = \frac{r^k}{1 - r^2}$$

For a fixed k, the function $f_k(r)$ is increasing on the interval $(0, 1/\sqrt{2})$. So for all $r < 1/\sqrt{2}$, we have

$$f_k(r) < f_k\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{1}{\sqrt{2}}\right)^k$$

Consider $k = 4$.

$$f_4\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

So for any $r < 1/\sqrt{2}$, $f_4(r) < 1/2$. Moreover, for a fixed r, $f_k(r)$ is a decreasing function of k. This implies that for any $k \geq 4$, $f_k(r) < 1/2$. So for any ϵ for which there are level 0 holes, there are no holes of level 4 or higher. That is, there could be holes at levels 0 through 3. By definition of complexity, this means that the tree has complexity at most 4. $\square$

Note. We have proved that 4 is an upper bound for the complexity. We used a rather generous upper bound for the collapse value of level 0 holes, so we might wonder if this complexity upper bound is too high. However, the tree $T(0.707, 135°)$ can be shown to have complexity 4. We do not present complete details for this tree here, but the main idea to show that the complexity is indeed 4 is to find a lower bound for the collapse value of level 0 holes. When $\epsilon = x_1/2$ (half the distance from P_1 with address $RR(RL)^\infty$ to the trunk), then there is still a level 0 hole. One can show that for $r = 0.707$, $r^3 x_1/2 > x_c/2$, hence there are holes at levels 0 through 3 when $\epsilon = x_c/2$, and thus the complexity equals 4 (since we have already shown that it is at most 4).

The Angle 45°

Proposition 7.1.3.7 *For self-avoiding trees with branching angle* 45°, *there is no upper bound on the complexity.*

Proof. To prove the proposition, we will just prove that there is no limit to the complexity for the main hole classes. The complexity of a hole class is less than or equal to the complexity of the tree, so if there is no upper limit for the complexity of the main hole classes, then there is no upper limit to the complexity of the trees. The main hole class is located by the pair $(\mathbf{A}_0, RL^3(LR)^\infty)$. Let $P_c = (x_c, y_c)$ denote the point with address $RL^3(LR)^\infty$. The contact value for the main class is equal to x_c. For a given r, we have

$$\begin{aligned} x_c &= r\sin\theta - \frac{r^3\sin\theta + r^4\sin(2\theta)}{1-r^2} \\ &= \frac{r}{\sqrt{2}} - \frac{r^3}{\sqrt{2}(1-r^2)} - \frac{r^4}{1-r^2} \\ &= \frac{r}{\sqrt{2}(1-r^2)}(1 - 2r^2 - \sqrt{2}r^3) \end{aligned} \tag{7.1.40}$$

As discussed in the subsection dealing with persistence, for a given tree $T(r, 45°)$, a square of side length r is contained within the region that forms the main hole. So the collapse value for the square is less than or equal to the collapse value for the actual hole. The collapse value of the square is $r/2$.

Let k be an arbitrary positive integer. We wish to show that there exists $r' < r_{sc}$ such that the tree $T(r', 45°)$ has complexity at least k. Let M denote the main hole class. Then

$$\begin{aligned} \underline{\epsilon_M} &= \frac{r}{\sqrt{2}(1-r^2)}(1-2r^2-\sqrt{2}r^3) \\ \overline{\epsilon_M} &\geq \frac{r}{2} \end{aligned} \tag{7.1.41}$$

The main hole class will have complexity at least k if

$$r^{k-1}\overline{\epsilon_M} > \underline{\epsilon_M}. \tag{7.1.42}$$

This inequality is satisfied if

$$r^{k-1}\frac{r}{2} > \underline{\epsilon_M} = \frac{r}{\sqrt{2}(1-r^2)}(1-2r^2-\sqrt{2}r^3) \tag{7.1.43}$$

We have the following:

$$\begin{aligned} & r^{k-1}\frac{r}{2} > \frac{r}{\sqrt{2}(1-r^2)}(1-2r^2-\sqrt{2}r^3) \\ \Rightarrow\ & r^{k-1} > \frac{2}{\sqrt{2}(1-r^2)}(1-2r^2-\sqrt{2}r^3) \end{aligned} \tag{7.1.44}$$

We are just considering self-avoiding trees, so $r < r_{sc}$. The self-contacting scaling ratio is the root of $1-2r^2-\sqrt{2}r^3$ (as discussed in the previous chapter), and for all $r < r_{sc}$, $1-2r^2-\sqrt{2}r^3$ is positive. Let $f(r)$ be defined as follows:

$$f(r) = \frac{2}{\sqrt{2}(1-r^2)} \tag{7.1.45}$$

Then $f(r)$ is an increasing function on $(0, r_{sc})$, so for all $r < r_{sc}$ we have $f(r) < f(r_{sc})$. Thus

$$f(r) < f(r_{sc}) \approx 2.1812 \tag{7.1.46}$$

Let $C = f(r_{sc})$. Now for $k \geq 2$, let $g_k(r) = r^{k-1}$. Then $g_k(r)$ is an increasing function on $(0, r_{sc})$, and $g_k(r) > 0$.

Let $k \geq 2$ be given. Consider scaling ratios in the interval $[0.5, r_{sc})$. For each $r \in$

$[0.5, r_{sc})$, we have $g_k(r) \geq g_k(0.5) > 0$. For any $\delta > 0$, we can find $r < r_{sc}$ such that $C(1 - 2r^2 - \sqrt{2}r^3) < \delta$, since

$$\lim_{r \to r_{sc}} C(1 - 2r^2 - \sqrt{2}r^3) = 0 \tag{7.1.47}$$

So let $\delta = g_k(0.5)$, and let r' be a value in $[0.5, r_{sc})$ such that $C(1-2(r')^2-\sqrt{2}(r')^3) < \delta$. Then

$$g_k(r') \geq g_k(0.5) > C(1 - 2(r')^2 - \sqrt{2}(r')^3) \tag{7.1.48}$$

Which means that the tree $T(r', 45°)$ has complexity at least k. This can be done for any k, so there is no upper bound to the complexity that a self-avoiding tree with branching angle 45° can have. $\square$

Remarks. Given a specific branching angle, we do not know in general if there is an upper bound to the complexity of self-avoiding trees with that branching angle. We have seen three examples of branching angles, and there is an upper bound in two cases and not in the third. In the first two cases, the angles are special (90° and 135°), and they can be distinguished from the other angles because they are the only ones whose self-contacting trees are space-filling. So one might conjecture that an upper bound on complexity of self-avoiding trees is unique to these two angles. Since we do not have a straightforward method to find the collapse value of any hole class, this conjecture is difficult to prove or find a counter-example. We are currently looking for a counter-example, because we do not believe the conjecture is true. Consider angles that are close to the special angles. The persistence of the main hole class just of the self-contacting trees seems to decrease as the branching angle gets closer to 90° or 135°. The smaller the persistence of a hole class, the lower its complexity. We are still trying to determine the connections between persistence and complexity.

An early conjecture was that for any θ, if $r_1 < r_2 < r_{sc}(\theta)$, then the complexity of the tree $T(r_1, \theta)$ is less than or equal to the complexity of the tree $T(r_2, \theta)$. In other words, the complexity is non-decreasing as r approaches the self-contacting scaling ratio. The angle 90° provides a counter-example to this conjecture.

Another early conjecture was that for any tree and for a fixed ϵ-value, if there are holes at level j and level k in the closed ϵ-neighbourhood for some $j > k \geq 0$, then there are holes in the closed ϵ-neighbourhood for any level l such that $k \leq l \leq j$. The golden tree $T(r_{sc}, 108°)$ provides a counter-example, as mentioned in the previous chapter.

To summarize, some open questions regarding complexity are:

- Is there a relationship between persistence and complexity, and if so, what is it?
- For a given branching angle, is there a finite upper bound to the complexity that the self-avoiding trees with that branching angle can have?
- Recall that the critical scaling ratios for a given branching angle θ are values that indicate a change in complexity. The k-complexity class of θ is the set of scaling ratios for which $T(r, \theta)$ has complexity equal to k. So the critical scaling ratios are endpoints of the components of the k-complexity classes. Is it possible for a k-complexity class to be disconnected? To date we have not found an example of such an angle, nor do we have a proof that the class is connected.
- If the k-complexity classes are connected, then we can define a sequence of values $r_k(\theta)$, where $r_k(\theta)$ is the upper bound of the k-complexity class. For a given k, how does $r_k(\theta)$ change as a function of θ? We already know that the functions would not be continuous in general, because for any $k \geq 3$, $r_k(90°)$ is not defined. Is the shape of the curve $r_k(\theta)$ related to the shape of the curve $r_{sc}(\theta)$?

7.1.4 Hole Partitions and Hole Sequences

The hole sequence is one way to characterize a fractal tree, and it reflects various features of the tree, such as persistence, type and location of holes, and the complexity of the tree. It gives a 'topological barcode', a term coined by Carlsson *et al.* [7]. Many of the examples in the previous chapter were such that the hole sequence was order-isomorphic to the naturals, that is, it could be indexed by the natural numbers.

However, this is not true in general, since a few other sequences were such that they could be indexed by non-standard ordinals, using two indices that each came from the natural numbers.

As demonstrated in the examples, the hole partitions and hole sequences have a wide range of complexity, where we mean complexity in the usual sense of the word, not in the sense of our definition pertaining to levels of holes. As ϵ decreases from ∞, the hole partition eventually comes to a block of intervals that is repeated (only scaled by a factor of r after one cycle of the block). For example, a self-contacting tree $T(r_{sc}, \theta)$ with only main holes has a straightforward hole partition. The first non-trivial interval is $[r_{sc}\overline{\epsilon_M}, \overline{\epsilon_M})$, and every other interval is equal to this interval scaled by some factor of r_{sc}, so the length of the block that is repeated is just 1. On the other hand, a specific example where the length of the repeated block is infinite is the tree $T(0.5, 90°)$. So perhaps the length of the repeated block is another way to characterize 'complexity' of a tree.

An important feature that the hole sequences indicate is that the growth rate of holes is always equal to the similarity dimension of the tree. For any non-simple tree, and for any sequence of ϵ-values where $\epsilon_n = r\epsilon_{n-1}$, we have the following growth rate:

$$\lim_{n\to\infty} \frac{\log(N([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

Now we provide a proof for this claim. In the Introduction Chapter to this thesis, we mentioned that a general conjecture for all self-similar fractals regarding growth rates of holes was put forth by Robins in [46]. Our proof is for the analagous result of non-overlapping non-simple symmetric binary fractal trees. The fact that it does not work for non-overlapping trees is not a counter-example, because they are not strictly self-similar. However, it does show that Robins' conjecture would not work for all fractals with condensation. First we give a lemma that deals with the growth rate of holes for a specific hole class.

Lemma 7.1.4.1 *Let $T(r, \theta)$ be a non-simple tree. Let $[H]$ be any level 0 hole class for the tree. Let $\epsilon_0 \in p([H])$ be such that $\epsilon_0 > 0$ and the only holes of type $[H]$ in $E(r, \theta, \epsilon_0)$ are level 0. Let $\{\epsilon_n\}$ be a sequence of ϵ-values for $n \geq 0$ such that $\epsilon_n = r^n\epsilon_0$.*

Then

$$\lim_{n\to\infty} \frac{\log(N_{[H]}([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

where $N_{[H]}([\epsilon_n])$ denotes the number of holes of type $[H]$ for any ϵ in the equivalence class $[\epsilon_n]$ (so the number of holes of the hole class $[H]$ and its descendant hole classes).

Proof. If H is above the line $y = 1$, then $N_{[H]}([\epsilon_0]) = 1$ and if H is below the line $y = 1$, then $N_{[H]}([\epsilon_0]) = 2$. Let $N'_n = N_{[H]}([\epsilon_n])$.
First suppose $[H]$ is a self-contacting hole class, so that $r^n \underline{\epsilon_H} = 0$ for all $n \geq 0$. For any $n \geq 0$, we have

$$\begin{aligned} N'_n &= N'_0(1 + 2 + \cdots 2^n) \\ &= N'_0(2^{n+1} - 1) \end{aligned} \tag{7.1.49}$$

Then

$$\begin{aligned} \lim_{n\to\infty} \frac{\log N'_n}{\log(1/\epsilon_n)} &= \lim_{n\to\infty} \frac{\log N'_0(2^{n+1}-1)}{\log(1/r^n\epsilon_0)} \\ &= \lim_{n\to\infty} \frac{\log(2^{n+1}-1) + \log(N'_0)}{\log(1/r^n) + \log(1/\epsilon_0)} \\ &= \lim_{n\to\infty} \frac{\log(2^{n+1}-1)}{\log(1/r^n) + \log(1/\epsilon_0)} + \frac{\log(N'_0)}{\log(1/r^n) + \log(1/\epsilon_0)} \\ &= \lim_{n\to\infty} \frac{\log(2^{n+1}-1)}{\log(1/r^n) + \log(1/\epsilon_0)} \\ &= \lim_{n\to\infty} \frac{\log(2^{n+1}-1)}{\log(1/r^n)} \\ &= \lim_{n\to\infty} \frac{\log 2^{n+1}}{n\log(1/r)} \\ &= \lim_{n\to\infty} \frac{n\log 2 + \log 2}{n\log(1/r)} \\ &= \lim_{n\to\infty} \frac{n\log 2}{n\log(1/r)} \\ &= \frac{\log 2}{\log(1/r)} \end{aligned} \tag{7.1.50}$$

Now suppose the hole class is not a self-contacting hole class. Then it has finite complexity. There is a finite integer d less than or equal to $C([H])$ (the complexity of the hole class) such that there are no level 0 holes of type H for any ϵ_n where $n \geq d$. So for any ϵ_n where $n \geq d$, there are holes of levels $n-d+1$ through n, and

$$N'_n = N'_0(2^{n-d+1} + \cdots + 2^n) \tag{7.1.51}$$

Then we can use a similar argument as for the self-contacting hole classes to show that

$$\lim_{n\to\infty} \frac{\log N'_n}{\log(1/\epsilon_n)} = \frac{\log 2}{\log(1/r)} \tag{7.1.52}$$

□

Corollary 7.1.4.2 *Let $T(r,\theta)$ be a non-simple tree. Let $[H]$ be any hole class for the tree. Let $\epsilon_0 \in p([H])$ be such that $\epsilon_0 > 0$. Let $\{\epsilon_n\}$ be a sequence of ϵ-values for $n \geq 0$ such that $\epsilon_n = r^n\epsilon_0$. Then*

$$\lim_{n\to\infty} \frac{\log(N_{[H]}([\epsilon_n]))}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

where $N_{[H]}([\epsilon_n])$ denotes the number of holes of type $[H]$ for any ϵ in the equivalence class $[\epsilon_n]$

Proof. This follows directly from the previous lemma. Suppose $[H]$ is level k, then it is the descendant of a unique level 0 hole class, $[H']$. Then $\epsilon' = r^{-k}\epsilon_0$ is in $p([H'])$. Consider the sequence $\{\epsilon'_n\}$ defined by $\epsilon'_n = r^n\epsilon'$. Then

$$\lim_{n\to\infty} \frac{\log(N_{[H]}([\epsilon_n]))}{\log(1/\epsilon_n)} = \lim_{n\to\infty} \frac{\log(N_{[H]}([\epsilon'_n]))}{\log(1/\epsilon'_n)} = \frac{\log 2}{\log 1/r}$$

□

Now we have the general theorem.

Theorem 7.1.4.3 *Let $T(r,\theta)$ be a non-simple tree. Let $\epsilon_0 > 0$ be such that $E(r,\theta,r^n\epsilon_0)$ has a finite number of hole classes for all $n \geq 0$. For the sequence $\{\epsilon_n\}$ defined by $\epsilon_n = r^n\epsilon_0$, the growth rate of holes is given by*

$$\lim_{n\to\infty} \frac{\log N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r} \tag{7.1.53}$$

Proof. First, there exists $m \geq 0$ such that for any $n \geq m$, the hole classes of $E(r, \theta, \epsilon_n)$ are all descendants of the hole classes of $E(r, \theta, \epsilon_m)$, otherwise the assumption that $E(r, \theta, \epsilon_n)$ has a finite number of hole classes for all n is contradicted. Let M be the number of hole classes in $E(r, \theta, \epsilon_m)$. We can label the hole classes $[H_i]$ for $1 \leq i \leq M$. For $n \geq m$, we have

$$N([\epsilon_n]) = \sum_{i=1}^{M} N_{H_i}([\epsilon_n]),$$

where $N_{H_i}([\epsilon_n])$ represents the number of holes that descend from $[H_i]$. Then

$$\lim_{n\to\infty} \frac{\log N([\epsilon_n])}{\log(1/\epsilon_n)} = \lim_{n\to\infty} \frac{\log \sum_{i=1}^{M} N_{H_i}([\epsilon_n])}{\log(1/\epsilon_n)}$$

We will evaluate this limit by using the Squeeze Theorem. Let N'_M denote $\sum_{i=1}^{M} N_{H_i}([\epsilon_m])$. For $n \geq m$, we have

$$2^{n-m} N'_M \leq \sum_{i=1}^{M} N_{H_i}([\epsilon_n]) \leq (1 + 2 + \cdots + 2^{n-m}) N'_M \tag{7.1.54}$$

The lower limit is the value that would correspond to each hole class having complexity equal to 1, while the upper limit corresponds to the hole classes all having infinite complexity. As in the proof of Lemma 7.1.4.1, we have

$$\lim_{n\to\infty} \frac{\log 2^{n-m} N'_M}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

and

$$\lim_{n\to\infty} \frac{\log(1 + \cdots + 2^{n-m}) N'_M}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\log N([\epsilon_n])}{\log(1/\epsilon_n)} = \frac{\log 2}{\log 1/r}$$

□

Other interesting items to note:

- It is possible for a closed ϵ-neighbourhood to have infinitely many holes for a non-zero ϵ-value.

- In general, the hole sequences are not monotonically increasing

The hole sequence is a detailed feature of a tree. It is interesting from a theoretical point of view, but may be difficult to use in applications. We can use the hole sequence as a kind of 'topological barcode' for a tree, as defined and studied by Carlsson *et al.* [7].

7.2 Conclusions

This thesis has presented a thorough study of symmetric binary fractal trees and their closed ϵ-neighbourhoods. This work was inspired by the work of Mandelbrot and Frame on the self-contacting symmetric binary fractal trees [31].

We have attempted to describe the taxonomy of holes in closed ϵ-neighbourhoods of symmetric non-overlapping binary fractal trees. Based on the work of Carlsson *et al.* [6], [7], we are led to determine the hole sequence of these trees together with the persistence intervals of the holes as the 'topological barcodes' of these trees. The first obstacle in doing this is that persistence has some interesting and perhaps unexpected properties. However, we also realize that this approach ignores other aspects of the holes, and so we classify holes more according to their shape and location.

The action of the free monoid on two generators on the tree brings a natural grading by level to these holes. For every higher level hole, there is precisely one level 0 hole that is mapped to this hole under the action of the monoid. So the level 0 holes form a kind of fundamental domain, and we can focus our attention on the level 0 holes. Moreover, the trees and their closed ϵ-neighbourhoods are symmetric about the y-axis, so we can further restrict our attention to level 0 holes that are not disjoint from the right side of the y-axis.

To describe the location of a hole, we have generalized the notion of contact address to hole locator address and hole locator pairs. This is a a nice feature of the symmetric binary trees. For self-contacting trees, self-contact has two cases for non-space-filling trees. If the angle is less than 90°, self-contact occurs above the trunk. If the angle is greater than 90°, self-contact occurs with the trunk. For closed ϵ-neighbourhoods of self-avoiding or self-contacting trees, the location of holes is not so straightforward. There seems to be a more continuous progression of the hole locations as the angle increases from 0° to 180°.

From these notions and properties we derive certain classifications of the symmetric binary fractal trees. These are the complexity, location, type and hole sequence classifications. We also obtain critical values based on these classifications. From the hole sequence we obtain critical ϵ-values for a specific tree. From the complexity classification we obtain critical values of the scaling ratios for a given branching angle. Finally, we obtain critical values of the branching angles based on the location of the holes. The type classification is coarser than the location classification. The classifications based on location, complexity and hole sequence are not comparable- it is possible to find two trees with the same hole location sets but different complexity, or same complexity but different hole locations, and so on. More work needs to be done to study the connections between the different classifications.

We presented a collection of examples of trees and their closed ϵ-neighbourhoods to demonstrate our theory and the geometrical techniques we use to obtain quantitative and qualitative information. The most surprising result of the thesis was the connection between the trees and the golden ratio. The four 'golden trees' were each discussed. These trees are particularly interesting because of their symmetrical properties, and we are currently continuing to study them.

Following the examples, we discussed specific critical values. Our work certainly supports the claim by Mandlebrot and Frame that the two angles 90° and 135° are topologically critical, but our work also presents a plethora of other critical values. Although they may not indicate as significant a change in topology as 90° and 135°, they are still noteworthy. For example, the angle $\theta \approx 57.0057^\circ$ is an important critical angle because it is the lower bound for hole locations with the trunk, and the angle 112.5° is important because it is the upper bound for hole locations above the trunk.

The fundamental part of the thesis is the introduction of new notations, definitions, and theory regarding closed ϵ-neighbourhoods of symmetric binary fractal trees. This includes various aspects of holes such as the notion of a hole class; persistence, complexity, level, location and type of a hole class. Our goal was to develop new techniques to characterize fractal trees, and this has been achieved.

7.3 Future Work

Throughout the discussion and conclusions, we have presented a collection of questions that are immediate consequence of our work. In this last section of the thesis, we present a broader sense of future work that could stem from our work. The main accomplishment of this thesis is the introduction of new notation, concepts, theory and geometrical techniques to study fractal trees and their closed ϵ-neighbourhoods. This is just the beginning, and we are excited to see where the theory will lead. Topics for future work include:

- Find a general definition of persistence for holes of closed ϵ-neighbourhoods of any subset of $\mathbb{R}^2$; study the nature of persistence and what it tells us about the underlying sets
- Study maps between one tree and its closed ϵ-neighbourhoods and another tree and its closed ϵ-neighbourhoods, and in particular, the homeomorphisms that preserve certain classifications that we have developed
- Study closed ϵ-neighbourhoods of finite trees, and compare with the results about fractal trees
- Study closed ϵ-neighbourhoods where ϵ scales according to level of a branch
- Extend theory to other classes of fractal trees: asymmetric trees, general binary trees, general n-ary trees, L-system trees, three-dimensional trees
- Extend theory to other classes of fractals
- Develop a suitable categorical framework in which to study fractal trees, and fractals in general
- Investigate possible applications of the theory to natural systems; perhaps critical values have physical or biological significance

Appendix A

Background in Topology and Fractals

For basic definitions, notations and theorems from topology, see [38].

A.1 Metric Spaces

A very important and useful concept in topology is the concept of a metric space, that is, a space that has some notion of distance.

A.1.1 Basic Definitions and Theorems

We follow Munkres [38] for notation and theory.

Definition A.1.1.1 *A* **metric** *on a set X is a function*

$$d : X \times X \longrightarrow \mathbb{R}$$

with the following properties:

1. *$d(x, y) \geq 0$ for all $x, y \in X$ (non-negative)*
2. *$d(x, y) = 0$ if and only if $x = y$*
3. *$d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)*
4. *$d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$ (Triangle inequality)*

The number $d(x, y)$ is usually called the **distance** *between x and y in the metric d.*

Definition A.1.1.2 *The* **ϵ-ball centered at** *x is the set*

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}.$$

If d is a metric on X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

Definition A.1.1.3 *A topological space X is* **metrizable** *if there exists a metric d on the set X that induces the topology of X. A metric space is a metrizable space with a specific metric d that gives the topology of X.*

Definition A.1.1.4 *A function $f : X_1 \longrightarrow X_2$ from a metric space (X_1, d_1) into a metric space (X_2, d_2) is* **continuous** *if, for each $\epsilon > 0$ and $x \in X_1$, there is a $\delta > 0$ such that*

$$d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon.$$

Definition A.1.1.5 *A sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is called a* **Cauchy sequence** *if for any $\epsilon > 0$ there exists an integer $N > 0$ such that*

$$d(x_n, x_m) < \epsilon \quad \textit{for all} \quad n, m > N.$$

Definition A.1.1.6 *A sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to* **converge** *to a point $x \in X$ if for any $\epsilon > 0$ there is an integer $N > 0$ such that*

$$d(x_n, x) < \epsilon \quad \textit{for all} \quad n > N.$$

In this case the point x is called the limit of the sequence, denoted

$$x = \lim_{n \to \infty} x_n.$$

Theorem A.1.1.7 *If a sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) converges to a point $x \in X$, then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.*

Definition A.1.1.8 *A metric space (X, d) is* **complete** *if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit $x \in X$.*

Definition A.1.1.9 *Let $S \subset X$ be a subset of a metric space $X, d)$. A point $x \in X$ is called a* **limit point** *of S if there is a sequence of points $\{x_n\}_{n=1}^{\infty}$ where $x_n \in S \backslash \{x\}$ such that $\lim_{n \to \infty} x_n = x$.*

Definition A.1.1.10 *Let $S \subset X$ be a subset of a metric space (X, d). The* **closure** *of S, denoted $\overline{S}$, is defined to be $\overline{S} = S \cup \{\textit{limit points of } S\}$. S is* **closed** *if it contains all of its limit points. S is* **perfect** *if it is equal to the set of all its limit points.*

Note that a metric space will be compact if every infinite sequence $\{x_n\}_{n=1}^{\infty}$ in S contains a subsequence having a limit in S.

Definition A.1.1.11 *For the space $\mathbb{R}^n$ with the standard topology, the euclidean metric d is defined by*

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}.$$

Theorem A.1.1.12 *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d.*

A.1.2 The Metric Spaces For Fractals

What metric space is appropriate for the study of fractals? Generally, we assume fractals are compact and are subsets of some complete metric space (X, d). Here we follow the notation and theory of Barnsley [4].

Definition A.1.2.1 *Let $\mathcal{H}(X)$ denote the space whose points are the compact subsets of X, other than the empty set.*

Definition A.1.2.2 *Let (X, d) be a complete metric space, $x \in X$, and $B \in \mathcal{H}(X)$. The* **distance from the point** x **to the set** B*, denoted $d(x, B)$ is defined by*

$$d(x, B) = \min\{d(x, y) : y \in B\}.$$

Definition A.1.2.3 *Let (X, d) be a complete metric space. Let $A, B \in \mathcal{H}(X)$. The* **distance from the set** A **to the set** B*, denoted $d(A, B)$, is defined by*

$$d(A, B) = \max\{d(x, B) : x \in A\}.$$

In general, $d(A, B) \neq d(B, A)$, so this d does not provide a metric. We symmetrize as follows to obtain a metric:

Definition A.1.2.4 *Let (X, d) be a complete metric space. The* **Hausdorff distance** *between points $A, B \in \mathcal{H}(X)$ is defined by*

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

h is called the **Hausdorff metric**.

Theorem A.1.2.5 *(The Completeness of the Space of Fractals) Let* (X, d) *be a complete metric space. Then* $(\mathcal{H}(X), h)$ *is a complete metric space. Moreover, if* $\{A_n \in \mathcal{H}(X)\}_{n=1}^{\infty}$ *is a Cauchy sequence then*

$$A = \lim_{n\to\infty} A_n \in \mathcal{H}(X)$$

can be characterized as

$$A = \{x \in X | \textit{there is a sequence } \{x_n \in A_n\} \textit{ that converges to } x\}.$$

The notion of a **metric inverse limit** was introduced in [36]. This concept connects the idea of Hausdorff limit and inverse limit.

A.1.3 Transformation Mappings

Let $f : X \to X$ be a transformation on a metric space (X, d). The **forward iterates** of f are transformations $f^n : X \to X$ defined by

$$f^0(x) = x, \quad f^1(x) = x, \quad f^{n+1}(x) = f \circ f^n(x), \quad n > 0.$$

Definition A.1.3.1 *A transformation* $f : X \to X$ *on a metric space* (X, d) *is called a* **contraction mapping** *if there is a constant* $0 \le s < 1$ *such that*

$$d(f(x), f(y)) \le s \cdot d(x, y),\ \forall x, y \in X. \tag{A.1.1}$$

The lower bound of constants satisfying A.1.1 is called the **contractivity factor**.

Theorem A.1.3.2 The Contraction Mapping Theorem. *Let* $f : X \to X$ *be a contraction mapping on a complete metric space* (X, d)*. Then f possesses exactly one fixed point* $x_f \in X$ *and for any point* $x \in X$*, the sequence* $\{f^n(x) : n = 0, 1, 2, ...\}$ *converges to* x_f*. That is*

$$\lim_{n\to\infty} f^n(x) = x_f, \quad \forall x \in X.$$

Note that contraction mappings are continuous. Also note that if $w : X \to X$ is a continuous mapping on the metric space (X, d) then w maps $\mathcal{H}(X)$ onto itself.

Lemma A.1.3.3 *Let $w : X \to X$ be a contraction mapping on the metric space (X, d) with contractivity factor s. Then $w : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by*

$$w(B) = \{w(x) : x \in B\} \ for B \in \mathcal{H}(X)$$

is a contraction mapping on $(\mathcal{H}(X), h)$ with contractivity factor s.

A.2 Fractals and Fractal Dimensions

As mentioned in the Introduction of this thesis, Benoit Mandelbrot first used the word "fractal" to describe objects that were too irregular to fit into traditional geometric settings [30]. A classic example of a fractal is the *Cantor set*, which we will denote C. We shall present some details about the Cantor set because it possesses many features that are associated with fractals, and also because generalized Cantor sets are important for the trees that we study. See Figure 1.1.

The Cantor set C is the set obtained by deleting a sequence of open sets, known as the middle thirds, from the closed unit interval. Let E_1 denote the closed unit interval $[0, 1]$, and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ to obtain $E_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. From the remaining intervals in E_2, remove the middle thirds to obtain E_3, which is the union of four closed intervals. Thus $E_3 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Continue to remove the middle thirds of the remaining intervals, so that E_i is the union of 2^{i-1} closed intervals each of length $\left(\frac{1}{3}\right)^{i-1}$, for $i \geq 1$. The Cantor set C is the intersection of the successive closed remainders:

$$C = \bigcap_{i=1}^{\infty} E_i.$$

We shall present some of the interesting properties of the Cantor set. It is compact because it is the intersection of closed subsets of the unit interval, which is compact. C is a complete metric space. C is dense-in-itself since every open set containing a point $p \in C$ contains points of C distinct from p. Thus C is a perfect set because it is closed. C is nowhere dense in $[0, 1]$ since it is closed and no open interval that is a subset of $[0, 1]$ is disjoint from all the deleted open intervals of $[0, 1]$. C is uncountable. We can define a function f from C onto the uncountable set $[0, 1]$. Here one uses the fact that the Cantor set consists of all points in the closed unit interval which can be expressed to the base 3 without using the digit 1 (see [52]). So if $x \in C$ is written uniquely to the base 3 without the digit 1, define $f(x)$ to be the point in $[0, 1]$ whose binary expansion is obtained by replacing each digit "2" in the ternary expansion of x by the digit 1. This shows that all points of $[0, 1]$ can be obtained. The components of C are single points. Hence C has zero length and is totally separated.

A typical fractal, such as the Cantor set, possesses the following features [15]:

- It has fine structure (detail at arbitrary scales).
- It is too irregular to be described in traditional geometrical language, both locally and globally.
- It has some sort of self-similarity, possibly approximate or statistical.
- The 'fractal dimension' of the fractal, defined in some way, usually exceeds its topological dimension.
- It can be defined in a simple, possibly recursive, way.

The properties listed above are often present in fractals, but not necessarily. They do not provide a definition or a description of every set that one might consider 'fractal', but they offer a starting place. The fractal trees that we study in this thesis can be described in simple, recursive ways. They do possess a kind of self-similarity, but not in the strict sense described below, because a symmetric binary fractal tree is the union of two smaller versions of itself and its trunk (the residue).

Until now, the main tools used for studying and characterizing fractals have been the many forms of dimension. Dimension will be discussed in detail in the next section. Although dimension can be a powerful tool, it is possible for two fractals to have the same fractal dimension and yet be topologically distinct. This fact has led to the study of other aspects of fractals, such as lacunarity. Lacunarity was briefly mentioned in the Introduction of this thesis. Lacunarity gives a measure of the degree of translational invariance within a fractal, so it is considered a texture parameter. For a symmetric binary fractal tree, we can associate a fractal dimension based on the scaling ratio for the branches. This is a similarity dimension (discussed in the next subsection). However, the fractal dimension does not fully describe a fractal tree from a topological point of view.

A.2.1 Iterated Function Schemes and Similarity Dimension

A special class of fractals are fractals that are self-similar; they consist of smaller parts that resemble the whole. Here we present some relevant mathematical background. Self-similar fractals are useful because their dimension is easy to calculate. As well, any compact set can be approximated arbitrarily closely by a self-similar set (see below). One way to describe self-similar sets is in terms of iterated function schemes or iterated function systems, which are often referred to as IFS. Here we follow the notation and theory of Falconer [15].

Recall that we defined the notion of a contraction mapping in A.1.1. A similarity is a special kind of contraction where equality in A.1.1 holds. Note that $|\cdot|$ is the Euclidean norm.

Definition A.2.1.1 *Let D be a closed subset of $\mathbb{R}^n$. A mapping $S : D \to D$ is called a* **similarity** *on D if there is a real number c with $0 < c < 1$ such that*

$$|S(x) - S(y)| = c|x - y|, \quad \forall x, y \in D \tag{A.2.1}$$

Definition A.2.1.2 *Let $S_1, ..., S_m$ be contractions. A subset F of D is* **invariant** *for the the family of transformations S_i if*

$$F = \bigcup_{i=1}^{m} S_i(F). \tag{A.2.2}$$

Such invariant sets are often fractals. As an example, consider the two maps $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ given by

$$S_1(x) = \frac{1}{3}x; \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

It is easy to show that these two mappings are similarities, with $c = \frac{1}{3}$ in both cases. The Cantor set C is invariant for the two mappings, and in fact these two mappings represent the self-similarities of the Cantor set.

Definition A.2.1.3 *Iterated function schemes (IFS) are finite sets of contractions.*

IFS define unique, non-empty compact invariant sets, as the following theorem will demonstrate.

Theorem A.2.1.4 *Let $S_1, ...S_m$ be contractions on $D \subset \mathbb{R}^n$ so that*

$$|S_i(x) - S_i(y)| \leq c_i|x - y|$$

with $c_i < 1$ for each i. Then there exists a **unique, non-empty, compact** *set F that is invariant for the S_i. That is, F satisfies*

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Define a transformation S on the class $\mathcal{L}$ of non-empty compact sets by

$$S(E) = \bigcup_{i=1}^{m} S_i(E).$$

Write S^k for the kth iterate of S given by $S^0(E) = E$, $S^k(E) = S(S^{k-1}(E))$ for integers $k \geq 1$. Then

$$F = \lim_{k\to\infty} S^k(E)$$

for any set $E \in \mathcal{L}$ such that $S_i(E) \subset E \ \forall i$.

Definition A.2.1.5 *The* **similarity dimension** *s of an iterated function scheme is the unique value of s defined by:*

$$\sum_{i=1}^{m} c_i^s = 1. \tag{A.2.3}$$

where the c_i are the contractivity factors of the family of contractions that form the IFS.

For example, the Cantor set C has similarity dimension s that satisfies

$$2\left(\frac{1}{3}\right)^s = 1.$$

This is because each interval splits into two intervals that are each one third the length of the original interval. Thus the similarity dimension of C is $\log 2/\log 3$.

One can generalize the traditional Cantor set to obtain fractals with other similarity dimensions. Let $m \geq 2$ be an integer and let λ be such that $0 < \lambda < 1/m$. Let F be the set obtained by the construction in which each basic interval I is replaced by m equally spaced subintervals of lengths $\lambda|I|$, with the endpoints of I coinciding with the endpoints of the extreme subintervals. Then $s = -\log m/\log \lambda$. We now present the Collage Theorem, along with an important and interesting corollary.

Theorem A.2.1.6 *Let $S_1, ...S_m$ be contractions on $\mathbb{R}^n$ and suppose that*

$$|S_i(x) - S_i(y)| \leq c|x - y|, \quad \forall x, y \in \mathbb{R}^n \text{ and } \forall i,$$

where $c < 1$. Let $E \subset \mathbb{R}^n$ be any non-empty compact set. Then

$$d(E, F) \leq d\left(E, \bigcup_{i=1}^{m} S_i(E)\right) \frac{1}{(1-c)}$$

where F is the invariant set for the S_i, and d is the Hausdorff metric.

Corollary A.2.1.7 *Let E be a non-empty compact subset of $\mathbb{R}^n$. Given $\delta > 0$ there exists an integer m and contracting similarities $S_1, ..., S_m$ with invariant set F satisfying $d(E, F) < \delta$.*

In other words: any compact subset of $\mathbb{R}^n$ can be approximated arbitrarily closely by a self-similar set.

The fractal trees that we study are not technically self-similar because they have a residue (the trunk): each tree is the union of two smaller trees that are similar to the tree, along with the trunk. The two smaller trees are similar to the whole tree with contraction factor given by the scaling ratio.

A.2.2 Other Fractal Dimensions

Not all fractals are self-similar. Other fractal dimensions besides the similarity dimension are needed. We first discuss the general notion of dimension, then present two other fractal dimensions. These are the Hausdorff dimension and the box-counting dimension. The first is mathematically rigourous, but often difficult to calculate, while the latter is more useful in applications. In the case of a self-similar fractal without overlap, the Hausdorff, box-counting and similarity dimensions are all equal.

What is dimension? The development of a mathematical concept of dimension was a major undertaking in the nineteenth and early twentieth century. See the first chapter of [14] for a thorough, and entertaining, presentation of the history of topological dimension theory. Dimension theory began as an understanding of the dimensions of the physical world. The previously uncontested notion of physical space being three-dimensional was old and accepted, and the idea of dimension itself had an intuitive basis. There seemed to be no reason to study the character of dimension itself. In 1877, Georg Cantor looked at dimension differently. He was able to show that the points of a 2-dimensional square could be put into one-to-one correspondence with the points of a 1-dimensional line segment. This paradox led to questions such as:

- In what sense is dimension a geometric invariant?
- Can the dimension of a space and the dimension of its image under a mapping be different?

As Mandelbrot states in “The Fractal Geometry of Nature” [30], “a proper understanding of irregularity or fragmentation (as of regularity and connectedness) cannot be satisfied with defining dimension as a number of coordinates.” Mandelbrot distinguishes between topological dimension (denoted D_T) and fractal dimension (denoted D). The more intuitive notion is the topological dimension according to Brouwer, Lebesgue, Menger and Urysohn, see [21], [14] and [30]. The topological dimension is always defined. D_T is always an integer, and it is 0 if the set is totally disconnected, 1 if each point has arbitrarily small neighbourhoods with boundary of dimension 0, and so on.

In the following discussion, we follow the notation and theory of Falconer [15] To begin our presentation of fractal dimensions, we give the Hausdorff(-Besicovitch) dimension.

Definition A.2.2.1 *If U is any non-empty subset of $\mathbb{R}^n$, then the* **diameter** *of U is defined as*

$$|U| = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in U\}.$$

Definition A.2.2.2 *If for some set F and some countable collection $\{U_i\}$ of sets of diameter at most δ we have*

$$F \subset \bigcup_{i=1}^{\infty} U_i \quad \textit{with} \quad 0 < |U_i| \leq \delta, \ \forall i$$

*then we say $\{U_i\}$ is a δ-***cover** *of F.*

Definition A.2.2.3 *Let $F \subset \mathbb{R}^n$ and let s be a non-negative integer. For any $\delta > 0$ we define*

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \textit{ is a } \delta\textit{-cover of } F\right\}.$$

Consider all covers of F by sets of diameter at most δ. As δ decreases, the class of permissable covers is reduced. Thus the infimum $\mathcal{H}^s_\delta(F)$ increases, and so it approaches a limit as $\delta \to 0$.

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

This limit exists for any subset F, but it may be 0 or ∞. $\mathcal{H}^s(F)$ is called the **s-dimensional Hausdorff measure of** *F.*

Definition A.2.2.4 *There is a critical s-value for which the value of $\mathcal{H}^s(F)$ jumps from ∞ to 0, and this value is called the* **Hausdorff dimension** *of F.*

See [15] for more details about the Hausdorff dimension. The Hausdorff dimension is generally difficult to calculate, so for many applications other fractal dimensions are used. We present the dimension most commonly used in applications, namely the box-counting dimension. The widespread use of the box-counting dimension for applications is due to the relative simplicity of the mathematical calculation and empirical estimation of the box-counting dimension.

Definition A.2.2.5 *Let F be any non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. Then we define the* **lower box-counting dimension** *to be*

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad (\textit{the} \lim\ \inf) \tag{A.2.4}$$

and we define the **upper box-counting dimension** *to be*

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad (\textit{the} \lim\ \sup) \tag{A.2.5}$$

If these two values are equal, we refer to the common value as the **box-counting dimension** *of F*

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{A.2.6}$$

The box-counting dimension is useful because there are several equivalent definitions, see [15]. Thus the most suitable definition can be chosen for a given set.

For self-similar fractals without overlap, the Hausdorff, box-counting and similarity dimensions are all equal.

Definition A.2.2.6 *Let $S_1, ..., S_m : \mathbb{R}^n \to \mathbb{R}^n$ be contracting similarities. Then the S_i satisfy the* **open set condition** *if there exists a non-empty, bounded, open set V such that*

$$V \supset \bigcup_{i=1}^{m} S_i(V)$$

with the union disjoint.

Theorem A.2.2.7 *Suppose that the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios c_i. If F is the invariant set satisfying*

$$F = \bigcup_{i=1}^{m} S_i(F)$$

then $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = s$, where s is the similarity dimension as defined in A.2.3 Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$. [15]

Appendix B

Some Details and Calculations for Chapter 3

In this appendix, we provide further details to a few calculations from Chapter 3 of the thesis.

The Coordinates of P_{c1} (see Section 3.3)

To find the coordinates of P_{c1} (so $0° < \theta < 90°$):

$$\begin{aligned}
x_{c1} &= r\sin(\theta) + r^2\sin(0) + r^3\sin(-\theta) + r^4\sin(-2\theta) + ... \\
&\quad +r^{N+1}\sin(-(N-1)\theta) + r^{N+2}\sin(-N\theta)r^{N+3}\sin(-(N-1)\theta) + r^{N+4}\sin(-N\theta) + ... \\
&= r\sin(\theta) - \left[\sum_{k=1}^{N-2} r^{k+2}\sin(k\theta)\right] \\
&\quad -r^{N+1}\sin((N-1)\theta)\left[1 + r^2 + r^4 + ...\right] - r^{N+1}r\sin(N\theta)\left[1 + r^2 + r^4 + ...\right] \\
&= r\sin(\theta) - \sum_{k=1}^{N-2} r^{k+2}\sin(k\theta) - \frac{r^{N+1}}{1-r^2}\left[\sin((N-1)\theta) + r\sin(N\theta)\right] \qquad \text{(B.0.1)}
\end{aligned}$$

and

$$\begin{aligned}
y_{c1} &= 1 + r\cos(\theta) + r^2\cos(0) + r^3\cos(-\theta) + r^4\cos(-2\theta) + ... \\
&\quad +r^{N+1}\cos(-(N-1)\theta) + r^{N+2}\cos(-N\theta) + \\
&\quad r^{N+3}\cos(-(N-1)\theta) + r^{N+4}\cos(-N\theta) + ... \\
&= 1 + r\cos(\theta) + r^2 + \left[\sum_{k=1}^{N-2} r^{k+2}\cos(k\theta)\right] \\
&\quad +r^{N+1}\cos((N-1)\theta)\left[1 + r^2 + r^4 + ...\right] \\
&\quad +r^{N+1}r\cos(N\theta)\left[1 + r^2 + r^4 + ...\right] \\
&= 1 + r\cos(\theta) + r^2 + \sum_{k=1}^{N-2} r^{k+2}\cos(k\theta) \\
&\quad +\frac{r^{N+1}}{1-r^2}\left[\cos((N-1)\theta) + r\cos(N\theta)\right] \qquad \text{(B.0.2)}
\end{aligned}$$

The coordinates of P_{c2} (see Section 3.3)

To find the coordinates of P_{c2} (so $90 < \theta \leq 135°$):

$$\begin{aligned} x_{c2} &= r\sin(\theta) + r^2\sin(2\theta) + r^3\sin(3\theta) + r^4\sin(2\theta) + r^5\sin(3\theta) + \cdots \\ &= r\sin(\theta) + \frac{r^2}{1-r^2}[\sin(2\theta) + r\sin(3\theta)] \qquad \text{(B.0.3)} \end{aligned}$$

and

$$\begin{aligned} y_{c2} &= 1 + r\cos(\theta) + r^2\cos(2\theta) + r^3\cos(3\theta) + r^4\cos(2\theta) + r^5\cos(3\theta) + \cdots \\ &= 1 + r\cos(\theta) + \frac{r^2}{1-r^2}[\cos(2\theta) + r\cos(3\theta)] \qquad \text{(B.0.4)} \end{aligned}$$

Self-contacting scaling ratio in second angle range

To find r_{sc} in the second angle range, we set $x_{c2} = 0$. Using trigonometric identities for $\sin(2\theta)$ and $\sin(3\theta)$, we have

$$\begin{aligned} & r\sin\theta + \frac{r^2}{1-r^2}[\sin(2\theta) + r\sin(3\theta)] = 0 \\ \Rightarrow\quad & (1-r^2)\sin\theta + r\sin(2\theta) + r^2\sin(3\theta) = 0 \\ \Rightarrow\quad & (1-r^2)\sin\theta + 2r\sin\theta\cos\theta + r^2\sin\theta[3\cos^2\theta - \sin^2\theta] = 0 \\ \Rightarrow\quad & 1 - r^2 + 2r\cos\theta + r^2[3\cos^2\theta - \sin^2\theta] = 0 \\ \Rightarrow\quad & 1 - r^2 + 2r\cos\theta + r^2[4\cos^2\theta - 1] = 0 \\ \Rightarrow\quad & r^2[4\cos^2\theta - 2] + r[2\cos\theta] + 1 = 0 \end{aligned}$$

When $\theta = 135°$, this reduces to $-\sqrt{2}r_{sc} + 1 = 0$, and thus $r_{sc} = 1/\sqrt{2}$. When $90° < \theta < 135°$, we solve the quadratic equation to obtain

$$r_{sc} = \frac{-\cos\theta - \sqrt{2 - 3\cos^2\theta}}{4\cos^2\theta - 2}$$

Proof of Proposition 3.4.0.7

Proposition. For angles θ such that $90° < \theta < 135°$, the maximal height of the self-contacting tree $T(r_{sc}, \theta)$ is greater than 0, and as a result, the line $y = y_{max}$ is above the endpoint of the trunk. When $\theta = 135°$, the maximal height of the self-contacting tree $T(r_{sc}, 135°)$ is 0, the line $y = y_{max}$ is the line $y = 0$, and the tip points at addresses of the form $\mathbf{A}$ where $\mathbf{A} \in \mathcal{AL}_\infty$ are also on the line $y = 0$.

Proof. Recall from (3.3.8) that

$$r_{sc} = \frac{-\cos\theta - \sqrt{2 - 3\cos^2\theta}}{4\cos^2\theta - 2}$$

It suffices to show that $r_{sc} > -\cos\theta$ for $90° < \theta < 135°$. Note that in this angle range we have $0 < -\cos\theta < 1/\sqrt{2}$.

$$
\begin{aligned}
& r_{sc} > -\cos\theta \\
\Rightarrow\quad & \frac{-\cos\theta - \sqrt{2-3\cos^2\theta}}{4\cos^2\theta - 2} > -\cos\theta \\
\Rightarrow\quad & \frac{-\cos\theta}{4\cos^2\theta - 2} > -\cos\theta + \frac{\sqrt{2-\cos^2\theta}}{4\cos^2\theta - 2} \\
\Rightarrow\quad & \frac{-\cos\theta}{4\cos^2\theta - 2} > \frac{-\cos\theta}{4\cos^2\theta - 2}\left[4\cos^2\theta - 2 - \frac{\sqrt{2-3\cos^2\theta}}{\cos\theta}\right] \\
\Rightarrow\quad & 1 > 4\cos^2\theta - 2 - \frac{\sqrt{2-3\cos^2\theta}}{\cos\theta} \\
\Rightarrow\quad & \frac{\sqrt{2-3\cos^2\theta}}{\cos\theta} > 4\cos^2\theta - 3 \\
\Rightarrow\quad & \sqrt{2-3\cos^2\theta} > (4\cos^2\theta - 3)\cos\theta \\
\Rightarrow\quad & 2 - 3\cos^2\theta > (4\cos^2\theta - 3)^2\cos^2\theta \\
& (\text{ that is valid for this particular angle range}) \\
\Rightarrow\quad & 16\cos^6\theta - 24\cos^4\theta + 12\cos^2\theta - 2 < 0
\end{aligned}
$$

Now let $x(\theta) = \cos^2\theta$ and consider the function $f(x) = 16x^3 - 24x^2 + 12x - 2$ on $(0, 1/2)$. $\cos^2\theta$ is monotonically increasing on $(90°, 135°)$. $f(x)$ is also increasing, because $f'(x) = 48x^2 - 48x + 12 = 12(2x-1)^2 > 0$ on $(0, 1/2)$. Thus $f(x) < f(1/2)$ for all $x \in (0, 1/2)$. $f(1/2) = 0$ and hence we have $16\cos^6\theta - 24\cos^4\theta + 12\cos^2\theta - 2 < 0$ as required.

When $\theta = 135°$, $r_{sc} = -1/\sqrt{2} = \cos 135°$, and $y_{max} = 0$. The height of the tip points at addresses of the form $\mathbf{A}$ where $\mathbf{A} \in \mathcal{AL}_\infty$ are all 0, and thus they are on the line $y = 0$.

Coordinates of P_{c3}

To find the coordinates of P_{c3} (so $135° < \theta$):

$$\begin{aligned}
x_{c3} &= r\sin(\theta) + r^2\sin(2\theta) + r^3\sin\theta + r^4\sin(2\theta) + \cdots \\
&= \frac{r}{1-r^2}[\sin(\theta) + r\sin(2\theta)] \\
&= \frac{r}{1-r^2}[\sin(\theta) + r2\sin\theta\cos\theta] \\
&= \frac{r\sin(\theta)}{1-r^2}[1 + 2r\cos\theta] \qquad \text{(B.0.5)}
\end{aligned}$$

and

$$\begin{aligned}
y_{c3} &= 1 + r\cos(\theta) + r^2\cos(2\theta) + r^3\cos\theta + r^4\cos(2\theta) + \cdots \\
&= 1 + \frac{r}{1-r^2}[\cos(\theta) + r\cos(2\theta)] \qquad \text{(B.0.6)}
\end{aligned}$$

Appendix C

List of Notation

In this appendix, we provide a list of notation for the thesis. This list contains the new notations that we have introduced along with some other relevant symbols, though we have not included every symbol used. For some symbols we have included a reference to the definition.

M_{LR}	free monoid on two generators
r	scaling ratio
θ	branching angle
m_L, m_R	generator maps, see 2.1.1 and 2.1.2
$\mathcal{A}_k$	set of addresses of length k
$\mathcal{A}_\infty$	set of infinite addresses
$\bar{\mathcal{A}}$	set of all addresses
$m_{\mathbf{A}}$	address map defined by $\mathbf{A}$
$\sim_r$, $\sim_r^k$	r-similarity relation, see 2.1.3.3
T_0	trunk
$b(\mathbf{A})$	branch equal to $m_{\mathbf{A}}(T_0)$
$P_{\mathbf{A}}$	point with address $\mathbf{A}$
$\mathrm{Tip}(r,\theta)$	collection of points with infinite addresses
*	mirror image
$T_k(r,\theta)$	level k approximation tree
$T(r,\theta)$	symmetric binary fractal tree with scaling ratio r and branching angle θ
$\mathcal{T}$	collection of all symmetric binary fractal trees
$lin(\mathbf{A})$	linear extension of the branch $b(\mathbf{A})$
$p_{\mathbf{A}}(\mathbf{A}')$	path starting at $b(\mathbf{A}')$ defined by $\mathbf{A}$
$\mathcal{AL}_{2k}$	alternating addresses of level $2k$, see 2.3.5
$\mathcal{AL}_\infty$	alternating infinite addresses, see 2.3.6
$S_{\mathbf{A}}(r,\theta)$	subtree equal to $m_{\mathbf{A}}(T(r,\theta))$

$N(\theta)$	turning number of θ, see 3.0.1.1
θ_N	special angle equal to $90°/N$
$h(r,\theta)$, h	height of tree $T(r,\theta)$
$w(r,\theta,$ w	width of tree $T(r,\theta)$
$y_{\max}(r,\theta)$, $y_{\max}$	maximal y-value of $T(r,\theta)$
$y_{\min}(r,\theta)$, $y_{\min}$	minimal y-value of $T(r,\theta)$
$x_{\max}(r,\theta)$, $x_{\max}$	maximal x-value of $T(r,\theta)$
$BR(r,\theta)$, BR	bounding rectangle of $T(r,\theta)$
$\mathcal{T}_{sa}$	collection of self-avoiding symmetric binary fractal trees
$\mathcal{T}_{sc}$	collection of self-contacting symmetric binary fractal trees
$\mathcal{T}_{so}$	collection of self-overlapping symmetric binary fractal trees
$O(\gamma)$	inside of a curve γ
$\Gamma(r,\theta)$	collection of simple closed curves of $T(r,\theta)$
$T^C(r,\theta)$	complement of $T(r,\theta)$ in $\mathbb{R}^2$
$r_{sc}(\theta)$, r_{sc}	unique scaling ratio for θ that yields a self-contacting tree
$\mathbf{y}$	the y-axis
$\mathbf{y}_I$	subset of $\mathbf{y}$ where $y \in I$
$\mathbf{A}_c$	contact address, see Table 3.1
$P_{c1} = (x_{c1}, y_{c1})$	point with address $RL^{N+1}(LR)^\infty$
$P_{c2} = (x_{c2}, y_{c2})$	point with address $R^3(LR)^\infty$
$P_{c3} = (x_{c3}, y_{c3})$	point with address RR
$\mathbf{A}_s$	secondary contact address, see Table 3.2
$\mathbf{C}_R$	address $RL(LR)^\infty$ (for right endpoint of degree 0 canopy interval)
$\mathbf{C}_L$	address $LR(RL)^\infty$ (for left endpoint of degree 0 canopy interval)
I_{tc}	degree 0 top canopy interval
$\mathcal{I}_{tc}$	collection of top canopy intervals

$\overline{U_\epsilon}$	closed ϵ-neighbourhood of U
$\partial\overline{U_\epsilon}$	boundary of the closed ϵ-neighbourhood of U
$\overline{U_\infty}$	closed ϵ-neighbourhood of U at ∞
$E(r,\theta,\epsilon)$, $E(\epsilon)$, E	closed ϵ-neighbourhood of $T(r,\theta)$
$\partial E(r,\theta,\epsilon)$, $\partial E(\epsilon)$, ∂E	boundary of closed ϵ-neighbourhood of $T(r,\theta)$
$E_{\mathbf{A}}(r,\theta,\epsilon)$, E_S	closed ϵ-neighbourhood of subtree $S = S_{\mathbf{A}}$
$E^C(r,\theta,\epsilon)$, E^C	complement of closed ϵ-neighbourhood of $T(r,\theta)$
$\Gamma(r,\theta,\epsilon)$	simple closed curves of $E(r,\theta,\epsilon)$
$\mathcal{H}(r,\theta,\epsilon)$	collection of holes of $E(r,\theta,\epsilon)$
$N(r,\theta,\epsilon)$, $N(\epsilon)$	number of holes in $\mathcal{H}(r,\theta,\epsilon)$
$[H]$	hole class of a hole H
$p(H)$, $p([H])$	persistence interval of H
$\|p(H)\|$, $\|p([H])\|$	persistence of H
$\underline{\epsilon_H}$, $\underline{\epsilon_{[H]}}$	contact value for $[H]$
$\overline{\epsilon_H}$, $\overline{\epsilon_{[H]}}$	contact value for $[H]$
$[H]_{max}$	maximal hole of $[H]$
$\mathcal{H}_k(r,\theta,\epsilon)$	level k holes of $\mathcal{H}(r,\theta,\epsilon)$
H_{-k}	level 0 ancestor of a level k hole H
$C(H)$, $C([H])$	complexity of $[H]$
P_{top}, P_{bot}	top and bottom point of $[H]$, see 4.6.2.3
P_{TOP}, P_{BOT}	corresponding points on the tree, see 4.6.2.4
$(\mathbf{A}_{TOP}, \mathbf{A}_{BOT})$	hole locator pair, see 4.6.2.8
$\mathcal{HL}(r,\theta)$	hole location set of (r,θ)
$\mathcal{HL}(\theta)$	hole location set of θ
$\sim_{Loc}$	location relation
$\mathcal{TY}(r,\theta)$	type set of (r,θ)
$\mathcal{TY}(\theta)$	type set of θ
$\sim_{Type}$	type relation
$AR(\mathbf{A}_1,\mathbf{A}_2)$	angle range of the pair $(\mathbf{A}_1,\mathbf{A}_2)$

$Con(r, \theta)$	set of ϵ contact values for $T(r, \theta)$
$Col(r, \theta)$	set of ϵ collapse values for $T(r, \theta)$
$Crit(r, \theta)$	set of critical ϵ values for $T(r, \theta)$
$\sim_{r,\theta}$	hole congruence relation of the pair (r, θ)
$\mathcal{HP}(r, \theta)$	hole partition of the pair (r, θ)
$\{N_\epsilon(r, \theta)\}$	hole sequence of the pair (r, θ)
$\sim_{HS}$	hole sequence relation
$LS(r, \theta, \epsilon)$	level set of ϵ for $T(r, \theta)$
$LR(r, \theta, \epsilon)$	level range of ϵ for $T(r, \theta)$
$C(r, \theta)$	complexity of the tree $T(r, \theta)$
$\sim_\theta$	complexity relation on scaling ratios
$C_k(\theta)$	k-complexity class for θ
$\sim_C$	complexity relation on pairs (r, θ)
ϕ	golden ratio

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